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ABSTRACT

This two-volume teacher's guide is to be used with a second-semester, third-grade mathematics program. The program is in large part the outgrowth of involvement with children and was influenced by information gathered in an extended pilot project conducted nationwide during 1975-79. The manner of presentation and the pedagogical ideas and tools are based on the works of Georges and Frederique Papy. They are recognized as having introduced colored arrow drawings ("papygrams") and models of our numeration system (the Papy "minicomputer"). This two-volume set contains five sections: (1) General Introduction; (2) World of Whole Numbers; (3) Languages of Strings and Arrows; (4) Geometry and Measurement; and (5) Workbooks. Each section has its own table of contents. A series of lesson plans include capsule lesson summaries, lists of materials needed for both teacher and student, and lesson descriptions which suggest possible classroom dialogue and indicate the use of worksheets and manipulative materials. (Author/MK)

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prehensive School Mathematics Program
CEMREL, Inc.
Final Experimental Version
3-44101

CSMP Mathematics for the Upper Primary Grades

Part IV

Teacher's Guide

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General Introduction
The World of Numbers

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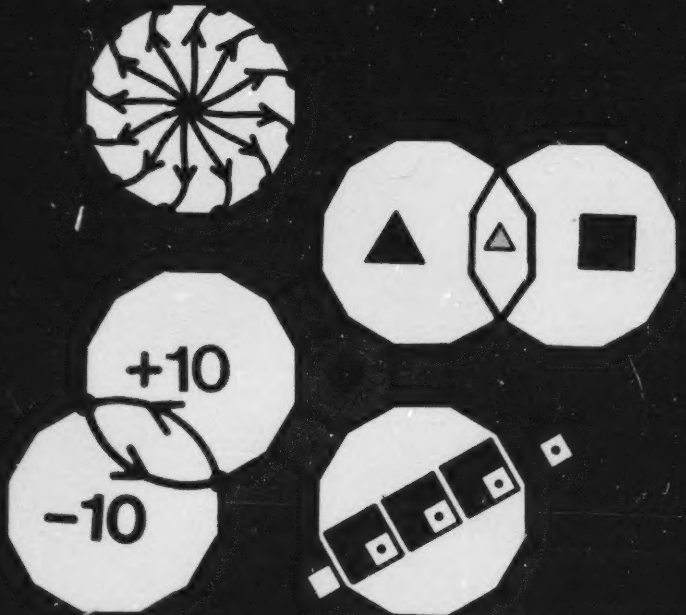

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CEMREL, Inc.
Comprehensive School Mathematics Program
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Master Schedule for the CSMP Upper Primary Grades, Part IV

Week	World of Numbers	Languages of Strings and Arrows	World of Numbers	Workbooks	Geometry and Measurement
1	N1 Subtraction Problems #1 N p. 1	L1 Permutations and Common Multiples L p. 1	N2 Composition of Functions #1 N p. 6	W1 GALAXY OF PROBLEMS #1 (Lesson 1) W p. 1	ADJUSTMENT DAY
2	N3 Addition of Decimals N p. 21	L2 A Detective Story L p. 8	N4 Subtraction and Composition #1 N p. 31	W2 GALAXY OF PROBLEMS #1 (Lesson 2) W p. 5	G1 Taxi-Circles G p. 1
3	N5 One-Third of 200 N p. 42	L3 The String Game #1 L p. 17	N6 An Arrow Road with 10× and +1 N p. 53	W3 Who is Ply? W p. 19	G2 Fractional Parts of Shapes and Area G p. 9
4	N7 Subtraction Problems #2 N p. 60	L4 Common Multiples/ Composition Games L p. 19	N8 Roads #1 N p. 66	W4 GALAXY OF PROBLEMS #2 (Lesson 1) W p. 29	G3 Concentric Taxi-Circles G p. 21
5	N9 Hand-Calculator Games #1 N p. 69	L5 DANCING FRIENDS (Lesson 1) L p. 28	N10 Return Arrows #1 N p. 76	W5 GALAXY OF PROBLEMS #2 (Lesson 2) W p. 30	ADJUSTMENT DAY
6	N11 Subtraction Problems #3 N p. 83	L6 DANCING FRIENDS (Lesson 2) L p. 34	N12 Division Problems #1 N p. 92	W6 Who are Kit and Kat? W p. 45	G4 Trees G p. 33
7	N13 Subtraction Problems #4 N p. 103	L7 Three Boxes #1 L p. 48	N14 Roads #2 N p. 115	W7 TO PICTURE (Lesson 1) W p. 59	G5 Length in Centimeters and Decimeters G p. 44
8	N15 Minicomputer Golf #1 N p. 119	L8 Basketball Bar Graphs L p. 61	N16 Subtraction and Composition #2 N p. 125	W8 TO PICTURE (Lesson 2) W p. 64	G6 Taxi-Distance G p. 55
9	N17 Subtraction Algorithm #1 N p. 131	L9 The String Game #2 L p. 65	N18 **Division Problems** #2 N p. 143	W9 GALAXY OF PROBLEMS #3 (Lesson 1) W p. 71	ADJUSTMENT DAY

10	N19 Decimals on the Number Line N p. 151	L10 I AM NOT MY NAME L p. 78	N20 Return Arrows #2 N p. 164	W10 GALAXY OF PROBLEMS #3 (Lesson 2) W p. 75	G7 Touring Houses #1 G p. 64
11	N21 Subtraction Algorithm #2 N p. 170	ADJUSTMENT DAY	N22 Composition of Functions #2 N p. 176	W11 Spinner Game Detective Story W p. 85	G8 Length of Spirals G p. 71
12	N23 Comparing Prices #1 N p. 185	L11 Multiplication Modulo 10 #1 L p. 84	N24 Composition of Functions #3 N p. 198	W12 GALAXY OF PROBLEMS #4 (Lesson 1) W p. 96	G9 Touring Houses #2 G p. 81
13	N25 Subtraction Algorithm #3 N p. 206	L12 Talkative Numbers #1 L p. 94	N26 Roads #3 N p. 211	W13 GALAXY OF PROBLEMS #4 (Lesson 2) W p. 99	ADJUSTMENT DAY
14	N27 Mystery Arrows N p. 217	L13 Who is Max? L p. 102	N28 Division Problems #3 N p. 225	W14 Who are Zin and Zan? W p. 109	G10 Rectangles of a Given Area G p. 91
15	N29 Multiplication Problems N p. 232	L14 Three Boxes #2 L p. 112	N30 Division Problems #4 N p. 241	ADJUSTMENT DAY	G11 Rectangles of a Given Perimeter G p. 102
16	N31 Hand-Calculator Games #2 N p. 249	L15 The String Game #3 L p. 122	N32 Division Problems #5 N p. 256	W15 FISHING FOR NUMBERS (Lesson 1) W p. 116	G12 Touring Houses #3 G p. 113
17	N33 Comparing Prices #2 N p. 261	L16 Talkative Numbers #2 L p. 129	N34 Roads with Cords N p. 271	W16 FISHING FOR NUMBERS (Lesson 2) W p. 119	ADJUSTMENT DAY
18	N35 Minicomputer Golf #2 N p. 278	L17 Multiplication Modulo 10 #2 L p. 137	ADJUSTMENT DAY	W17 A Short Story about Three Friends W p. 133	G13 Pencil Tracings G p. 122

MASTER SCHEDULE FOR THE CSMP UPPER PRIMARY GRADES, PART IV (continued)

PREFACE

CSMP Mathematics for the Upper Primary Grades (UPG) owes its existence to the coming together under the leadership of Burt Kaufman (CSMP Director) of a special group of people who have been concerned not only about the kind of mathematics that is being taught to children but also about the manner in which it is taught. This is even more important in the crucial early years of a child's life than in later years.

The CSMP elementary school curriculum is in large part the outgrowth of a great deal of active involvement with children in ordinary classroom settings, and its present form was strongly influenced by information gathered from classes participating in an Extended Pilot Test conducted nationwide during 1975-79. But its manner of presentation and its main pedagogical ideas and tools, which stand out from all other curricula, are based on the pioneering works of Georges and Frédérique Papy acknowledged here with deep gratitude.

Use of colored arrow diagrams[†] -

Georges Papy, Die Erste Elemente der Moderne Mathematik: Shriftenreihe zur Mathematik, Hefte 10-11 (Frankfurt and Hamburg: Otto Saile Verlag, 1962-63)
Mathématique Moderne, Vols. 1, 2, 3, 5, 6 (Brussels and Paris: Didier, 1964-67)
Modern Mathematics 1, 2 (London: Collier-Macmillan Ltd., 1968)

Frédérique and Georges Papy, L'Enfant et les Graphes (Brussels, Montreal and Paris: Didier, 1968)
Graphs and the Child (Montreal: Algonquin, 1970

[†] In 1959, Georges Papy introduced colored arrow diagrams into the teaching of mathematics at the elementary and the secondary school level. For this reason, they are sometimes referred to as "papygrams" in his honor.

Les Enfants de 4 Ans et le Langage des Graphes (Paris: Classiques Hachette, 1973)

Origin of the Papy Minicomputer -

Georges Papy, Minicomputer (Brussels: IVAC, 1968) in collaboration with Frédérique Papy

Use of graphs and the Papy Minicomputer with young children -

Frédérique Papy, Les Enfants et la Mathématique[†], Vols. 1, 2, 3, 4 (Brussels, Montreal and Paris: Didier, 1970-1976)

Mathematics and the Child 1, (Montreal: Algonquin, 1971)

By making this wealth of experience available to CSMP, Frédérique and Georges Papy have made an invaluable contribution to the production of this program. It must also be acknowledged that the fundamental idea of presenting mathematics to children via "pedagogical situations" is due to the mathematician and psychologist Caleb Gattegno.

Various prototype versions of the UPG Program dating back to 1974 were due to the efforts of former CSMP staff members José Agusti, Phyllis Klein, Edward Martin, and Christiane Vandeputte, and staff members Burt Kaufman and Frédérique Papy.

The large amount of information gleaned, thanks to the interest and concern of many teachers, in the Extended Pilot Tests since 1975 led to the revision of the program that culminated in this final version. During the 1977-78 school year, this final version of Part IV was tried out by CSMP staff members Clare Heidema and Janis Schweitzer in West Richmond Elementary School, Maplewood-Richmond Heights School District, St. Louis, Missouri. In this effort they received valued collaboration and support of third grade teacher Dorothy Daily and the motivating responses of her students.

[†] A series of books reporting on an experiment in teaching mathematics in grades 1 through 6 in Brussels, using the languages of strings, arrows, and the Papy Minicomputer. The reports of the first four years have been published by Didier; the reports of the last two years exist only in manuscript.

NOTE ON AUTHORSHIP

This printed copy of the final experimental version of CSMP Mathematics for the Upper Primary Grades, Part IV, is the result of the work of the following people:

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GENERAL INTRODUCTION

Comprehensive School Mathematics Program

CSMP Mathematics
for the
Upper Primary Grades
Part IV

General Introduction

CEMREL, Inc.

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SECTION ONE

NOTES TO THE TEACHER

WELCOME

You are about to teach the fourth semester (Part IV) of CSMP Mathematics for the Upper Primary Grades (grades 2 and 3), perhaps for the first time. We wish you success in your teaching and hope that both you and your students will enjoy the activities suggested in this teacher's guide. It is the purpose of these Notes to the Teacher to help you use this book effectively, to describe the materials and to give suggestions for organizing the program, and to answer some frequently asked questions which may also be your questions about CSMP.

You will find, as you and your students take part in the activities during the year, that there is a definite CSMP viewpoint about elementary school mathematics, a viewpoint that we hope you will come to understand and to share. We believe that learning takes place when children react to interesting situations (sometimes called a "pedagogy of situations"[*]), whether real life or fantasy, such as in stories or games. These situations involve children personally and allow arithmetic to take the form of "adventures in the world of numbers."

We believe that every child can and does learn something from each situation encountered in the program, sometimes suddenly and dramatically, sometimes latently. For this reason, CSMP teachers do not stick to one topic until it is "mastered". Indeed our experience indicates that true learning often stops when the purpose of a lesson is merely the mastery of a skill. Instead, we believe that learning is more a "spiral" process than a "straight line" process, that intuitive leaps play as big a role as acquiring small successive pieces of information, and that a variety of situations can provide the interrelated experiences through which we learn.

The Question and Answer section of these Notes to the Teacher gives more explanation of what we mean by a "pedagogy of situations" and a "spiral development", but you will probably learn their meanings only by experiencing them yourself as you teach the curriculum. Meanwhile we assure you that a large number of CSMP teachers across the country have come to understand and embrace the CSMP style of teaching and learning and are greatly pleased with the results.[9]

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[*] This phrase is due to the mathematician and psychologist, Caleb Gattegno.

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[9] See evaluation Report 4-A-1 in the CSMP Evaluation Report Series, CEMREL, Inc., St. Louis, Missouri.

PROGRAM DESCRIPTION

CSMP Mathematics for the Upper Primary Grades (UPG) is designed as a program for students in second and third grades. There are four parts to the curriculum, each corresponding to one semester's work. This is Part IV of the four.

Since the learning process occurs through many interrelated experiences, we have developed this curriculum so that no single lesson is an end in itself. It is neither intended nor expected that every student will meet the full challenge of a particular situation in any given lesson. Furthermore, we have found that it is most effective to vary the situations from day to day rather than to continue one type of situation or topic until so-called mastery has occurred. This spiral development (see also Question 1 of Section Two) is reflected in the materials and scheduling of various topics in the curriculum. It is consistent with CSMP's pedagogy of situations since with such an approach, a child experiences each of several ideas a little at a time and then proceeds through increasing levels of sophistication as the situations become more challenging. We hope you will keep in mind the spiral approach as you teach the CSMP curriculum.

1. Description of Materials

a) Teacher's Guide

This set of two volumes (five books) which is directed specifically to you, the teacher, is known as the "Teacher's Guide to CSMP Mathematics for the Upper Primary Grades, Part IV". (By way of abbreviation, in what follows we shall refer to this collection simply as the "Teacher's Guide".) With the exception of this book, "General Introduction", each book of the Teacher's Guide addresses itself to a single area of the curriculum, and each of these areas is known as a "strand". The four strands with abbreviations given in parentheses are:

- World of Numbers - (N)

- Languages of Strings and Arrows - (L)
- Geometry and Measurement - (G)
- Workbooks - (W)

The five books comprising the Teacher's Guide are bound into two volumes, with a separation page inserted between the various books of a given volume. This volume contains both the General Introduction and the World of Numbers strand. The other volume contains the Languages of Strings and Arrows strand, the Geometry and Measurement strand, and the Workbooks strand. Each of these five books has its own table of contents and is paginated individually.

The strands have the following arrangement of material.

- A table of contents. This enables you to locate whatever you need in that particular strand.
- An introduction to the strand. This provides you with information explaining why certain material finds its place in that strand, what are the overall goals of study in that strand, and how this material fits in with that which is described in other strands.
- A series of lesson plans. These plans include Capsule Lesson Summaries, lists of needed materials for both teacher and student, and lesson descriptions which suggest possible classroom dialogue and indicate the use of worksheets and manipulative materials. The lesson descriptions also provide illustrations of diagrams which the teacher puts on the board or otherwise displays for the class and answer keys for worksheets or other individual work during the lessons.

In each of the strand books, N, L, G, and W, the lessons are consecutively numbered. Each lesson is designated to be used during one full class period. For example, Lesson N8 is the eighth lesson in the N (World of Numbers) strand. Since this lesson is surrounded by lessons from other strands, it might be taught on, say, Wednesday of the fourth week of the term and the next N lesson (N9) might be taught on Monday of the fifth week.

Each lesson is presented in a form that includes its strand and number, title, summary, materials and description. For example, here is the first page from the first lesson in the N (World of Numbers) strand.

Strand abbreviation and number

Title of lesson - indicates this is the first lesson in a series on subtraction

N1 Subtraction Problems #1

CAPSULE LESSON SUMMARY

Short summary of lesson

On the Minicomputer add positive numbers to negative numbers and write the corresponding addition and subtraction number stories.

MATERIALS

Teacher : Minicomputer set

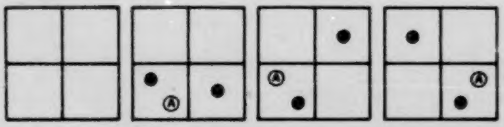
Material teacher uses for demonstration

Student : Minicomputer set* ; Worksheets N1*, **, *** and ****; paper ; pencil

Materials each student needs for individual work

DESCRIPTION OF LESSON

Invite students to put 369 and $\widehat{221}$ on the Minicomputer. Write "$369 + \widehat{221}$" to one side of the Minicomputer.



$= 369 + \widehat{221}$

T: What number is $369 + \widehat{221}$? Write it on a piece of paper.

Check many of the students' answers before asking a student to answer aloud.

S: $369 + \widehat{221} = 148$.

Write this number story on the board.

Detailed description of presentation

$$369 + \widehat{221} = 148$$

---

[*] A student's Minicomputer set consists of two individual Minicomputer sheets (each having two boards) and a sheet of individual checkers.

The organization of lessons in the teacher's guide for UPG-IV, as for UPG-I, II, and UPG-III, is a strand-to-strand presentation. By this we mean that on one day of the week a lesson is taught from a certain strand, and then on the next day of the week a lesson is taught from another strand, and so on. In this way we can achieve a spiraling effect and at the same time gain the benefit of organization of topics according to strands. For example, suppose you are interested in the way a topic in geometry develops in the third grade. With a day-to-day format you would need to search through the entire guide for geometry lessons, but with a strand-to-strand format all these lessons are found, in their sequential order, in one part of the guide, the G-strand. You can then read the lessons in their natural order to gain an overview of the Geometry strand in third grade (second semester). At the same time, the strand-to-strand arrangement allows you to anticipate what is coming on the next "spoke" of the spiral development of any topic simply by reading ahead in the strand book in which that topic is developed.

b) Teacher's Materials

In addition to the teacher's guide, there are some demonstration manipulative materials you will need. The following materials are available from CSMP and should be in your classroom set of materials.

- Demonstration Minicomputer Set
  (4 magnetic Minicomputer boards; 1 set of magnetic checkers)
- A-Block String Game Kit
  (2 sets of A-Blocks; 2 attribute posters; 2 sets of string game cards; magnetic material)
- Classroom Number Line
- UPG-IV Poster Set
  (One Language poster, and six Geometry posters)
- Poster Grid Sheets

- Three Geometry Transparencies
- One set of ⑩-checkers

Other materials needed for individual lessons are listed under MATERIALS at the beginning of each lesson. Usually such materials (those not in your classroom set) are readily available or can be easily prepared from usual classroom supplies.

c) Student's Materials

There are some individual materials that each student (each pair of students in the case of storybooks) will need. The following materials are available from CSMP and should be in your classroom set of materials.

- Worksheet Booklet for UPG-IV (with perforated pages which can be torn off sheet by sheet as needed)
- Workbooks
  (five titles: "Galaxy of Problems #1", "Galaxy of Problems #2", "Galaxy of Problems #3", "Galaxy of Problems #4", and "Fishing for Numbers (Part IV))
- Story-Workbook from the CSMP LIBRARY ("To Picture")
- Storybooks from the STORIES BY FREDERIQUE Collection ("Dancing Friends" and "I Am Not My Name")
- Subtraction Problems Booklet
- Individual Minicomputer Set
  (2 sheets of Minicomputer boards; 1 sheet of punch-out checkers)
- Basketball Data and Charts
- Metric Tape Measure

Other materials needed for a lesson are listed under MATERIALS at the beginning of the lesson. Materials not available from CSMP are readily available or can be easily prepared from usual classroom supplies.

Some teachers find it convenient to provide each student with a sturdy cardboard box for storing manipulative materials. Others find it more convenient to store all the materials of one kind together in a container of appropriate size and to distribute these materials to the students as they are needed. In any case, you should probably keep the worksheet booklets, workbooks, and storybooks in your storage area and distribute them as called for in the lessons.

Some storybooks in the STORIES BY FREDERIQUE collection are provided in your classroom set of materials although they are not used in any lessons of the UPG-IV Program. There are two copies of each such storybook; we suggest that these be available in a reading corner for students to read whenever they wish.

2. Role and Use of Worksheets, Workbooks, and Storybooks

a) Worksheets

With some lessons there are worksheets for individual work. Worksheets are designated with numbers corresponding to the lessons with which they are used and with stars indicating the order in which they are to be done by the students. The stars also suggest an increasing level of difficulty and, thereby, provide for various abilities of students. IMPORTANT: It is imperative that you do not insist on every student finishing every worksheet before going on to the next lesson. The worksheets are designed to accommodate differences in students' interests and abilities. Some students will finish more worksheets than others during the allotted time. But the same ideas emerge in new settings in later lessons, so that students will have other opportunities to work on the same topics.

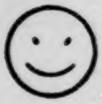
b) Workbooks and Story-Workbooks

Part IV of the CSMP Upper Primary Grades Program (UPG-IV) includes five workbooks titled: "Galaxy of Problems #1", "Galaxy of Problems #2", "Galaxy of Problems #3", "Galaxy of Problems #4", and "Fishing for Numbers" (Part IV) and one story-workbook, "To Picture". Each workbook is a thirty-two page booklet that reviews problems which have been encountered up to that point in the curriculum. Two lessons are scheduled for each workbook, and we suggest that all students be instructed to start at the beginning of the booklet and to work independently through as many pages as they can during the scheduled periods.

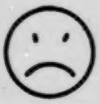
Ordinarily the first ten pages of a workbook are easy enough to guarantee some success for all the students; the next ten to twelve pages are average level difficulty; and the remaining pages are more difficult, often challenging even your best students. We would estimate that about two-thirds of your students should correctly finish the first ten pages during the two scheduled lessons, one-third should finish the first twenty pages, and a few might finish all or most of the pages. Of course, these proportions will vary from class to class.

Two full class periods are devoted to each workbook. At the end of the first period, you should collect the workbooks and check some or all of the pages completed. Then when you return the workbooks at the beginning of the next workbook period, the students can correct their mistakes before going on with new problems.

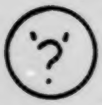
Notice that the cover page of each workbook includes spaces for marking your evaluation of the work on each page. One way to do this is to use these symbols:



Meaning "well done" or "correct"

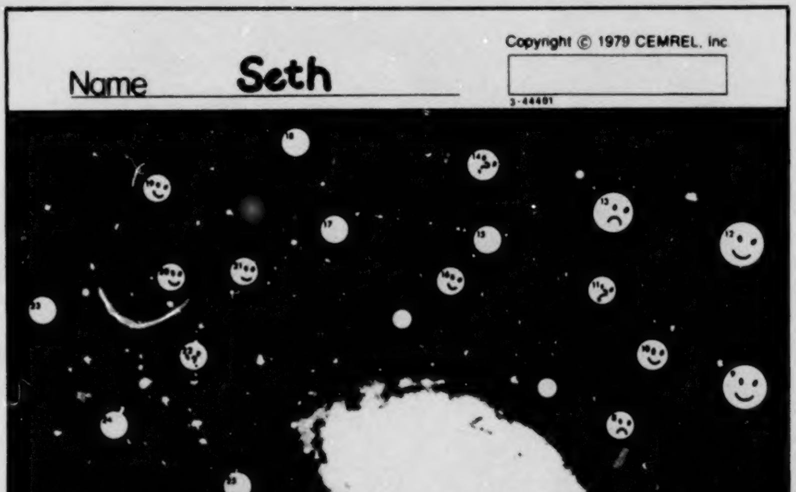


Meaning "something wrong, try again"



Meaning "did you forget this problem?" or "do you need a hint?"

Here is the front cover of one student's "Galaxy of Problems #1" workbook after it was checked following the first lesson devoted to this workbook. This is the way Seth's workbook was returned to him at the beginning of the second lesson. The empty bubbles numbered 15, 17, 18, and 23-32 indicate that he has not yet worked on those pages. Before Seth begins on those pages during the second lesson, he should correct his mistakes on pages 8 and 13 and finish his work on pages 11, 14, and 22.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IMPORTANT: Again we ask that you not require or even encourage every student to finish every problem in a workbook. If, at the end of the two lessons allotted for a workbook, some students have not completed as much of the workbook as you expect that they are capable of, you might suggest that they spend some extra time on it during the next adjustment day. It would not be useful to demand completion or "mastery" before permitting work to begin on the next workbook series.

After the second workbook period, and after you have checked the results, you might want to send the workbooks home with students. This could be one way to keep parents informed of their children's progress.

One of the story-workbooks from the CSMP LIBRARY, "To Picture", is used as the basis for two lessons in UPG-IV. A story-workbook has both the flavor of a storybook and the advantages of a workbook. In these two lessons the students read "To Picture" collectively, stopping periodically to do some individual work.

c) Storybooks

Presently, there are twenty-four storybooks for young readers in the collection STORIES BY FREDERIQUE. Each story is presented in a colorful newsprint booklet and can be used for individual or collective reading at school or at home. The stories are independent of each other and for the most part need no teacher explanations. The titles of the books in the collection and the intended age range for each story are:

Ages 5 to 8

THE PLAYFUL NUMBERS
81 ROSES
I AM A VERY HAPPY BOY
ONE OUT OF SEVEN
THE HAPPY PUPPET
TWO BY TWO

THE LITTLE DREAMER
THE LITTLE DONKEY
THE MAGIC BOX
THE BABY IS BORN
THE WEIRD STORY OF 24
THE OLD SHOEMAKER

Ages 8 to 12

WHERE'S MY NOSE?
SINGING FRIENDS
DANCING FRIENDS
THE LIVING LINES
I AM NOT MY NAME
NABU WINS AN AWARD
THE SQUARE TRAP
SUMMER SCHOOL IN THE OLD DAYS

Ages 10 to 14

THE HIDDEN TREASURE
A VERY STRANGE NEIGHBORHOOD
ELECTION IN THE NUMBER WORLD
A VALENTINE MYSTERY

Two of these storybooks are used as the basis of lessons in the UPG-IV Program. Young people can become personally involved in these stories; there are questions to answer, pictures to relate to the story, information to be found in the pictures, and new adventures to invent for the characters in the stories. They find that in the world of numbers there are emotions to be expressed and aesthetic senses to be developed just as in the world of people. Your students might enjoy rereading (or just looking at) a storybook many times after they have used it for some lessons. Other storybooks (additional titles from the STORIES BY FREDERIQUE collection have been included in your classroom set of materials although they are not used with lessons) could be made available for individual reading during free time.

3. Role and Use of Hand-Calculators

The electronic hand-calculator, if used creatively, can have a remarkably beneficial effect on the students' interest in and understanding of arithmetic. Its use can be a contributing factor in the improvement of their capabilities in that area. The reasons for this are many and varied, but among them are certainly the following:

- By removing the burden of mechanical computation from the children's shoulders, the calculator frees them to concentrate on the thinking that is a necessary part of the solution to any meaningful problem.

- The instant feedback provided by the calculator enables the children to indulge in making and testing conjectures, an area into which they would not otherwise venture because the quantity of computation involved would be overwhelming.

This pedagogical tool is particularly useful in the introduction of new numerical notions such as negative integers, decimal numbers, multiples of a given whole number, and so on, because through suitably chosen, rich situations these concepts can be made to arise in a natural way.

Two battery powered hand-calculators are recommended for the UPG Program[*] and many different kinds are available from CSMP as a distributor for National Semi-Conductor, Inc. and Texas Instruments, Inc. Any of the products of these two companies can be obtained at discount prices.

Two hand-calculators are adequate for most lessons in the UPG Program, although there are three UPG-IV lessons in which it is recommended that each student have a hand-calculator. Encourage the students to bring calculators from home. The hand-calculators that are available from CSMP have algebraic logic and function keys which are permanently in a constant mode. You should check to be sure that any calculators brought in by students also have these characteristics.

- Algebraic logic. This means that the calculator responds to instructions given in the order in which they are usually written. To check that this is the case, press $\boxed{2}\ \boxed{+}\ \boxed{3}\ \boxed{=}$. If you have "5" on the display, then all is well.
- Constant mode. Some calculators have to switch so that the calculator can be set either in a constant mode or in a chain mode. If

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[*] If you already have calculators, be sure to check them for algebraic logic and constant mode (as explained on this and the following page).

your calculator lacks such a switch, it may well be that the calculator has an automatic constant. For calculators with an automatic constant, the first number entered during multiplication is usually the constant, while the second number entered during the other three operations (addition, subtraction, division) is usually the constant.

To check your calculator for the constant mode operation of $\boxed{+}$ is a simple procedure. Press $\boxed{0}\,\boxed{+}\,\boxed{2}\,\boxed{=}\,\boxed{=}\,\boxed{=}$ and so on. If the number on the display is increasing by two at each depression of $\boxed{=}$, then $\boxed{+}$ is operating in a constant mode. Check the constant mode operation of $\boxed{-}$ in a similar fashion.

To check the constant mode operation of $\boxed{x}$ (where the first number entered will be the constant), press $\boxed{2}\,\boxed{x}\,\boxed{1}\,\boxed{=}\,\boxed{=}\,\boxed{=}$ and so on. If the number on the display doubles at each depression of $\boxed{=}$, then $\boxed{x}$ is operating in a constant mode.

While it would be possible to use calculators without these characteristics in the lesson, the adjustments that would be required in the instructions make the calculators unsuitable.

In several lessons, the hand-calculator is used to explore patterns in a sequence of numbers which appear on the display. For example, put 3 on the display and press $\boxed{+}\,\boxed{5}\,\boxed{=}\,\boxed{=}\,\boxed{=}$ and so on. The sequence 3, 8, 13, 18, 23, 28, ...should appear. Another example in which the positive multiples of some number are generated is to put 4 on the display and press $\boxed{+}\,\boxed{=}\,\boxed{=}\,\boxed{=}$ and so on. The sequence of positive multiples of 4 should appear; that is, 4, 8, 12, 16, 20, 24, 28, ... Even if your calculators have algebraic logic, it may be necessary for you to check that they operate in the way described in the lessons. If you find that your calculators do not operate in this manner, then try replacing (for example)

$\boxed{+}\,\boxed{5}\,\boxed{=}\,\boxed{=}\,\boxed{=}$ ... with $\boxed{+}\,\boxed{5}\,\boxed{=}\,\boxed{+}\,\boxed{5}\,\boxed{=}\,\boxed{+}\,\boxed{5}\,\boxed{=}$ ...

and

$\boxed{4}\ \boxed{+}\ \boxed{=}\ \boxed{=}\ \boxed{=}$ . . . with $\boxed{4}\ \boxed{+}\ \boxed{4}\ \boxed{+}\ \boxed{4}\ \boxed{+}\ \boxed{4}\ \boxed{+}\ \boxed{4}$ . . .

or

$\boxed{4}\ \boxed{+}\ \boxed{4}\ \boxed{=}\ \boxed{+}\ \boxed{4}\ \boxed{=}\ \boxed{+}\ \boxed{4}\ \boxed{=}$ . . .

SUGGESTIONS FOR ORGANIZING THE PROGRAM

1. Schedule of Lessons

The design and schedule of the lessons in this guide should be regarded as suggestions; they reflect the experiences of the CSMP staff in developing the curriculum. It is difficult for curriculum developers to predict which topics will be easy for your students and which lessons will require more or less time for your particular class. Only you will be able to judge that as you teach the lessons. Thus the program is not meant to be a rigid sequence of lessons to be followed exactly, but only to be a suggested guide.

The master schedule shown on pages 22 and 23 was developed as one semester course for a hypothetical third grade class of average ability. It illustrates some important ideas about organizing the program. You will notice that several different strands are scheduled each week and that even within a strand the topics are interspersed. This was done to carry out the idea of a spiral curriculum. If you organized your own schedule, it is best to follow a similar pattern rather than to spend two or three full weeks, for example, "covering" one particular topic such as composition of functions or negative numbers. In a usual five day week, two days should be devoted to the World of Numbers strand, one day to the Languages of Strings and Arrows strand, one day to the Geometry and Measurement strand, and one day to the Workbook strand.

In the master schedule, a block represents one full period of approximately 45-50 minutes duration. It may very well be, however, that you and your class will spend more than one period or, possibly, less than one period on some lessons. Exercise your own professional judgment in deciding when to stop one particular lesson and when or whether to take it up again. Just be sure you do not insist that every single child understands every single aspect of a lesson before you go on. It is better to stop a lesson at a point when students' interest is high and a problem remains unsolved than to teach so methodically that everyone in the class understands everything, but is completely bored.

Inside each block of the master schedule, the lesson of the day is designated in several ways. First, there is listed a number which indicates its suggested chronological order among the lessons in that strand. Hence, "N5" indicates that the lesson in question is suggested as the fifth lesson in the N (World of Numbers) strand. This identifying number also appears in the upper outside corner of each page in the description of that lesson, thus enabling a lesson to be located simply by flipping through the pages of the strand book. Second, you are told on which page of the strand book the description of the lesson in question begins. Third, you are given the title of the lesson.

Please bear in mind that the model schedule was designed for a hypothetical class with the usual range of abilities. In order to be successful with the program, you should adjust the schedule to the ability, maturity, and interests of your class. For example, you may wish to extend a lesson or use a workbook for an additional period beyond those scheduled.

2. Adjustment Days

From time to time an adjustment day appears in the schedule. Since we have intended that UPG-IV would be the mathematics curriculum for the second semester of the third grade and a usual school semester is eighteen weeks, the schedule has been designed to indicate eighteen weeks of lessons. However, it is also usual for there to be days when no mathematics lesson is taught; e.g., parent-teacher conference days, field trip days, snow days, and so on. These adjustment days should account for such non-class days, and should also give you opportunities to do many kinds of things. You could use an adjustment day to review a lesson you feel needs review, or to complete an unfinished lesson, or to extend some interesting ideas from completed lessons. You could also use an adjustment day to give your students time to become more familiar with manipulative materials in the program. The students may especially enjoy playing games such as the Number Line Game, the String Game, or Minicomputer Golf, and an adjustment day could give them another opportunity to play these games. You may wish to use an adjustment day to do some mental arithmetic exercises, or to teach a topic in mathematics that you think would be especially interesting or useful to your class. For suggestions on ways to engage in mental arithmetic with your class, see the next section and Appendix B: MENTAL ARITHMETIC in the World of Numbers strand.

3. Supplementary Activities and Mental Arithmetic

Throughout the lessons there are many activities that children enjoy doing over and over in a variety of forms. For example, the String Game and the Number Line Game (to mention only two such activities) can be played whenever an extra time period is available. Of particular importance and enjoyment are the activities dealing with mental arithemtic; fully one-third of the lessons involve such activities. Other general suggestions about mental arithmetic are given in Appendix B to the World of Numbers strand. Teachers can invent clever variations of these mental arithmetic activities to be used often during a school day as adjuncts to the natural school routines. Some of the most obvious are: checking class attendance (How many present? How many should be here? How many absent?); the opening exercises (What day of the month is it? How many days in this month? How many days left?); distributing materials (How many students? Two pencils for each. How many pencils?).

Some teachers might want to do some supplementary drill work on basic number facts. We would suggest that the drill be kept to a minimum so as not to suggest to children that arithmetic is a chore.

SUGGESTED SCHEDULE OF LESSONS

SECTION TWO

QUESTIONS AND ANSWERS ABOUT CSMP

As you teach the CSMP curriculum or even as you prepare to teach it, there will be some basic questions that come to your mind. Very likely they are the same kinds of questions that parents and supervisors will ask about a curriculum that appears to be different from traditional programs. Perhaps some of the questions that concern you are answered here.

1. Q: WHAT IS THE SPIRAL APPROACH? HOW DOES IT WORK?

A: We learn through many interrelated experiences. But no experience, particularly a third grade math lesson, is an end in itself. With this in mind, CSMP introduces an idea in one lesson, then another in the next lesson, then another. Later the first idea reappears in a slightly new setting or in a slightly expanded version, then a second topic reappears, and so on. Each of the many times that a particular idea or area of study reappears, it is treated in a different manner and at a slightly higher level than the preceding time. This is called a spiral development.

Children learn at different times and at different rates. Only part of the learning of a given topic actually takes place during a lesson on the topic. In between the times when the topic occurs on the learning "spiral", the child is mentally digesting the idea, sometimes consciously, usually unconsciously. In this way the spiral development gives each student a new chance to "catch on" at each stage.

We find it highly effective to follow this kind of development rather than continuing on and on with a topic until so called "mastery" has occurred.

Sticking with the spiral idea requires an act of faith on your part, particularly if you have been used to teaching a given topic until you are satisfied that all (or at least most) of the class knows it "cold". Belief in the spiral approach implies that some lessons may be stopped before some (or even a majority of) children appear to have caught on, or in a lesson involving worksheets or workbooks, before everyone has successfully completed all of the pages. It involves knowing that the topic will reappear again and again, and that varying degrees of understanding will come at varying rates and times. It entails a belief that an idea planted now may not sprout until much later, and that is is best not to force its development. All our experiences to date (including five years of extended pilot testing of the curriculum) indicates that this approach really works.

2. Q: I UNDERSTAND HOW THE SPIRAL DEVELOPMENT WORKS, BUT HOW CAN I ASSESS MY STUDENTS' PROGRESS UNDER SUCH AN APPROACH?

A: We don't believe it is desirable or reasonable to try to evaluate student mastery of skills from day to day or even from week to week. On the other hand, teachers, parents, and the children themselves benefit from some monitoring of their progress and their involvement in the activities. Most teachers have their own methods for doing this with which they are comfortable, and we suggest that you continue to use your own monitoring methods. You will find that group interaction in the lessons and the individual worksheets and workbooks in the materials provide some valuable information. The spiral development in fact may help rather than hinder your long term assessment of students' progress.

3. Q: FOR WHAT SORT OF STUDENT IS THE CSMP ELEMENTARY CURRICULUM DESIGNED?

A: It is designed for all children in the schools. Each learning situation is designed to involve all students at their own levels in such a way that even the slowest can succeed in dealing with some aspects of the situation that challenge them. In fact, it has been found that such a "pedagogy of situations" (see Question 4) is particularly effective with the so-called "slow" learners, and recent work suggests it might be effective also with educable mentally handicapped children.

4. Q: BRIEFLY, HOW DOES CSMP DIFFER FROM A TRADITIONAL MATHEMATICS CURRICULUM?

A: The core content of CSMP is not revolutionary. The program aims to inculcate ideas and facts that have always been considered important in mathematics. The major differences are in method. Traditional teaching takes pride in disciplining students to be submitted to the usual paper and pencil algorithms. CSMP students are involved as well in many other real mathematical activities, with understanding at their level. For that purpose, CSMP employs new teaching aids such as non-verbal languages (see Question 7), which are themselves part of mathematics.

A pedagogy of situations is used to permit children to participate in the learning process. The students are faced with situations and are led by their reactions to important ideas in mathematics. Thereby, students begin to mathematize situations, a process which is of fundamental importance whenever mathematics is applied.

CSMP also makes use of a spiral approach rather than the mastery approach found in most mathematics curricula (see Question 1).

5. Q: ISN'T THIS THE "NEW MATH"?

A: There is a current popular belief that all school mathematics today is either "new math" or "old math". Neither of these labels is well defined. "New math" usually denotes the programs that heavily emphasize the teaching of sets; that introduce technical mathematical language for concepts such as "commutativity", "associativity", "distributivity", and so on; and that teach "concepts" before "skills". The "old math" refers to programs of the "good old days": memorization of the multiplication tables and complete emphasis on paper and pencil drill in calculations.

The CSMP curriculum is definitely not the "new math" nor the "old math" by any definition. It uses no technical language, yet achieves the understandings sought by the "new math". It uses no repetitive drills, yet elicits the ability to calculate stressed by the "old math". But these achievements are just by-products of a more pervasive goal: to help children use their curiosity, their intuition, their powers of reasoning to formulate and think about interesting problems and situations.

6. Q: ARE ALL THE TOPICS IN THE CSMP CURRICULUM REALLY MATHEMATICS?

A: Yes. Mathematics is much more than calculation with numbers, and its methods include more than paper and pencil algorithms. The CSMP curriculum is filled with experiences that directly and indirectly relate to the wide world of numbers, their interrelationships and the kinds of logical reasoning about these relationships that make mathematics an activity of interest and concern.

7. Q: WHY USE THE PAPY MINICOMPUTER AND THE LANGUAGES OF STRINGS AND ARROWS?

A: It is surprising that the language of strings was not used earlier and more extensively in the teaching of mathematics. Mathematicians have used strings for a very long time as they support classification notions so fundamental and natural that they go back to Aristotle. The scope of the string diagrams is broader than mathematics, the string diagram being a marvelous instrument for logical thinking.

String diagrams permit the understanding of a situation at a glance; they generate math concepts and ideas through child-related discussion.

Arrow diagrams are probably the most important teaching aid in CSMP. They are used everywhere in our daily lives: signals, advertisements, directions, and so on. Why not use them in teaching mathematics? Arrows are a pictorial way to represent relations. Relations have always been important in mathematics, but in the past seemed to be something rather abstract, elusive, and mysterious. With arrows they become vivid, concrete, and easy to understand.

The Minicomputer allows very young children to become familiar with numbers and operations on them (especially large numbers and the more difficult calculations) before they are able to work with them on paper. Later, and more importantly, the Minicomputer is used as a vehicle for posing situations involving mental arithmetic and for examining the workings of the arithmetic operations. It provides experiences that either support or directly lend themselves to the development of usual algorithms.

8. Q: IF THE STUDENTS USE THE MINICOMPUTER, WILL THEY BE ABLE TO CALCULATE WITHOUT IT?

A: This question suggests that the only purpose of the Minicomputer in the CSMP curriculum is to perform calculations. To be sure, the Minicomputer does enable very young children to deal with "large" numbers and perform relatively difficult calculations long before they know the paper and pencil algorithms for the calculations. But very soon these experiences with the Minicomputer suggest how the paper and pencil algorithms come about, and from that moment the Minicomputer is no longer regarded as a calculating device. Later, for the remainder of the curriculum, the Minicomputer assumes its primary roles: as a source of very interesting problems and games, as an effective vehicle for stimulating mental arithmetic, as a teaching device for introducing more advanced topics such as the negative numbers and the decimal numbers, and, in general, as an alternative to the usual representation of numbers.

9. Q: WHAT IS THE CSMP POSITION ON HAND-CALCULATORS AND WHAT ROLE DO THEY PLAY IN THE PROGRAM?

A: Hand-calculators should be a part of any mathematics curriculum. As more and more children have hand-calculators, these tools will become incorporated in the elementary and secondary school curricula in a natural way with uses very similar to those of the Minicomputer. In some cases, they may be used as a tool for mechanical computation where the difficulty of the calculation might turn an otherwise interesting problem into an exercise in drudgery. In other cases, they may be used as the context for posing situations leading to the discovery of important mathematical properties or providing opportunities for interesting exercises in mental arithmetic.

Hand-calculators are of particular benefit to "slow" learners. Many slow learners have mechanical difficulties rather than problems in thinking about a situation. The hand-calculator alleviates this problem.

Some lessons in the CSMP curriculum require only two calculators, while some lessons suggest that each student have one. Most often, however, lessons of this latter sort also include suggestions on how they may be modified since not every student may have a hand-calculator. In either situation, students with access to hand-calculators should always be encouraged to bring them to class.

10. Q: ARE THERE GOALS FOR EACH LESSON?

A: At the beginning of each lesson plan there is a CAPSULE LESSON SUMMARY, which describes briefly the activities and experiences included in the lesson. Indirectly these summaries suggest goals but, of course, the goals of the curriculum spread over whole series of lessons and cannot be broken into specific goals for individual lessons. If you read the introduction to the strand in which a lesson appears as well as the Capsule Lesson Summary for that lesson, we expect you will easily see the direction and intent of the lesson. We decline to predict for any one lesson what will happen in terms of student learning.

11. Q: SHOULD I SUPPLEMENT THE PROGRAM WITH MORE PRACTICE IN NUMBER FACTS?

A: There is a surprising amount of such practice throughout the program. It is implicit in the activities and is not in the form of practice for its own sake. If you still feel it is necessary to give supplementary practice, we suggest that you keep repetitive drill to a minimum so as not to endanger the students' interest in numerical situations.

12. Q: WHY IS THERE A LACK OF DRILL AND LITTLE EMPHASIS ON "SKILLS"?

A: CSMP believes that training for rote skill should never precede understanding. The avowed purpose of drill in arithmetic is to acquire the "skills", that is, the skills of calculation. Certain numerical skills are essential but, unfortunately for school children, the usual drill techniques for acquiring them are dull, stultifying, and sometimes even counterproductive. The CSMP curriculum appears to place little emphasis on "skills" because it does not resort to drill. But as a matter of fact, the essential skills of both mental and written calculation are learned almost inadvertently in response to the many situations where the ability to calculate is needed in order to respond to the situation: to win the number game, to identify the secret number of the detective story, or to build a number road. Calculation is always considered as a tool for doing interesting things, rather than as a chore done for its own sake.

13. Q: HOW DO STUDENTS LEARN TO USE MATHEMATICS IN PRACTICAL SITUATIONS?

A: For students of elementary school age, a situation that intrigues them and catches their imagination or their curiosity will certainly motivate them to learn whatever is necessary in order to deal with the situation. But situations that adults consider "practical" very often are outside the realm of interest of school children and do not provide motivation for learning. CSMP has found that students will indeed learn to use mathematics in situations they consider worthy of their attention. As they approach adulthood they will soon consider "practical" situations to be most worthy of their attention, but in the meanwhile, they are more interested in games, fantasy, puzzles, stories, and so on.

14. Q: HOW ARE TELLING TIME, MONEY, AND THE METRIC SYSTEM TREATED?

A: Telling time is not included in the CSMP program. It is our belief that telling time is more naturally a part of a language arts program than of a mathematics program, although it is frequently included in the mathematics curriculum. Since there are many excellent materials available to help you teach telling time, we urge you to teach this topic using those methods and materials you particularly like or that have proved successful to you in the past.

The topic of money is first introduced in the CSMP curriculum in situations involving combinations of coins. It is treated again in the Upper Primary Grades and the Intermediate Grades as a support

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for the teaching of the decimal numbers. Although money is not treated as a separate topic, these situations provide experience with our monetary system. Should the students need more exposure to this topic, teachers should feel free to develop supplementary materials in order to provide extra practice.

Metric units are used whenever there is cause to measure in standard units. Activities involving the metric system are begun in the First Grade Program and continue throughout the Geometry and Measurement strand of the Upper Primary Grades Program. We recommend that you do not ask the students to convert from the English to the metric system or vice versa (inches to centimeters, yards to meters, Celsius to Fahrenheit, grams to ounces, and so on). It would be confusing for children to deal with two system simultaneously, especially at this early stage. Since the United States has not yet "gone metric", your students will still need to be familiar with the English system. We would suggest that you use whatever materials and methods you have found successful in the past for teaching this topic, but we repeat our recommendation that you not ask the students to convert from one system to the other.

15. Q: WHAT IS THE ROLE OF STORYBOOKS IN THE CURRICULUM?

A: The storybooks in the collection STORIES BY FREDERIQUE include twenty-four stories for young children, presented in colorful newsprint booklets. Two stories from this collection are used for lessons in UPG-IV, with fifteen copies of each storybook provided in a classroom set of materials. (If your school can afford it, we recommend that you order enough extra copies to provide one for each child.) Other storybooks, two copies of each, are provided for supplementary reading. Each of the stories is portrayed by the mathematical language of dots, arrows, and strings which reinforce the ideas of the stories. Young readers can become personally involved in the stories; there are questions to answer, pictures to relate to the story, and new adventures to invent for the characters of the stories. They find that emotions and aesthetic senses are involved in the world of mathematics just as in the world of people.

16. Q: WHAT PROVISION IS MADE FOR INDIVIDUALIZATION IN THE CSMP PROGRAM?

A: In many of the lessons there are worksheets to be done individually by the students. Sometimes the worksheets are done at the beginning of a lesson, followed by a full group discussion of them. But ordinarily a topic is first developed in a full group setting, followed by individual student work on related worksheets. If "individualization" means a procedure for allowing students to work alone at their individual rates and levels, then the worksheets (and also the workbooks, discussed below) do provide for (partial) individualization. A series of worksheets for a given lesson offers a whole spectrum of levels from very simple to quite challenging. Students all start with the first worksheet (one-star) and proceed at whatever pace is appropriate and to levels according to their capabilities. The spiral approach accommodates this kind of work in the same manner as the full group work; that is, no worksheet is a prerequisite for the content of any succeeding lesson.

Eleven lessons during the second semester of the third grade are devoted to the five UPG-IV workbooks. Each workbook is also designed to move through a spectrum of levels of difficulty. We believe that the worksheets and workbooks realize the best sense of "individualization".

17. Q: WHAT PROVISIONS ARE MADE FOR THE VARIETY OF ABILITIES IN A USUAL CLASS?

A: Every lesson includes something for all students; there are opportunities for the slowest students to participate in a meaningful way, and there are challenges for even the very brightest of students. This means that you should not exclude any students from a full group lesson. At the same time, you should not expect every student to understand everything during the course of the lesson. Furthermore, with the spiral development many situations are open-ended and allow students to experience them at levels as high as they are capable.

Evaluation has shown that students traditionally labeled "slow" are doing better in CSMP classes than in non-CSMP classes. We believe this is due to the spiral development, to full group work on mathematical activities, and to the presence in the CSMP curriculum of certain nonnumerical activities in which many supposedly "slow" students can excel.

Additionally, CSMP provides for a variety of ability levels in the student materials. Worksheets and workbooks offer problems in a range from very easy to quite challenging. Indeed, some of the "higher starred" worksheets and pages toward the ends of workbooks may be accessible only to your very best students.

In extreme cases, you might seriously consider the possibility of arranging for exceptionally bright (slow) students to take their math with a higher (lower) grade level CSMP class.

18. Q: I PREFER TEACHING SMALL GROUPS, BUT MOST OF THE LESSONS SEEM TO BE AIMED AT THE WHOLE GROUP. WHAT DO YOU SUGGEST?

A: If your school routine allows you to teach each lesson twice, once to each half of your class, that would be an alternative way to proceed. But if you do this, it might be wise to vary the composition of the groups from day to day so that you are not tempted to group your students according to your perception of their ability. Otherwise you would guarantee that half your students would not have as rich an experience as the other half, through no fault of their own. The purpose of the full group experiences is to allow interaction among all the children in every situation posed in the lessons. Students learn quite a bit from each others' reactions. You will be surprised to find that in some situations the key responses are made by students you had labeled "slow". We feel the spiral approach has maximum potential in full group teaching.

The above response might give the impression that CSMP teaching takes place only in full group settings. This is not the case. There is much time devoted to individual student work on worksheets and workbooks, and there are various games to be played. We also encourage you to arrange small group sessions whenever you have time, particularly with your slower students. Small group activities could supplement the full group lessons, but should not replace them. Extra small group work should not be simply for drill and for rehashing ideas, but rather should provide additional experiences.

19. Q: WHY IS A "DIALOGUE" STYLE USED AS FORMAT FOR THE LESSON PLANS? ARE TEACHERS EXPECTED TO FOLLOW THIS DIALOGUE EXACTLY?

A: Most of the lessons are written in considerable detail; step by step descriptions are frequently given, together with suggestions for the exact words a teacher might use (in the form of a possible dialogue between teacher and students). These details are given only for the teachers who feel the need for them. You should never feel obligated to follow any lesson word for word as it is written in the guide. By the same token, you should never insist that your students give the same responses as those given in the imaginary dialogues. Use your own creativity in drawing out students' responses as the situations suggest. Read the lesson plans very carefully beforehand to get an idea how the lessons are expected to proceed.

20. Q: IS IT NECESSARY TO FOLLOW THE SUGGESTED SCHEDULE EXACTLY?

A: This program is not meant to be a static sequence of exercises to be followed rigidly. The lessons in this guide are only suggestions based on our own experience teaching the materials. If this will be the first time you are teaching the program and you would like specific suggestions for organizing the lessons, we offer some guidelines and a suggested schedule of lessons. You may wish to follow it at the beginning of the school year and then deviate from it as the needs of your students may dictate. It is probably best to follow a one-day, one-strand pattern when you organize your own schedule, rather than to spend two or three full weeks "covering" one particular topic. In a usual five day week, two days should be devoted to the World of Numbers strand, one day to the Languages of Strings and Arrows strand, one day to the Geometry and Measurement strand, and one day to the Workbook strand.

Exercise your own professional judgment in deciding when to stop a particular lesson. It is better to stop a lesson when student interest is high than to teach so methodically that everyone in the class understands everything, but is completely bored. In order to be successful with this curriculum, you should adjust the schedule to the ability, maturity, and interests of your class.

21. Q: HOW DO I HANDLE STUDENTS WHO HAVE AN EXTENDED ABSENCE OR STUDENTS (NEW TO CSMP) WHO TRANSFER INTO MY CLASS DURING THE YEAR?

A: The spiral approach allows a student to miss one or two weeks of school without much "catch up" work needed. When the student returns, either you or classmates can help with any material that is deemed necessary.

If several months of classes have been missed, you might need to treat the student as you would a transfer student new to CSMP. Give the student a little extra attention during a lesson especially when the questions are review for most of the class.

In most cases, the differences between new and veteran students tend to fade rather quickly. One exception might be in lessons involving the Minicomputer. Veteran students who are unusually adept with the Minicomputer very often are good tutors for the transfer students in the use of this abacus.

It is probably best not to send worksheets and/or workbooks home to a child; without participating in the collective lessons that introduce the worksheets and workbooks, it would be very difficult for the student to proceed.

22. Q: HOW DO I HANDLE WHOLE CLASSES THAT ARE NEW TO CSMP?

A: CSMP does not assume that kindergarten and first grade students have any prior experience with CSMP. The Kindergarten and First Grade Programs will need no adaptation for classes that are new to CSMP.

Second grade classes that are new to CSMP will initially progress more slowly than veteran classes. The lessons at the beginning of the second grade program are designed to serve either as an introduction to CSMP languages for new classes or as a quick review for veteran classes.

Third grade classes that are new to CSMP will need to become fluent in the CSMP languages before beginning the lessons in UPG Part III. A special schedule has been prepared for these classes which consists of prerequisite lessons from a third grade entry supplement followed by a slightly modified UPG Parts III and IV Program.

23. Q: WHAT SHOULD I DO WITH MY CLASS IF THEY DID NOT FINISH THE PREVIOUS YEAR'S SCHEDULE?

A: No special measures are required. Each level of the CSMP curriculum begins with introductory lessons. In addition, the spiral approach in the lessons permits students to participate at their own level of understanding, even if that level is not quite as high as it might have been if the previous year's schedule had been completed.

24. Q: HOW DO CSMP STUDENTS DO ON STANDARDIZED TESTS?

A: While CSMP disagrees with the philosophy underlying standardized testing, test results are available to us from several sources. The results from standardized tests indicate that CSMP classes generally do as well as or better than do non-CSMP classes. It should be pointed out that standardized testing and traditional programs are correlated, and that CSMP students have been exposed to a great deal of mathematics that is not taught in more traditional programs or that is not considered in the construction of the tests. There are also many unmeasurable or unmeasured areas such as enjoyment of mathematics, strategic thinking, organizational ability, and depth of understanding of the methods and uses of mathematics where we feel CSMP students have a definite advantage. Evaluation Report 2-A-3 in the CSMP Evaluation Report Series gives more detailed answers to this question. Copies are available upon request.

25. Q: WHAT PROBLEMS WOULD CSMP STUDENTS HAVE IF THEY RETURN TO A MORE TRADITIONAL PROGRAM IN THE NEXT SCHOOL YEAR?

A: There should be no problems for CSMP students who return to a traditional program. Test results indicate that CSMP students are at or above the level of non-CSMP students in material covered in traditional courses. In addition, CSMP students have encountered many situations that are not introduced in traditional programs nor are tested for on standardized tests (see Question 24). In particular, their reasoning ability should be much sharper than their non-CSMP counterparts.

There are, of course, differences in timing as to when different programs introduce the standard algorithms and often CSMP's introduction will occur later than in some of the other programs. The reason for this is our belief that it is essential that the creative freedom of the student not be impaired by a premature emphasis on the standard algorithms for adding, subtracting, multiplying, and dividing. We believe that many of the present computational difficulties experienced by elementary and junior high school students were caused by their spending an inordinate amount of time trying to master the algorithms too early in their mathematical education. This shuts the students off from other experiences necessary for their mathematical growth. Students need plenty of time to explore the world of numbers. Mastering an algorithm should be the last step of this exploration, a sort of synthesis of previous experience. Furthermore, as electronic hand-calculators become available to nearly everyone, algorithmic mastery as a prerequisite to other numerical experiences should become less important.

CSMP students usually can solve a problem before having learned the standard algorithm. If they enter a program that expects an algorithm they have not learned, they are still more than adequately prepared for the algorithm and should have no trouble quickly learning to use it.

CSMP introduces the usual addition algorithm in the second grade, one of the two standard subtraction algorithms in the third grade, the multiplication algorithm in the fourth grade, and the division algorithm in the fifth grade.

26. Q: WHAT ARE SOME WAYS TO ORGANIZE MATERIALS AND MANAGE MY CLASS?

A: It is very important to organize your CSMP materials before school begins. Decide where you will store the materials once they are organized. Filing drawers, shelves, boxes, or closets are all satisfactory as long as you can easily get to the materials. Take then all out of their shipping boxes and spread them on several tables and/or desk tops in your classroom so they can be organized into the order you will be using them.

Workbooks: The teacher's guide is needed to find the order in which the workbooks are used. Locate the Workbooks strand column in the Master Suggested Schedule of Lessons and collect all the workbooks in the order in which they are listed. After organizing the workbooks in order, place them in an upright position in your file drawer, box, shelf, etc., so that the first workbook to be used is at the front of the stack.

Worksheets: There is a separate worksheet booklet for each student and you should decide whether you will store these booklets and distribute them when a particular lesson calls for a worksheet, or whether you will let the students keep their own worksheet booklet in their desks.

Each worksheet can be removed from the booklet or left in depending on how you wish to manage worksheets. However, once a worksheet is removed from the booklet it cannot be replaced, so make sure students are careful about tearing off the correct sheet.

Manipulatives: Storage of each manipulative in a separate container, properly marked, is one way to make everything accessible. Another way is to provide a sturdy box (for example, a shoe box) for each student and to put individual manipulative materials in the box, properly labeled with the student's name.

Additional Materials: Be sure that you have plenty of high quality colored chalk, large sheets of unlined white paper, a set of felt-tip markers, and that your students have colored pencils, pens, or crayons.

Plan well ahead to be sure you have any needed materials that are not supplied by CSMP. Keep all teacher materials where they are easily accessible. You may find the guides easier to use if they are put in looseleaf binders; the books are perforated so that pages can be removed easily. The demonstration Minicomputer boards and the number line should always be on display in the room.

27. Q: SOME OF THE CSMP MATERIAL IS NEW TO ME. HOW DO I LEARN ENOUGH TO TEACH A CLASS?

A: Every CSMP teacher should have been trained by his or her coordinator. Kindergarten and first grade teachers receive a minimum of eighteen hours training, teachers of the Upper Primary Grades Program receive up to thirty-two hours training, and teachers of the Intermediate Grades Program receive up to fifty-six hours training. During the training sessions the CSMP content is taught to new teachers and suggestions are given for organization of their classes.

If the suggested training schedule has been followed, many lessons taken from the CSMP curriculum, or very similar lessons, have been presented. The training schedule also makes use of the spiral approach so that training sessions aim not only to teach CSMP mathematics but to familiarize teachers with the spirit and the pedagogy of the CSMP curriculum.

28. Q: WHAT SHOULD BE DONE WHEN THERE IS A SUBSTITUTE TEACHER?

A: Some school systems have included people who frequently substitute in their CSMP training program. When this is not the case, it would be very difficult for the untrained substitute to teach from CSMP materials. One solution is to arrange for another CSMP teacher to teach math to your class while the substitute teacher teaches another subject to that teacher's class. If that cannot be arranged and if you will only miss one or two days, you may find certain CSMP material, uncompleted workbooks for example, that the substitute can manage. Otherwise, use those days to cover material not included in the CSMP curriculum, such as telling time. If your absence will be an extended one, arrangements should be made with your coordinator to train a substitute, to rearrange schedules to put a CSMP teacher in your class, or to otherwise provide for your students.

29. Q: THE PROGRAM MAKES USE OF THE STUDENT'S CREATIVE FREEDOM. HOW MUCH ROOM IS THERE FOR THE TEACHER'S INGENUITY?

A: CSMP not only allows for your ingenuity but depends on it. While the lessons may appear to be very rigid by presenting dialogue for teacher and student, it should be remembered that these are suggested questions, answers, and comments based on our experience with development classes. Even if you were to follow your part of the "script," your students are likely to offer responses which differ from those we have listed. It is the student's response that will determine the content and the direction of your next question or comment.

In a traditional program your creativity is called upon the first time you develop a particular lesson, and to a lesser extent, as you revise lessons in succeeding years. There is little room for spur-of-the-moment creativity once a lesson begins. The CSMP curriculum, by creating a free and open situation emphasizing the student's creative freedom, continually presents you with unanticipated student response and calls on your ingenuity at all times.

30. Q: HOW CAN I EXPLAIN CSMP TO PARENTS?

A: The best way would be for your coordinator to organize a workshop for interested parents. The parents can then see the languages of the program in use and have some of the same experiences their children have had. Workshops can be of almost any length but anything short of two hours does not allow for much familiarity with the material. In such a workshop we would suggest that the language of the Minicomputer be explained with particular emphasis on its use in presenting situations where mental arithmetic is called upon and in developing algorithms (for example, physically carrying a checker from the ones' to the tens' board in the "8 + 2 = 10" play, paralleling "carrying" in the paper and pencil operation). A lesson using the storybook "Where's My Nose?" provided a good example of the power of the language of strings. A nonnumerical example, such as the brother-sister graph, and a numerical example, such as building a road from 2 to 23 using +5 and - 2 arrows, will provide an introduction to the language of arrows. The lessons that are the favorites of the children would be excellent for a presentation to parents as well.

You might also extend an invitation to parents to visit your class during the math period. Many times such an invitation leads eventually to a closer parent-teacher relationship, with the parent offering voluntary services as an aide.

You will find a letter to parents at the end of this section which may be reproduced and sent home with the student.

TO THE PARENTS OF CSMP STUDENTS

Your child is participating in the Comprehensive School Mathematics Program's (CSMP) elementary school curriculum, developed under a contract with the National Institute of Education. While the program cannot be fully explained in a brief note, we hope that this letter will give you some feeling for the substance of your child's mathematics education.

The goals of the CSMP K-6 program include a development of the standard number systems and their operations (including an early introduction to the integers, rational numbers and decimal numbers) interwoven with a development of geometry and measurement. On an equal footing with the above, the CSMP program includes a strong component of probability, statistics, and combinatorics which extends throughout grades 2-6. Powerful pedagogical devices are used, such as several nonverbal languages. These make the unifying ideas of the program accessible to very young children.

These languages include:

1) The Languages of Strings. This "language" mimics the basic methods of collecting and classifying data applying to the problems at hand. That is, it deals with the fundamentally useful and important mathematical notion of set.

2) The Language of Arrows. This graphical language models the process of comparing and analyzing sets and operations on them. That is, it deals with the mathematical notions of relations and functions including, but not restricted to, ordering relations and the numerical processes of adding to, subtracting from, multiplying by, and sharing equally among.

3) The Language of the Papy Minicomputer. The Papy Minicomputer, an extremely effective paper abacus invented by Georges Papy, models the positional structure of our system of numeration and lends itself to a multitude of numerical algorithms, including the standard algorithms for the basic numerical operations.

Among other things, these languages allow for the development of the basic operations on numbers in a natural way based on previous experiences with the languages. They also provide a context in which situations arise that call for the repeated use of arithmetic operations without resorting to drill exercises, which most children find boring.

The content is learned in an atmosphere of constant applications utilizing a "pedagogy of situations".[*] These vary from simple story situations in the early primary grades to challenging applications and nontrivial simulations of real world problems in the upper grades.

One of the tenets of CSMP's development philosophy is that mathematics is a unified whole and should be learned as such. As a result, along with CSMP's concern with the variation in ability levels in the typical heterogeneous class, the curriculum is structured in a spiral format throughout the K-6 program. The content is completely sequenced in spiral form so that each student is brought into contact with each area of content continuously throughout the program. This consciously avoids atomizing the content and mastering each bit before continuing to the next. Rather, each student spirals through repeated exposures to the content, building interlocking experiences of increasing sophistication.

Another tenet of CSMP's development philosophy, based on years of experimentation in the classroom, is that no single method of classroom management suffices to meet the needs of every student. Hence, the program is constructed to allow numerous opportunities for whole class participation, small group participation, and individual independent experiences. Individualization throughout the K-6 program is achieved through the independent work on workbooks and worksheets of varying levels of difficulty. Due to the spiral nature of the curriculum, each student can engage in a workbook without losing touch with the progress of the class while at the same time working to the limits of his or her ability.

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[*] Inspired by the work of the mathematician and psychologist, Caleb Cattegno.

A feature unique to the CSMP program is a collection of storybooks written by Frédérique Papy (STORIES BY FREDERIQUE) and a collection of story-workbooks (THE CSMP LIBRARY). These probe the affective domain of learning and bring children into contact with areas of mathematics until now unexplored by them.

CSMP does not have its own system of student assessment and progress reporting. While standardized testing indicates that CSMP students develop the "basic skills" as early or earlier than do non-CSMP students, we hope you will not be obsessed with your child's development of these "skills". Each child is an individual and should not be judged against norms but on what he or she can and does accomplish. Many extremely important areas--development of self-confidence, ability to think and reason, etc.--are very difficult to measure and grade. If you have any questions regarding the progress of your child, his or her teacher is in the best position to answer those questions.

THE CSMP STAFF

THE
WORLD
OF
NUMBERS

Comprehensive School Mathematics Program

CSMP Mathematics for the Upper Primary Grades

Part IV

The World Of Numbers

CEMREL, Inc.

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THE WORLD OF NUMBERS

INTRODUCTION

Numbers are among the most important things that mathematics (at all levels) is about. Mathematicians are interested in numbers just as astronomers are interested in stars, botanists in plants, and sociologists in the organization and dynamics of human societies. Surely everyone agrees that one of the primary goals of any elementary school mathematics curriculum should be to introduce students to the world of numbers — to give them the opportunity to become familiar with numbers, their properties, and the relations between them. The ability to calculate with numbers is an important part — but not the only part — of being familiar and comfortable with numbers. The World of Numbers strand of the CSMP Upper Primary Grades Program is designed to provide the students with a wide variety of challenging experiences with numbers so that gradually they will become not only familiar, but comfortable with numbers; they will, so to speak, get to know numbers on a "first name basis".

The Minicomputer and the World of Numbers

The CSMP Upper Primary Grades program uses the Papy Minicomputer* as a support for the positional system of numeration; for calculations and estimation; for number patterns and mental arithmetic; and for modeling the basic operations involving whole numbers, integers, and decimal numbers. Although the Minicomputer can be seen as a tool for calculation and as a device to help students learn routine methods for calculations, its more exciting use is as a vehicle for posing interesting problems which challenge a child's intellectual curiosity about numbers and for presenting situations which dually encourage strategic thinking and reinforce numerical skills. The lessons which make use of the Minicomputer are often intended to be explorations into the world of numbers.

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* See also Questions 7 and 8 of Section Two, Questions and Answers About CSMP, of the General Introduction and APPENDIX A: STANDARD CONFIGURATION AND STANDARD MINICOMPUTER MOVES.

Standard Algorithms of Arithmetic

CSMP seeks to develop basic numerical skills as well as an understanding of the underlying mathematical ideas. We are fully in agreement with the thesis that, along with the growth of understanding of the world of numbers, there must be a concomitant growth of familiarity and facility with numbers and operations on them. But facility should not be confused with understanding; they are partners in the growth of mathematical maturity. A balanced growth of each must be maintained, neither being sacrificed for the other.

Students must eventually learn mechanical algorithms for the basic operations (addition, subtraction, multiplication, division). However, premature presentation of these algorithms may actually stunt a student's ability to develop alternative algorithms, to do mental arithmetic, or to estimate.

Consider the problem of calculating "194 - 57". A third grader may have difficulty performing a paper and pencil subtraction algorithm. An easy way to proceed is to subtract 60 from 194 and then to add 3
(194 - 60 = 134 and 134 + 3 = 137). To insist on a mechanistic response to such a problem would be to encourage inefficiency and might also inhibit the development of the flexibility necessary for problem solving. On the other hand, a rich array of situations in which students interact with numbers provides them with opportunities to gain the necessary facility with standard algorithmic procedures while retaining the openness required to respond creatively to new situations in the world of numbers.

Numerical Relations

One of the main aims of the World of Numbers strand is to familiarize students with numbers by studying explicitly and in a variety of contexts relations between numbers. (For more general comments about mathematical relations, see the introduction to the Languages of Strings and Arrows strand). Arrow diagrams represent relations in a simple, suggestive, and pictorial way -- usually more conveniently than the same information could be given in words or other symbols.

Students are brought into contact with an assortment of challenging situations, many of which would be totally inaccessible to them were it not for the arrow diagrams. The problems and activities of this strand include many examples: solving linear equations presented in terms of arrows, studying iterated processes and patterns in sequences of numbers, tackling problems which may have many solutions or no solution, estimating or testing that a solution is reasonable, and exploring properties of operations on numbers.

In summary, what is most important in the study of numbers is to confront the students with a variety of problems and situations which capture their interests, challenge their abilities to reason, and stimulate their curiosities about numbers -- and, at the same time, provide them with tools for coping with these situations and problems.

N1 Subtraction Problems #1

CAPSULE LESSON SUMMARY

On the Minicomputer add positive numbers to negative numbers and write the corresponding addition and subtraction number stories.

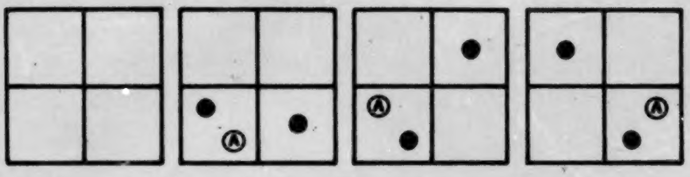
MATERIALS

Teacher: Minicomputer set,

Student : Minicomputer set[*]; Worksheets N1*, **, *** and ****; paper; pencil

DESCRIPTION OF LESSON

Invite students to put 369 and $\widehat{221}$ on the Minicomputer. Write "$369 + \widehat{221}$" to one side of the Minicomputer.



$= 369 + \widehat{221}$

T: What number is $369 + \widehat{221}$? Write it on a piece of paper.

Check many of the students' answers before asking a student to answer aloud.

S: $369 + \widehat{221} = 148$.

Write this number story on the board.

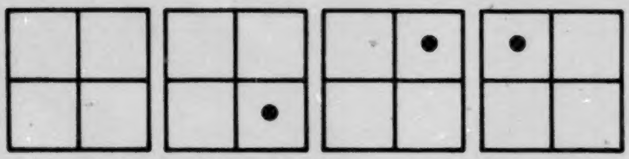
$$369 + \widehat{221} = 148$$

---

[*] A student's Minicomputer set consists of two individual Minicomputer sheets (each having two boards) and a sheet of individual checkers.

T: How did you find the answer?

Encourage several students to explain their calculations. If no one suggests removing the pairs with one negative checker and one regular checker from the Minicomputer, suggest this yourself. Emphasize that when a negative checker is on the same square as a regular checker, both checkers can be removed from the Minicomputer.



$= 369 + \widehat{221} = 148$

Point to the number story "$369 + \widehat{221} = 148$".

T: What subtraction story could we write for this?

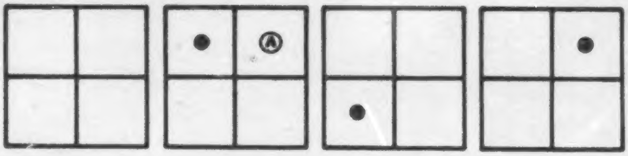
S: "369 - 221 = 148".

$$369 + \widehat{221} = 148$$

$$369 - 221 = 148$$

Repeat this activity to calculate "$824 + \widehat{400}$ ". After the students have had an opportunity to write the answer on their papers, complete the number story on the board and ask several students to explain how they did the calculation.

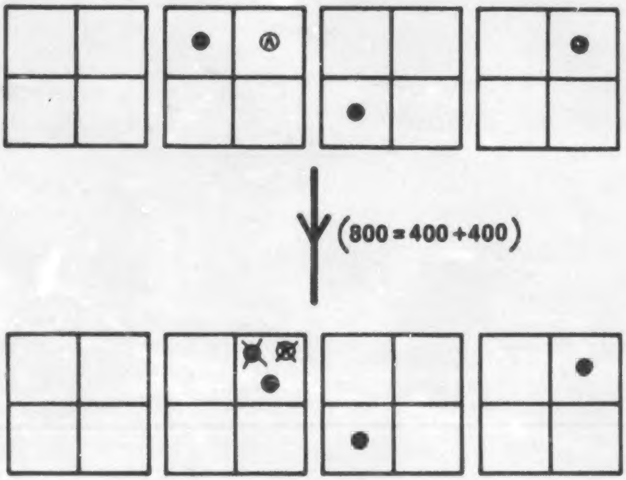
For example,



$= 824 + \widehat{400} = 424$

S: I made the "800 = 400 + 400" play in my head.

Make this play yourself, pair the negative checker with a regular checker on the 400-square, and then remove those two checkers.



S: I subtracted "800 - 400" and then added 24 to that answer; 400 + 24 = 424.

T: What subtraction story could we write?

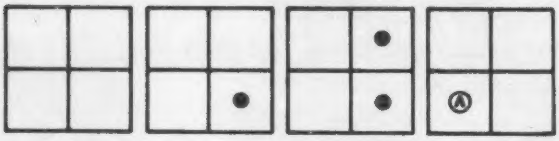
S: 824 - 400 = 424.

---

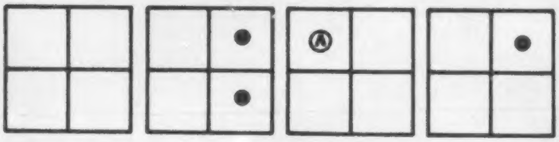
[*] An "✕" through a checker indicates that the checker has been removed from the Minicomputer.

N1

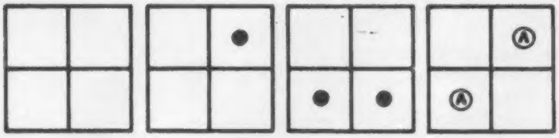
Continue this activity with these calculations.



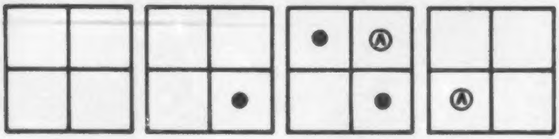
= $150 + \widehat{2}$



= $504 + \widehat{80}$



= $430 + \widehat{6}$



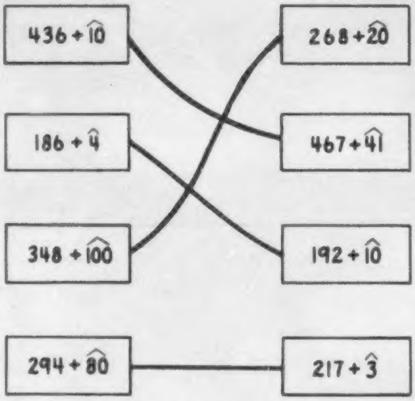
= $190 + \widehat{42}$

Perhaps someone will comment that the answer to each of these calculations is either 148 or 424. Encourage students to notice that many calculations can have the same answer.

Worksheets N1*, **, ***, and **** are available for individual work. Suggest the use of the individual Minicomputers to solve these problems, but allow the students to use whatever methods they prefer.

Name ____________________ **N 1 ***

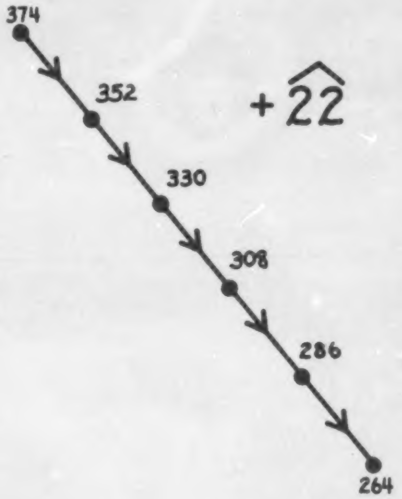
Pair names for the same number. Two are paired for you.



436 + $\widehat{10}$
268 + $\widehat{20}$
186 + $\widehat{4}$
467 + $\widehat{41}$
348 + $\widehat{100}$
192 + $\widehat{10}$
294 + $\widehat{80}$
217 + $\widehat{3}$

Name ____________________ **N 1 ****

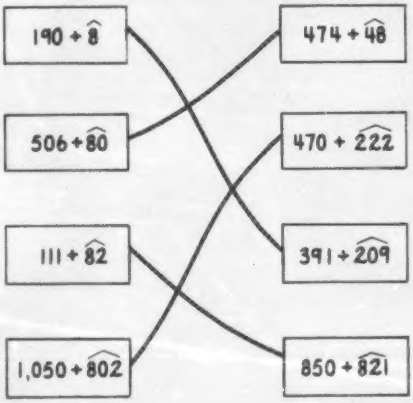
Label the dots.



374
352
330
308
286
264
+ $\widehat{22}$

Name ____________________ **N 1 *****

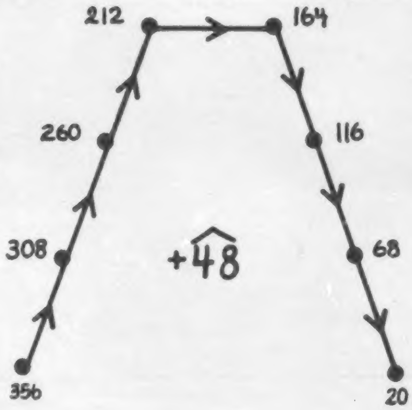
Pair names for the same number.



190 + $\widehat{8}$
474 + $\widehat{48}$
506 + $\widehat{80}$
470 + $\widehat{222}$
111 + $\widehat{82}$
391 + $\widehat{209}$
1,050 + $\widehat{802}$
850 + $\widehat{821}$

Name ____________________ **N 1 ******

Label the dots.



212
164
260
116
308
+ $\widehat{48}$
68
356
20

N2

N2 Composition of Functions #1

CAPSULE LESSON SUMMARY

Support that the composition of 3x followed by 2x is 6x using a dot picture. Support that the composition of 2x followed by 3x is 6x using the Minicomputer. Notice that to multiply a number by 6, one could multiply it by 3 and then multiply the answer by 2 or vice versa.

MATERIALS

Teacher: Colored chalk; Minicomputer set
Student : Paper; colored pencils[*]; Worksheets N2* and **

DESCRIPTION OF LESSON

Draw a row of eight dots on the board.

T: How many dots are in this row? (Eight)

I want to draw a dot picture with 3 x 8 dots. How could I draw the dots so we can easily see that there are 3 x 8 dots?

Consider all suggestions until someone says to draw two more rows with eight dots.

T: If I draw two more rows of eight dots, will there be 3 x 8 dots in this picture?

S: Yes, because you will have three rows of eight dots.

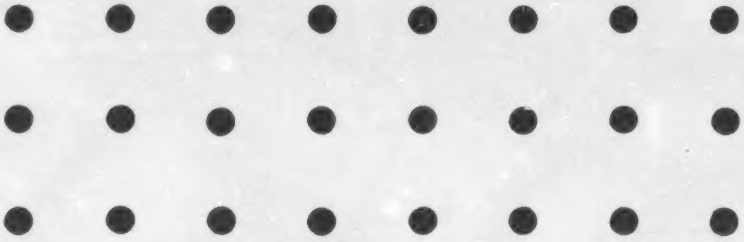
T: How many dots will there be in my picture then? How do you know?

S: Twenty-four, because 3 x 8 = 24.

---

[*] Crayons may be substituted for colored pencils in any of the lessons.

Complete the dot picture and ask for a volunteer to count the dots by threes to confirm that there are twenty-four dots in your picture.



T: What number stories could we write about this dot picture?

Record number stories as they are mentioned and ask the students to show how each number story is suggested by the dot picture. Several appropriate number stories are given here.

$8 \times 3 = 24$ $\qquad \frac{1}{2} \times 24 = 12$

$8 + 8 + 8 = 24$ $\qquad \frac{1}{3} \times 24 = 8$

$6 \times 4 = 24$ $\qquad 6 + 6 + 6 + 6 = 24$

$3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 = 24$

T: I would like to draw an arrow picture about this dot picture. How many dots are in each row?

S: Eight.

Draw a dot and label it "8".

T: How many dots are there in total?

S: Twenty-four.

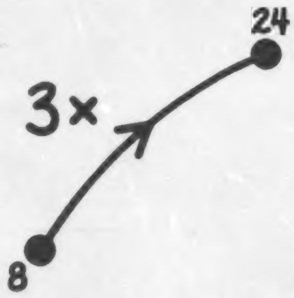
Draw another dot and label it "24". Draw a red arrow from 8 to 24.

T: What could this arrow be for?

S: 3 x.

NOTE: The red arrow could also be for + 16 and several other relations, but we are only interested in 3 x for the purpose of this lesson. Accept other correct suggestions, but do not record them on the board.

Label the red arrow "3 x".

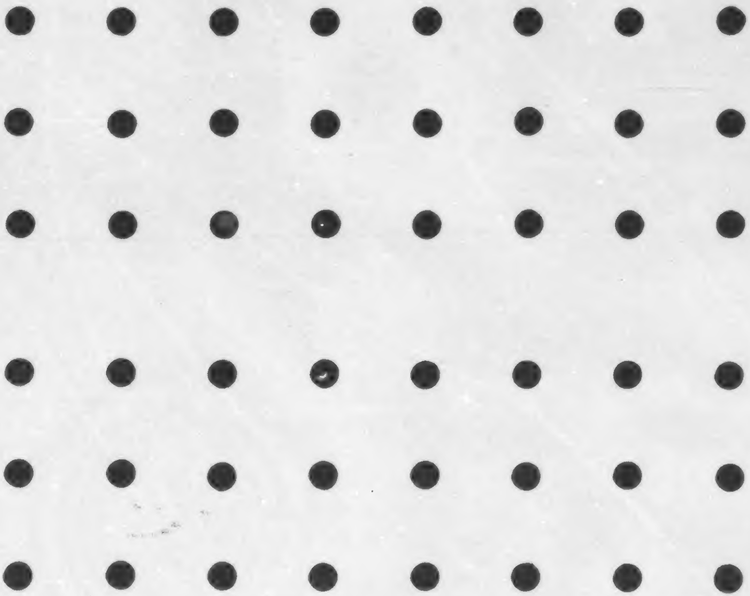


T: Now I would like to continue drawing dots in my dot picture until there are twice as many as there are now. How could I do this?

S: Draw three more rows of dots.

T: There are three rows of eight dots in the picture now. So if I want twice as many dots, I need to draw three more rows of eight dots.

Draw three more rows of eight dots but leave extra space between the first and second groups of twenty-four dots.



T: How many dots are there in each of these rows? (Eight)
How many rows of eight dots are there? (Six)
How many dots are there altogether in this dot picture? How do you know?

S: Forty-eight, because there are twenty-four dots in the first picture and 2 × 24 = 48.

Erase the number stories on the board and write "2 × 24 = 48".

S: I knew that 6 × 8 = 48.

Record "6 × 8 = 48" on the board.

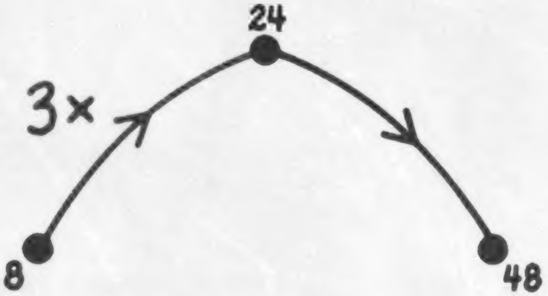
N2

S: I counted the dots and there are forty-eight of them.

T: What other number stories could we write about this dot picture?

Record number stories as they are mentioned and ask the students to show how each number story is suggested by the dot picture.

Draw another dot in your arrow picture and label it "48". Draw a blue arrow from 24 to 48.



T: What could this blue arrow be for?

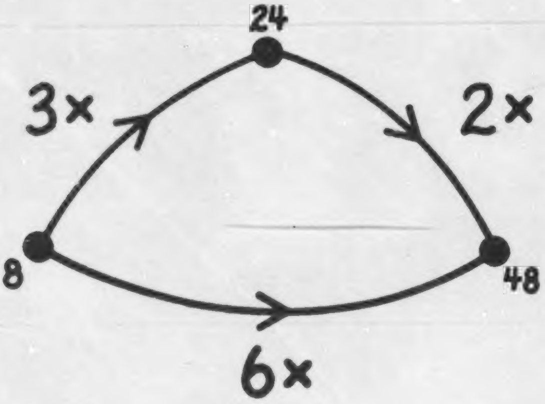
S: 2×.

Label the blue arrow "2 ×" and draw a green arrow from 8 to 48.

T: I drew a green arrow from 8 to 48. What could this arrow be for?

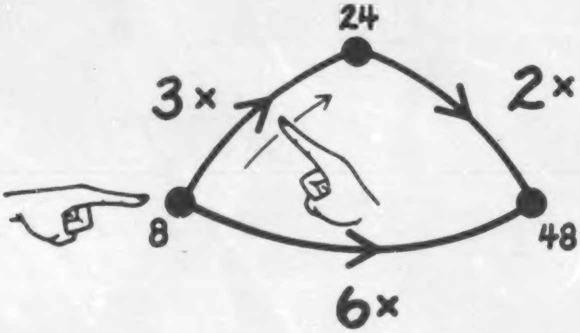
S: 6×.

Label the green arrow "6 ×".

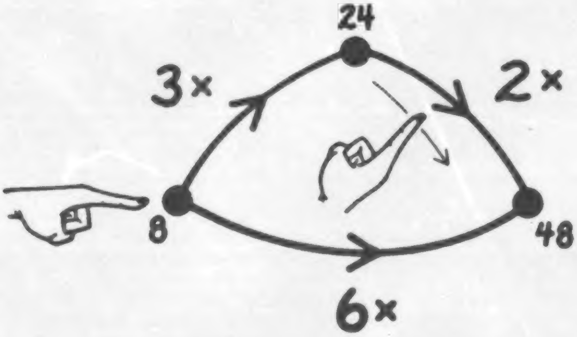


With the following dialogue, use motions as described in the Composition Games lessons. See UPG-III Lesson L2 Composition Games or UPG-IV Lesson L4 Common Multiples/Composition Games.

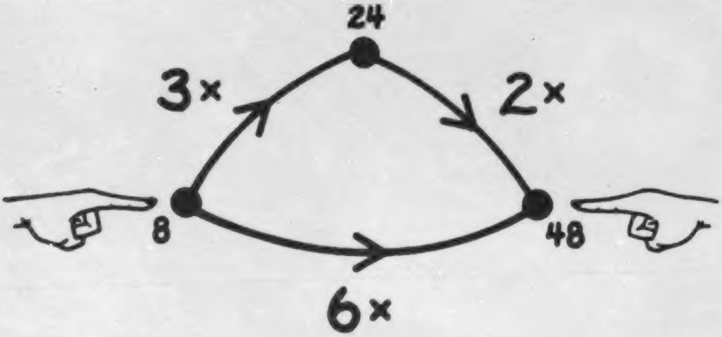
T: A 3 × arrow (trace the red arrow) . . .



followed by a 2 × arrow (trace the blue arrow) . . .

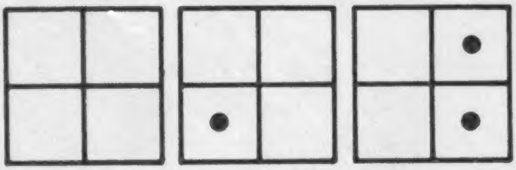


. . . is the same as a 6 × arrow (trace the green arrow).



Erase everything on the board except the arrow picture. Erase the labels for the dots.

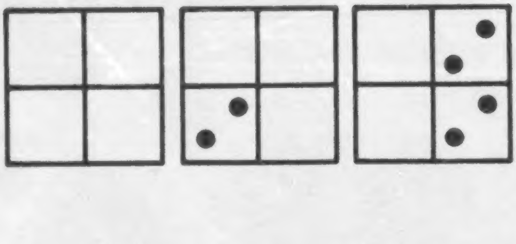
Put 25 on the Minicomputer.



T: What number is on the Minicomputer? (25)
I would like to double the number on the Minicomputer. What is an easy way to make the number on the Minicomputer twice as large as it is now?

S: Put another checker with each checker already on the Minicomputer.

Ask a student to double the number on the Minicomputer.

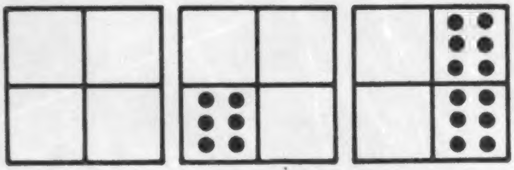


T: We have a new number on the Minicomputer. What is an easy way to make the number on the Minicomputer three times as large as it is now?

Encourage the students to explain their suggestions before doing anything on the Minicomputer. If necessary, ask the students to focus on the 20-square and ask how many checkers there will be on that square after the number is multiplied by 3.

S: After we multiply by 3, there will be six checkers on each square where there are two checkers now.

Ask for a volunteer to make the number on the Minicomputer three times as large using the method suggested above.



T: What calculation is suggested by the checkers on the Minicomputer?

S: "6 × 25".

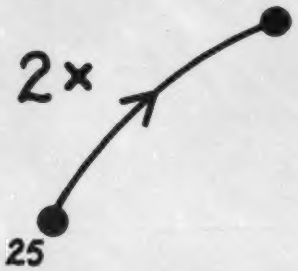
T: I would like to draw an arrow picture about what we have done on the Minicomputer. We started with 25 on the Minicomputer.

Draw a dot and label it "25".

T: Then we doubled the number on the Minicomputer.

N2

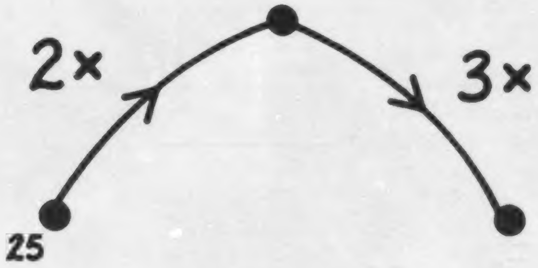
Draw a blue arrow starting at 25 and label it "2 x".



T: What did we do after we multiplied by 2?

S: We multiplied by 3.

Continue the arrow picture.



T (pointing to the dot on the right): What number is this? How do you know?

S: It's the number on the Minicomputer. There are six checkers on the 20-square. 6 × 20 = 120, so there is 120 on the ten's board. 6 × 5 = 30, so there is 30 on the one's board. 120 + 30 = 150.

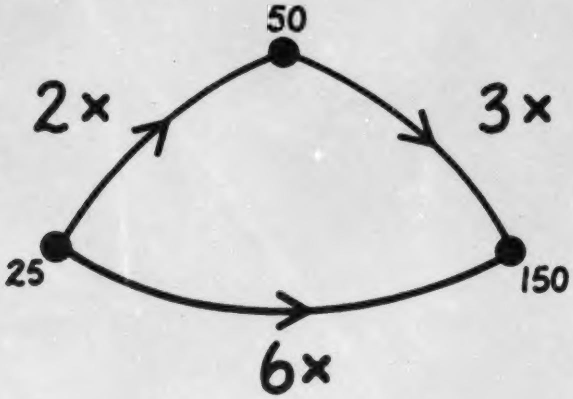
If no one multiplied "2 × 25" and then "3 × 50", suggest this as another approach.

Ask a student to label the dots in the arrow picture. Draw a green arrow from 25 to 150.

T: What could this green arrow be for?

S: 6 ×.

Label the green arrow "6 ×".



T: 6 × 25 = 150. 6 × 25 is the calculation we are doing on the Minicomputer (point to it).

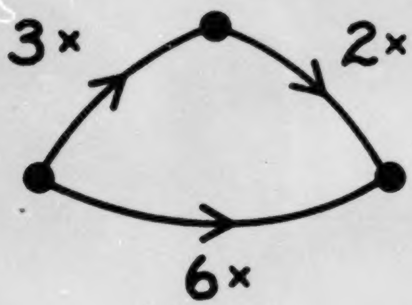
Ask a student to make plays on the Minicomputer until the standard configuration for 150 is obtained.

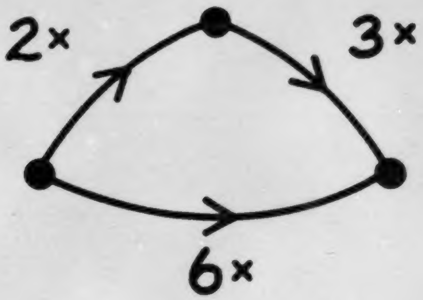
Point to the arrow picture and use motions as described in the Composition Games lessons.

T: A 2× arrow (trace the blue arrow) followed by a 3× arrow (trace the red arrow) is the same as a 6× arrow (trace the green arrow).

N2

Erase the labels for the dots. You should have these two arrow pictures on the board.





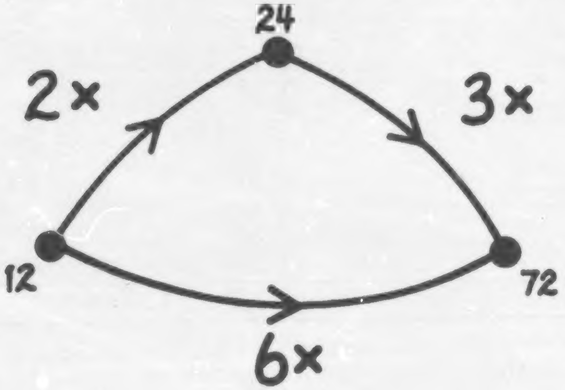
T: Let's see if we can use these arrow pictures to solve some multiplication problems. What number is 6 × 12? Write your answer on a piece of paper.

Write "6 × 12 = ?" on the board. Look at many of the students' papers before asking a student to complete the number story. Invite several students to explain the various methods that they used to calculate the answer. Whenever possible, relate a method to one or the other of the arrow pictures on the board. A sample dialogue is given here.

S: 72. I knew that 2 × 12 = 24, so I added "24 + 24 + 24".

T: You knew that 2 × 12 is 24 and then you added "24 + 24 + 24", which is the same as multiplying "3 × 24". Your method is very similar to this arrow picture which has a 2× arrow followed by a 3× arrow.

Point to the appropriate arrow pictures and label the dots with the student's help.



T: Who calculated "6 x 12" another way?

S: I added "12 + 12 + 12 + 12 + 12 + 12".

Write this addition problem on the board and invite a student to do the addition.

$$\begin{array}{r} {}^{1} \\ 12 \\ 12 \\ 12 \\ 12 \\ 12 \\ +12 \\ \hline 72 \end{array}$$

S: I wrote the same problem on my paper but then I saw that 3 x 12 = 36, so I put the first three twelves together and the last three twelves together and I calculated "36 + 36 = 72".

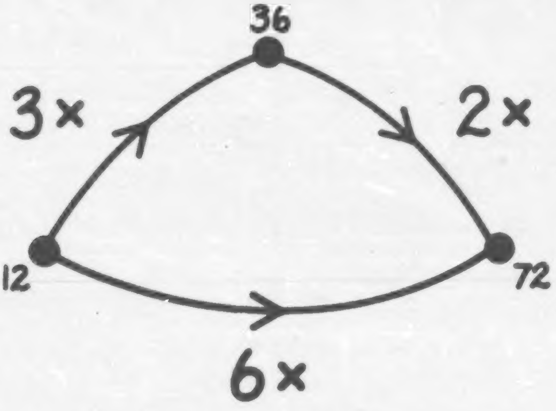
N2

Show this method on the board.

$$\begin{array}{rl}
\left.\begin{array}{r}\overset{1}{1}2\\12\\12\end{array}\right\} & \overset{1}{3}6\\
\left.\begin{array}{r}12\\12\\+12\end{array}\right\} & 36\\
\hline 72 & 72
\end{array}$$

T: Your method is very much like the other arrow picture. Adding three 12's is the same as multiplying "3 × 12". Then you added that answer, 36, to itself, which is the same as multiplying "2 × 36".

Point to the appropriate arrow picture and label the dots with the students' help.



S: I knew that 6 × 10 = 60 and 6 × 2 = 12, so I just added "60 + 12".

Emphasize that multiplying by 3 and then by 2 gives the same answer as multiplying by 2 and then by 3. Erase everything except the arrow pictures. Erase the labels for the dots.

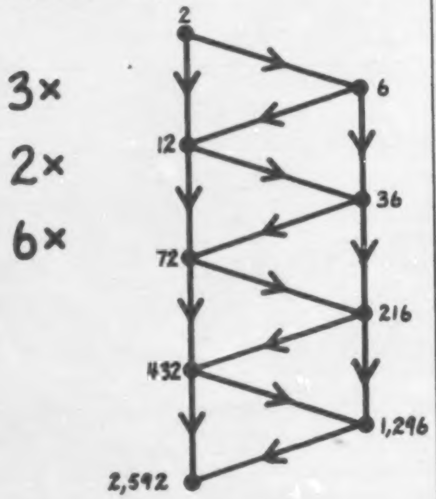
Repeat this activity to calculate "6 x 17".

Worksheets N2* and ** are available for individual work. Encourage the students who finish quickly to draw their own 2 × and 3 × arrow pictures.

Name ____________________ N 2 **

Label the dots.

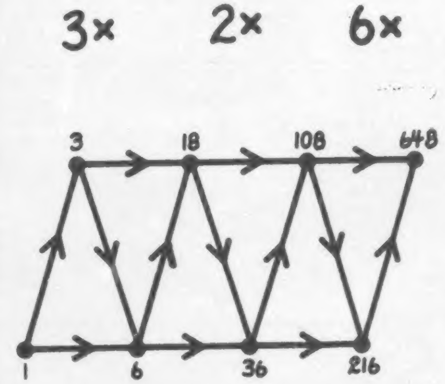
Draw all the possible 6x arrows in green.


3x
2x
6x
2
6
12
36
72
216
432
1,296
2,592

Name ____________________ N 2 *

Label the dots.

Draw all the possible 6x arrows in green.


3x
2x
6x
3
18
108
648
1
6
36
216

N3 Addition of Decimals

CAPSULE LESSON SUMMARY

Decide which of several items can be purchased within the limits of a certain amount of money. Use the Minicomputer to add costs.

MATERIALS

Teacher: Minicomputer set; colored chalk
Student: Worksheets N3*, **, ***, and ****; pencil

DESCRIPTION OF LESSON

Exercise 1

Tell the following or a similar story to your class. Choose one of your students to star in this story. Substitute that student's name for "William" throughout the story.

T: William is visiting a toy store. He has exactly $2 which he would like to spend. Which coins or bills could William have to make exactly $2?

Encourage the students to suggest many ways to have $2 in cash. Combinations your class might suggest include:

- twenty dimes;
- two $1 bills;
- four 50¢ pieces;
- ten dimes and four quarters;
- one $2 bill;
- two hundred pennies.

T: There are many things in the toy store which William would like to buy.

N3

Ask the student who is the star of your story to name three things he/she would like to buy in a toy store. In this lesson description the student chooses a jig-saw puzzle, a model boat, and a racing car track. Assign a price to each object the student chooses.

T: The jig-saw puzzle has a price of $1.25, the model boat costs $0.90 (read: ninety cents), and the price of the racing car track is $1.45.

Record each item and its price on the board.

jig-saw puzzle	$1.25
model boat	$0.90
racing track	$1.45

T: Which toy is the most expensive?

S: The racing car track.

T: Which toy is the least expensive?

S: The model boat.

T: William has exactly $2. Are there any of these toys which he can't afford to buy?

S: Each toy costs less than $2, so he can afford any one of them.

Point to each price in turn and ask if it is less than $2. Write "< $2" to the right of each price.

jig-saw puzzle	$1.25 < $2
model boat	$0.90 < $2
racing track	$1.45 < $2

T: William can afford to buy any one of these toys, but he would like to buy two different toys. Is that possible?

Encourage the students to comment on this problem. A sample dialogue is given here in which a student suggests that William can afford to buy two of the toys.

S: William could buy the jig-saw puzzle and the model boat.

S: No, $1.25 + $0.90 is more than $2.

Write the addition problem on the board.

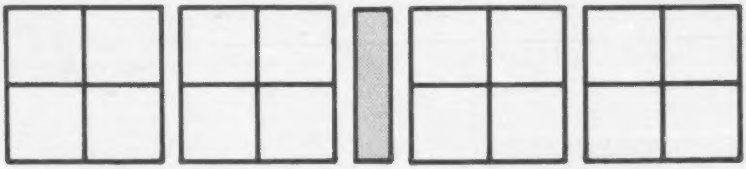
$$1.25 + 0.90 = ?$$

T: How much is $1.25 + $0.90?

Allow several students to whisper their answers to you.

T: Let's calculate "1.25 + $0.90" on the Minicomputer.

Display four Minicomputer boards with a green bar (the gray bar should be drawn in green on the board) between the second and third boards.

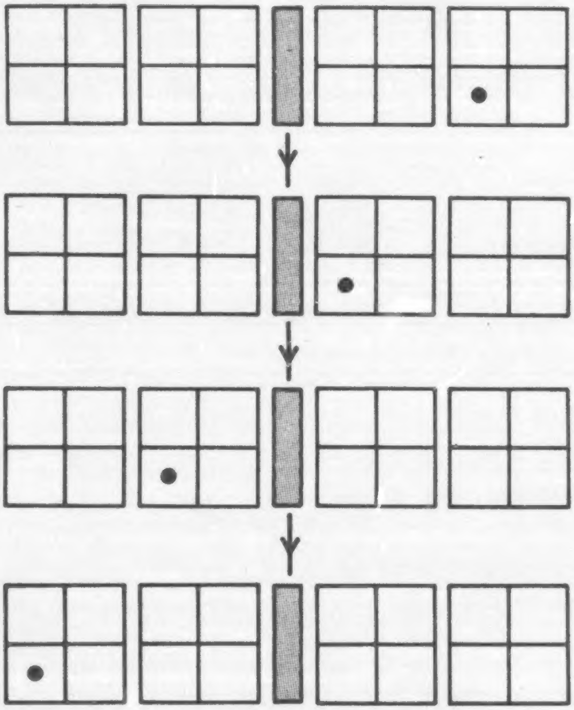


T: What does this green bar tell us?

N3

S: The boards on the left of the green bar are for dollars and the boards on the right of the green bar are for cents.

Briefly review the value of the boards by moving a checker from one red square to the next red square of higher value and asking the students to say each number.



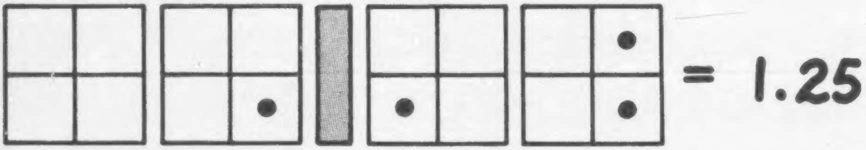
S: 2¢.

S: 20¢.

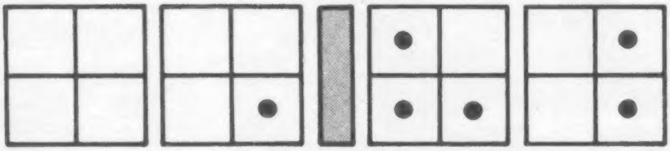
S: $2.

S: $20.

Ask someone to put $1.25 on the Minicomputer.

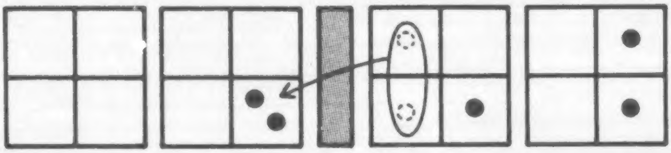


Ask another volunteer to add $0.90 to the number on the Minicomputer.



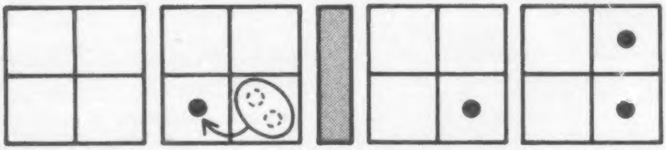
$= 1.25 + 0.90$

Students should suggest making plays at this time.



$= 1.25 + 0.90$

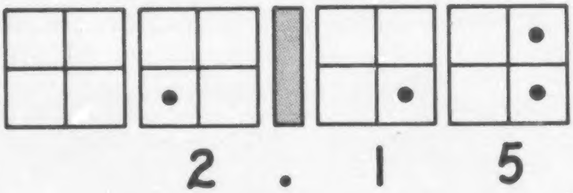
S: 80¢ + 20¢ = $1.



$= 1.25 + 0.90$

S: $1 + $1 = $2.

Ask another student to write the number below the Minicomputer and to complete the number story.



$= 1.25 + 0.90 = 2.15$

If any of the students had correctly calculated the answer mentally before the plays were made on the Minicomputer, ask them which calculations they had done to get the answer.

T: Is there another way we could calculate "$1.25 + $0.90" without using the Minicomputer?

If necessary, suggest this calculation yourself.

$$\begin{array}{r} {}^{1} \\ 1.25 \\ +0.90 \\ \hline 2.15 \end{array}$$

Emphasize that the decimal points must be lined up so that the cents will be added together and the dollars will be added together.

T: $1.25 + $0.90 = $2.15, so William can't afford to buy these two toys. Is it possible for William to buy two different toys?

Continue this activity until your students conclude that it is not possible for William to buy two different toys. Some students might propose alternative toys, but you should stress that William wants to buy two different toys of the three that he chose earlier.

Allow the student who is the star of your story to choose which toy of the three he/she would prefer to buy. Ask the students to determine how much change there will be from the $2 if this toy is purchased. If several students are able to calculate the amount of change mentally, ask them to explain their calculations. You can calculate the amount of change on the Minicomputer if your class is uncertain. If the toy costing $1.45 is selected, you would calculate "2.00 + $\widehat{1.45}$" on the Minicomputer.

Erase the board and remove the checkers from the Minicomputer.

Exercise 2

Tell the following story or a similar one to your class. Choose one of your students to star in this story.

T: Andrea is visiting a stationery store. What could she buy at a stationery store?

You may need to tell your class that stationery stores sell paper, envelopes, cards, pens, and other objects associated with letter writing.

T: Andrea is going on a trip and she wants to buy some stationery so she can write to her friends while she is away. She has $5 to spend. There are four things that Andrea is interested in buying.

List the following items and their prices on the board as they are mentioned.

T: Andrea would like to buy some stationery with clown faces at the top.
A box of this stationery sells for $3.50.
She would also like to buy a package of initialed envelopes that costs $1.10.
There is a box of plaid stationery she likes that costs $1.75.
Andrea also would like to buy a special set of colored pens.
The price of the set is $2.15.

You should have this information on the board.

clown face stationery	$3.50
envelopes	$1.10
plaid paper	$1.75
pens	$2.15

T: Which item is the most expensive?

S: The stationery with the clown faces.

T: Which item is the least expensive?

S: The envelopes.

T: Are there any of these items which Andrea cannot afford to buy? (No) She can afford each of them, but she would like to buy as many different things as possible. What do you suggest?

Encourage the students to comment. A possible dialogue is given here.

S: If Andrea buys the stationery with clown faces, the only other thing she can afford to buy is the envelopes.

T: How much will it cost to buy the stationery with clown faces and the envelopes?

S: \$4.60.

S: She could buy everything except the stationery with the clown faces.

T: How can we be sure? What calculation should we do?

S: "\$1.10 + \$1.75 + \$2.15".

Ask a student to write this calculation on the board and to solve it.

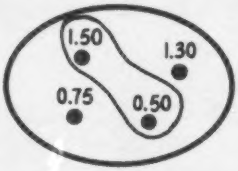
$$\begin{array}{r} 1\;\;1 \\ 1.10 \\ 1.75 \\ +2.15 \\ \hline 5.00 \end{array}$$

Conclude that Andrea can buy everything except the stationery with the clown faces for exactly $5.

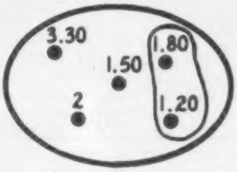
Worksheets N3*, **, ***, and **** are available for individual work.

Name ____________________ **N 3 ***

Carmen buys two different games and spends exactly $2.
Draw one string around the prices of these two games.

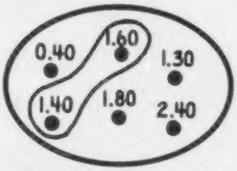

1.50
1.30
0.75
0.50

Anthony buys two different books and spends exactly $3.
Draw one string around the prices of these two books.

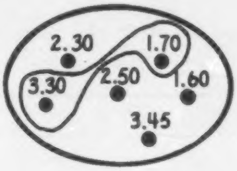

3.30
1.80
1.50
2
1.20

Name ____________________ **N 3 ****

William buys two different magic tricks and spends exactly $3.
Draw one string around the prices of these two magic tricks.

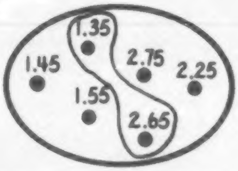

0.40
1.60
1.30
1.40
1.80
2.40

Sharon buys two different paint sets and spends exactly $5.
Draw one string around the prices of these two paint sets.

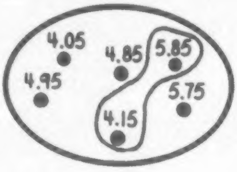

2.30
1.70
3.30
2.50
1.60
3.45

Name ____________________ **N 3 *****

Elizabeth buys two different scarfs and spends exactly $4.
Draw one string around the prices of these two scarfs.

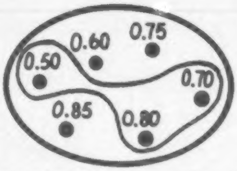

1.35
1.45
2.75
2.25
1.55
2.65

Scott buys two different hats and spends exactly $10.
Draw one string around the prices of these two hats.

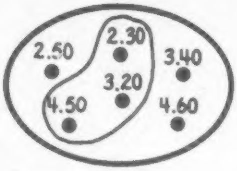

4.05
4.85
5.85
4.95
5.75
4.15

Name ____________________ **N 3 ******

Pat buys three different whistles and spends exactly $2.
Draw one string around the prices of these three whistles.


0.60
0.75
0.50
0.70
0.85
0.80

Elliot buys three different records and spends exactly $10.
Draw one string around the prices of these three records.


2.30
2.50
3.40
3.20
4.50
4.60

N4 Subtraction and Composition #1

CAPSULE LESSON SUMMARY

Subtract one number from another by doing a sequence of subtractions. Record the sequence using an arrow picture.

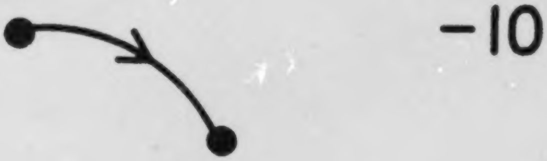
MATERIALS

Teacher: Colored chalk
Student : Paper; colored pencils

DESCRIPTION OF LESSON

Exercise 1: Mental Arithmetic

Draw this arrow picture on the board.



T: Which of these dots is for the larger number?

The students should point to the starting dot.

T: How much larger is this number (point to the starting dot) than the ending number? How do you know?

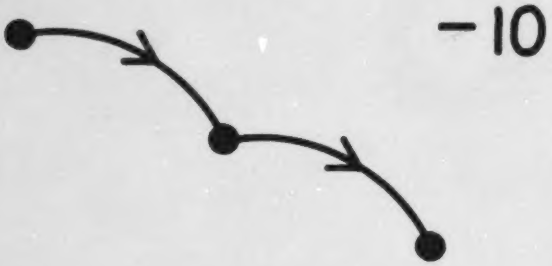
S: It is 10 more than the other number because the arrow is for -10.

T: If this number were 14, what would the ending number be? (4)
If this number were 25, what would the ending number be? (15)

N4

Continue this activity with starting numbers appropriate for the abilities of your students including numbers less than 10 and more than 100.

Extend the arrow picture to include two - 10 arrows.



T (pointing to the starting dot): If this number were 56, what would the ending number be? How do you know?

S: 36, because 56 - 10 = 46 and 46 - 10 = 36.

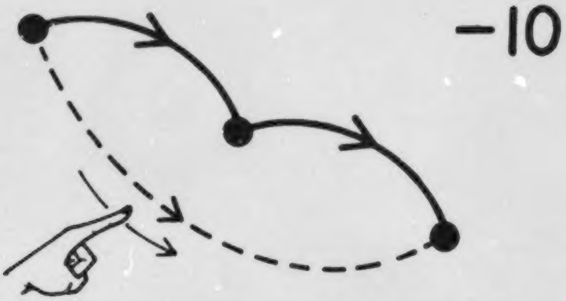
S: 36, because 56 - 20 = 36. Instead of subtracting 10 twice, I just subtracted 20.

Write this number story on the board.

$$56 - 20 = 36$$

T: There are five tens in 56; so if we take away two tens, then there will only be three tens left. And of course there are six ones left because none were taken away.

Trace (but do not draw) an arrow from the starting dot to the ending dot.



T: What could this arrow be for? (- 20)

If we subtract 10 and then subtract 10 again, it is the same as subtracting 20.

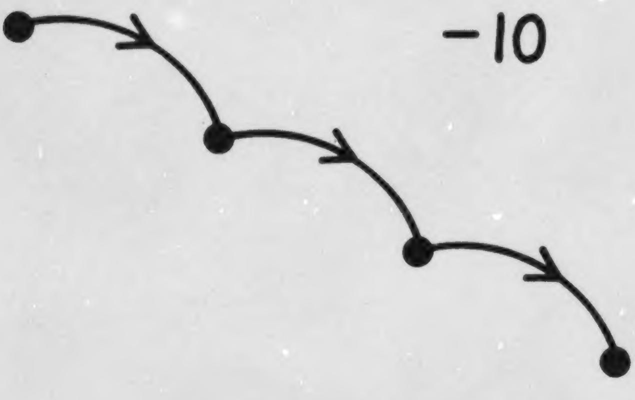
Point to the starting dot.

T: If this number were 43, what would the ending number be? (23)

If this number were 71, what would the ending number be? (51)

Continue this activity with starting numbers appropriate for the abilities of your students. Include several numbers which are more than 100.

Extend the arrow picture to include three - 10 arrows.



T: How much smaller is the ending number than the starting number?

S: 30 smaller.

Trace (but do not draw) an arrow from the starting dot to the ending dot.

T: What could this arrow be for? (-30)
If we subtract 10 three times, it is the same as subtracting 30.

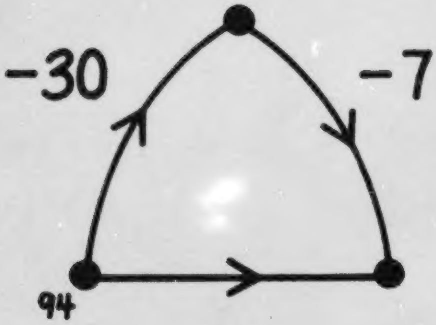
Point to the starting dot.

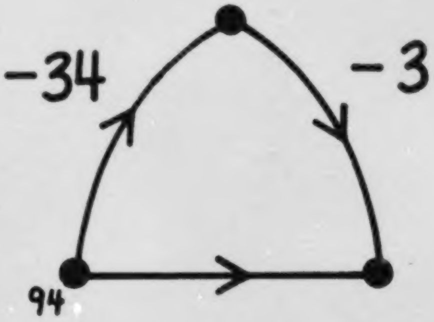
T: If this number were 80, what would the ending number be? (50)
If this number were 85, what would the ending number be? (55)

Continue this activity with starting numbers appropriate for the abilities of your students. Include several numbers which are more than 100. Occasionally write the appropriate number story on the board. Erase the board before going on to the next exercise.

Exercise 2: Composition of Functions

Draw these arrow pictures on the board.





T: What can you tell me about these two arrow pictures? In each of them, what could the green arrow be for?

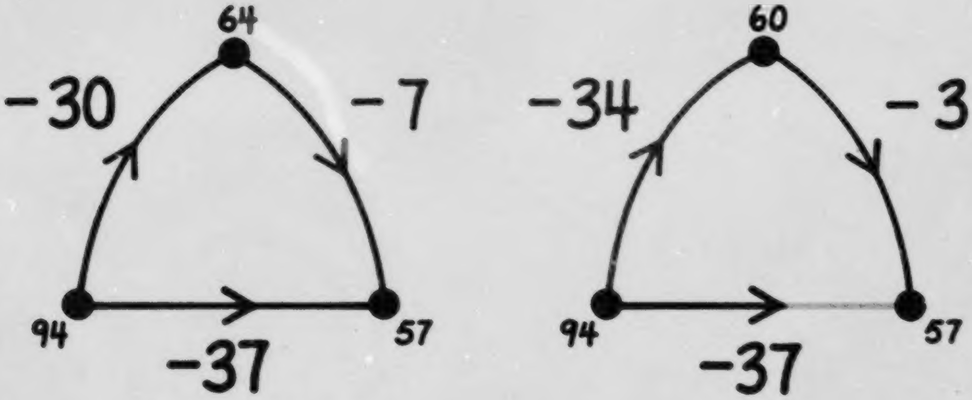
Encourage the students to express themselves.

S: The green arrow could be for - 37 in both arrow pictures.

Write "- 37" in green near each of the green arrows.

S: The ending dot is for 57 in both of these pictures.

Ask students to label the dots.



T: Each of these arrow pictures shows a way that we could calculate "94 - 37".

Point to the arrow picture on the left and trace the appropriate arrows as you say:

T: This arrow picture shows that subtracting 30 and then subtracting 7 is the same as subtracting 37.

N4

Write this problem on the board.

$$\begin{array}{r} 94 \\ -30 \\ \hline \end{array}$$

T: What number is 94 - 30?

S: 64.

$$\begin{array}{r} 94 \\ -30 \\ \hline 64 \end{array}$$

T: There are nine tens in 94, so if we take away three tens there will only be six tens left. And of course there are four ones left since none are taken away.

Now we need to subtract 7 more. What number is 64 - 7?

S: 57.

$$\begin{array}{r} 94 \\ -30 \\ \hline 64 \\ -\ 7 \\ \hline 57 \end{array}$$

T: What number is 94 - 37?

S: 57.

$$\begin{array}{r} 94 \\ -30 \\ \hline 64 \\ -\ 7 \\ \hline 57 \end{array} \qquad \begin{array}{r} 94 \\ -37 \\ \hline 57 \end{array}$$

T: This is one way to calculate "94 - 37".

Point to the arrow picture on the right and trace the appropriate arrows as you say:

T: This arrow picture shows that subtracting 34 and then subtracting 3 is the same as subtracting 37.

Write this problem on the board.

$$\begin{array}{r} 94 \\ -34 \\ \hline \end{array}$$

T: What number is 94 - 34?

S: 60.

$$\begin{array}{r} 94 \\ -34 \\ \hline 60 \end{array}$$

N4

T: There are nine tens in 94. When we take away three tens, there will only be six tens left. There are four ones in 94. When we take away all of them, there will be no ones left.

How much more do we need to subtract? (3 more)
What number is 60 - 3?

S: 57.

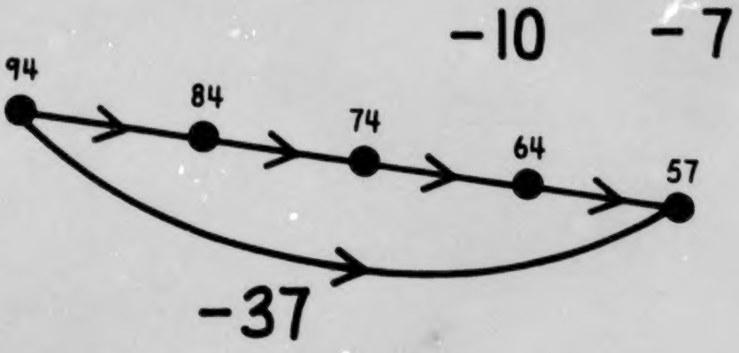
$$\begin{array}{r} 94 \\ -34 \\ \hline 60 \\ -\ 3 \\ \hline 57 \end{array}$$

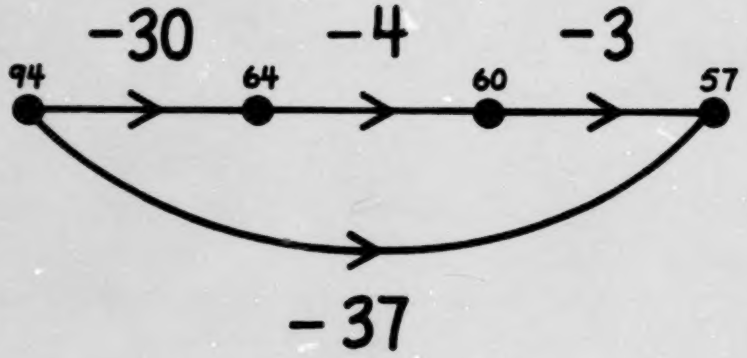
T: What number is 94 - 37?

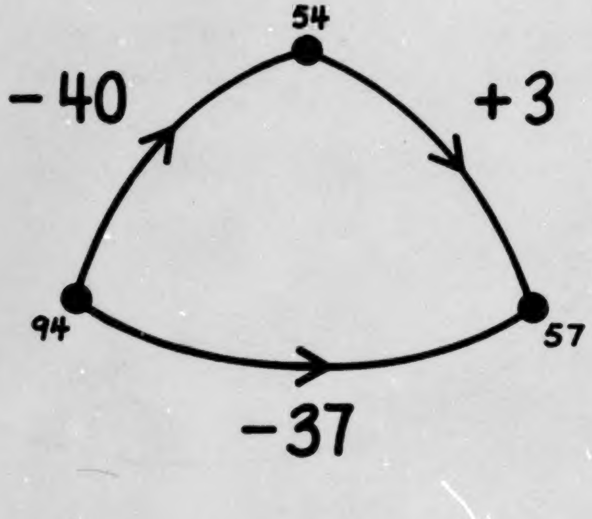
S: 57.

T: Is there another way to calculate "94 - 37"?
How could we show this calculation in an arrow picture?

Record the students' suggestions on the board. A few of the possible arrow pictures are shown below and on the next page.




-30
-4
-3
94
64
60
57
-37


54
-40
+3
94
57
-37

Conclude that there are many ways to calculate "94 - 37". Erase the board before going on to the next exercise.

Exercise 3

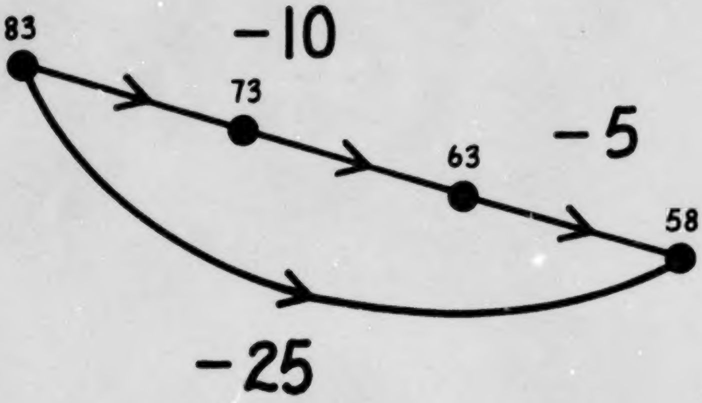
Write this problem on the board.

$$83 - 25 = ?$$

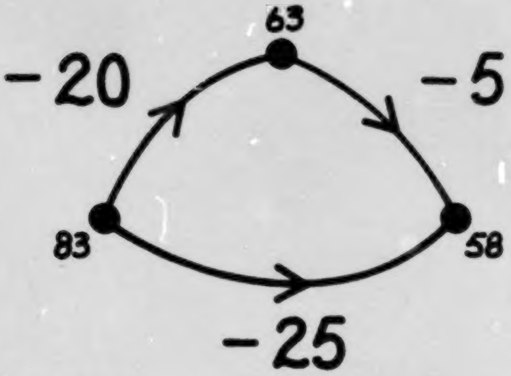
T: Solve this problem in your own way and then draw an arrow picture to show how you did the calculation.

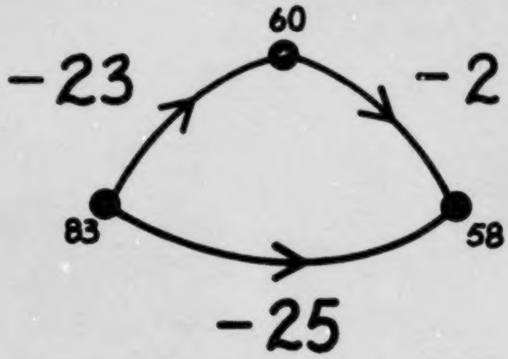
Some students may need help drawing an arrow picture. If these students are able to explain what calculation they did, you can describe for them the corresponding arrow picture. Suggest to students who are unable to calculate "83 - 25" that they start at 83 and draw five -5 arrows.

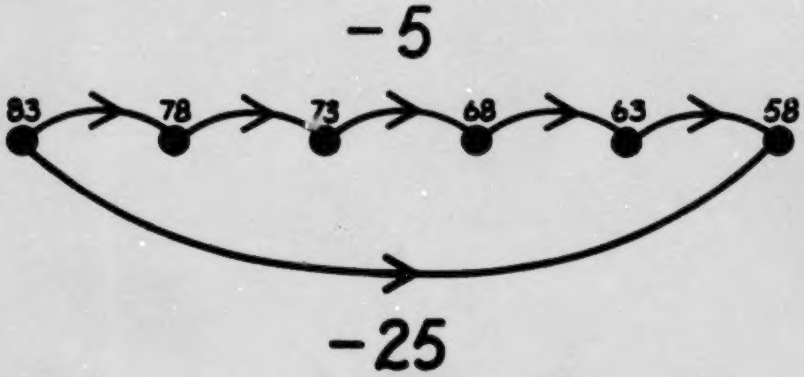
Ask students who finish quickly to draw their arrow pictures on the board. Try to get at least three different arrow pictures on the board. A few of the many possibilities are shown below and on the next page.



107


63
-20
-5
83
58
-25


60
-23
-2
83
58
-25


-5
83
78
73
68
63
58
-25

Discuss each arrow picture briefly and conclude that 83 - 25 = 58. Erase the board and repeat this activity with "105 - 49". If some of the students finish quickly, give them a subtraction problem such as "354 - 297" to solve with an arrow picture.

N5 One-Third of 200

CAPSULE LESSON SUMMARY

Estimate "$\frac{1}{3} \times 135$" and then do the calculation on the Minicomputer. Decide how to share \$200 between three friends.

MATERIALS

Teacher: Minicomputer set[#] ; colored chalk
Student : Worksheets N5* and **; pencil

DESCRIPTION OF LESSON

Begin this lesson with a mental arithmetic activity. These are some suggested sequences of calculations. (Read down the columns.)

2×10
$\frac{1}{2} \times 20$
2×14
$\frac{1}{2} \times 28$
2×18
$\frac{1}{2} \times 36$

$\frac{1}{2} \times 50$
$\frac{1}{2} \times 48$
$\frac{1}{2} \times 52$
$\frac{1}{2} \times 6$
$\frac{1}{2} \times 4$
$\frac{1}{2} \times 5$

$\frac{1}{2} \times 10$
$\frac{1}{2} \times 12$
$\frac{1}{2} \times 11$
$2 \times 5\frac{1}{2}$
$\frac{1}{2} \times 9$
$2 \times 4\frac{1}{2}$

3×10
$\frac{1}{3} \times 30$
3×13
$\frac{1}{3} \times 39$
3×15
$\frac{1}{3} \times 45$

T: What number is 3×33? (99)
$3 \times 33 = 99$, so what number is $\frac{1}{3} \times 99$? (33)

Approximately what number is $\frac{1}{3} \times 135$?

Write "$\frac{1}{3} \times 135$" on the board. Use whatever answers the students suggest to demonstrate a strategy for estimating "$\frac{1}{3} \times 135$". A possible dialogue is given here.

[#] More than thirty regular checkers might be needed for this lesson.

S: We know that $\frac{1}{3} \times 135$ is more than 33 because 135 is more than 99.

Indicate the appropriate inequality.

$$\frac{1}{3} \times 135 > 33$$

S: I think $\frac{1}{3} \times 135 = 42$.

T: How could we determine whether or not $\frac{1}{3} \times 135 = 42$?

S: Calculate "42 + 42 + 42".

Write the addition problem on the board and invite a student to solve it.

$$\begin{array}{r} 42 \\ 42 \\ +42 \\ \hline 126 \end{array}$$

T: $3 \times 42 = 126$. Is $\frac{1}{3} \times 135$ more or less than 42?

S: More.

Change the inequality appropriately.

$$\frac{1}{3} \times 135 > 42$$

S: I think $\frac{1}{3} \times 135 = 49$.

T: If $\frac{1}{3} \times 135 = 49$, then $3 \times 49 = 135$. Let's see if that's true.

N5

Invite a student to check "3 × 49 = 135" by doing the corresponding addition problem at the board.

$$\begin{array}{r} 49 \\ 49 \\ +\,49 \\ \hline 147 \end{array}$$

T: 3 × 49 = 147, so is $\frac{1}{3}$ × 135 more or less than 49?

S: Less.

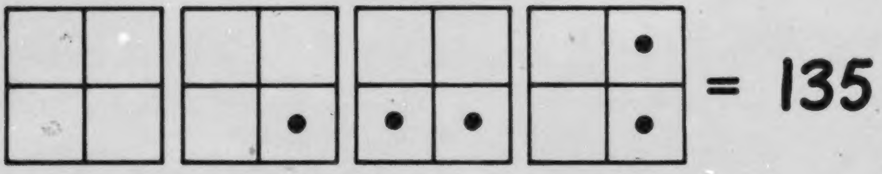
Record this new information on the board.

$$\frac{1}{3} \times 135 > 42$$

$$\frac{1}{3} \times 135 < 49$$

T: We know that $\frac{1}{3}$ x 135 is more than 42 and is less than 49.

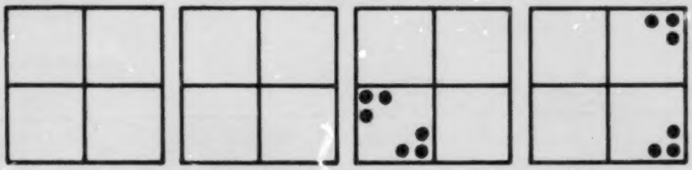
Continue this activity until your class concludes that $\frac{1}{3}$ × 135 = 45. Record this number story on the board and ask a student to put 135 on the Minicomputer.



T: We already know that $\frac{1}{3}$ × 135 = 45, but is there a way to calculate "$\frac{1}{3}$ × 135" on the Minicomputer?

S: We could make plays until all the checkers are in groups of three.

Invite students to make plays on the Minicomputer. When a group of three checkers on the same square is formed, push the three checkers to a corner of that square. Do not permit a student to break up a group of three checkers. Continue until all the checkers on any square of the Minicomputer are in groups of three. Perhaps you will have this configuration on the Minicomputer.



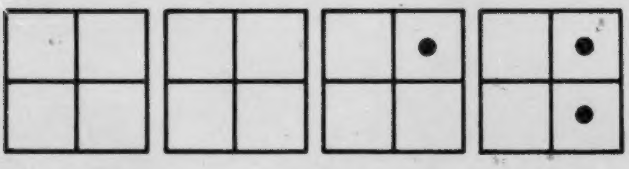
$= 135$

T: All the checkers are in groups of three, so how do we find $\frac{1}{3} \times 135$ on the Minicomputer?

S: Take off checkers and leave just one checker from each group of three.

S: Remove two of the checkers from each group of three.

Ask for a volunteer to remove checkers so that $\frac{1}{3} \times 135$ will be on the Minicomputer. If necessary, make some plays yourself to obtain the standard configuration for 45.



$= \frac{1}{3} \times 135 = 45$

Erase the board and remove the checkers from the Minicomputer.

Exercise 3

T: I have \$200 in cash. What coins or bills could I have?

N5

Encourage the class to offer many combinations. For example,

- two \$100 bills ;
- two hundred \$1 bills ;
- four hundred half-dollars ;
- five \$20 bills and ten \$10 bills.

T: I have three friends among whom I would like to share this \$200.

Choose three students to be your friends in this story.

T: I wish to give each of my three friends exactly one-third of the \$200. How much should I give each of them?

S: \$50.

T: If I give Cameron, Marian, and Rod each \$50, how much money will I have given away? (\$150)
I want to give each of my friends one-third of the \$200.
Is $\frac{1}{3} \times 200$ more or less than 50?

S: More.

Write this inequality on the board.

$$\frac{1}{3} \times 200 > 50$$

T: We know that each of my friends will receive at least \$50 because $\frac{1}{3} \times 200$ is more than 50.

S: I think that you can give each of them \$70.

T: What number is 3 × 70?

Write the corresponding addition problem on the board and ask a student to solve it.

$$\begin{array}{r} 70 \\ 70 \\ +\,70 \\ \hline 210 \end{array}$$

T: Do I have enough money to give each of my friends \$70?

S: No, you would need \$210 to give each of them \$70.

T: $\frac{1}{3} \times 200$ is less than 70.

Record this information on the board.

$$\frac{1}{3} \times 200 > 50$$

$$\frac{1}{3} \times 200 < 70$$

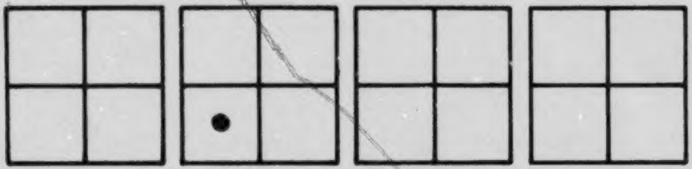
T: We know that $\frac{1}{3} \times 200$ is more than 50 and less than 70.

Continue this activity until your class concludes that each of your friends should receive between \$66 and \$67.

T: Let's calculate $\frac{1}{3} \times 200$ on the Minicomputer.

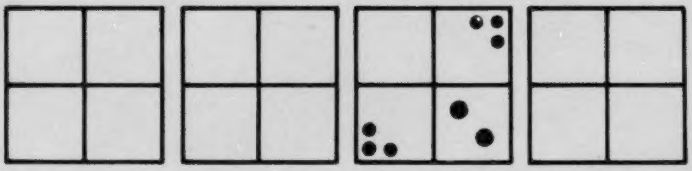
N5

Put 200 on the Minicomputer.



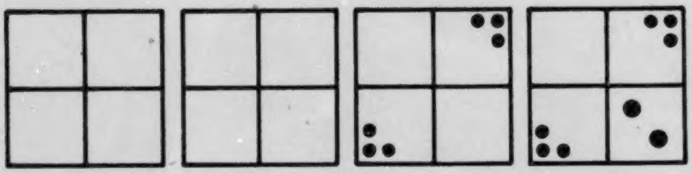
Invite students to make plays. Remind the students that they need to get the checkers in groups of three checkers on the same square. Since there are many plays to be made, you can hasten this activity by encouraging each volunteer to make several plays.

Before the jump is made from the tens' board to the ones' board, ...



... mention that all the checkers are in groups of three except the two extra checkers on the 10-square.

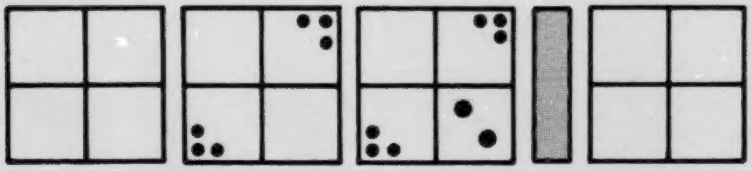
Continue this activity until all the checkers are in groups of three except for two extra checkers on the 1-square. Perhaps you will have this configuration on the Minicomputer.



T: All the checkers are in groups of three except for two extra checkers on the 1-square. These checkers are for $2 which needs to be shared. What could we do now?

Encourage the students to suggest solutions, then move a Minicomputer board to the right of the ones'-board leaving extra space between these two boards.

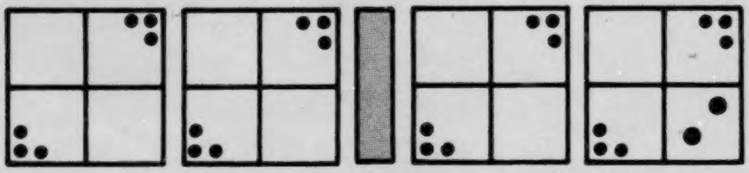
If no one mentions drawing the green bar between the two boards on the right, ask if you have changed the number on the Minicomputer, and then draw the green bar yourself.



T: The green bar will remind us that the Minicomputer boards on the left are for dollars and the boards on the right are for cents.

Invite the students to continue making plays until all the checkers are in groups of three except the two checkers on the white square of the dimes' board. Some students may predict that to continue making plays onto the pennies' board would only result in the same situation because there will still be two extra checkers on the white square of the pennies' board after the appropriate plays have been made.

Move a Minicomputer board to the right of the dimes' board and continue inviting students to make plays until all the checkers are in groups of three except the two extra checkers on the white square of the pennies' board. Perhaps you will have this configuration on the Minicomputer.



T: What would happen if I put another Minicomputer board on the right?

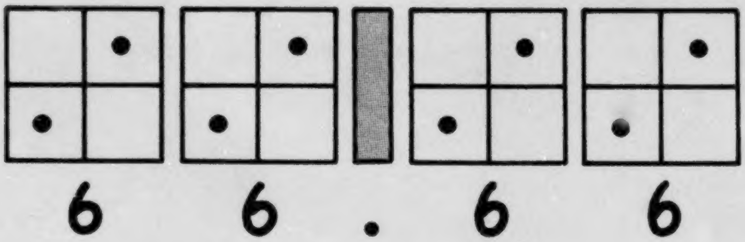
S: If we make plays, we would have two extra checkers on the white square.

T: No matter how many boards we use, there will always be two extra checkers on the last (furthest to the right) white square. Two checkers on this white square (point to the extra checkers) are for what? (Two pennies)

I don't know how to share two pennies among three people so I will keep them.

Remove the two extra checkers from the 1¢-square.

Ask a volunteer to remove some checkers so that one-third of this number will be on the Minicomputer. This student should remove two checkers from each group of three. Invite another student to write the number below the Minicomputer.



T: How much money should I give each of my three friends?

S: $66.66.

T: If I give each of them $66.66, how much money will I give away altogether?

Write this problem on the board and ask for a volunteer to solve it.

$$\begin{array}{r} \overset{1}{6}\overset{1}{6}.\overset{1}{6}6 \\ 66.66 \\ +\,66.66 \\ \hline 199.98 \end{array}$$

T: Will I be able to give each of my friends $66.66?

S: Yes, because $199.98 is less than $200.

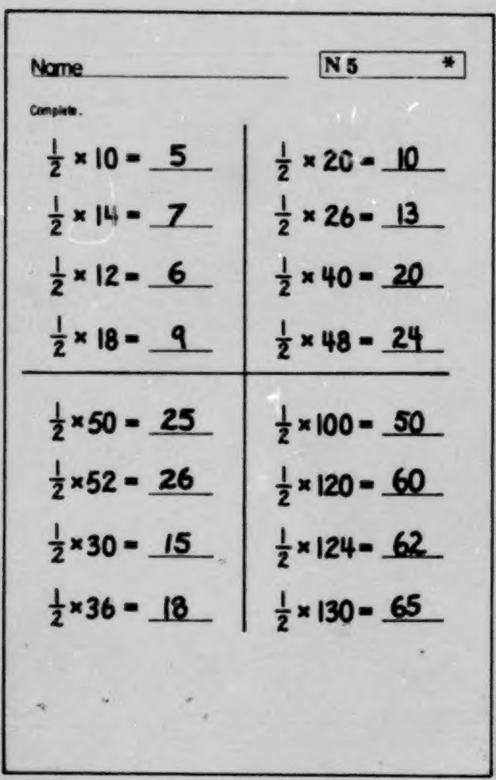
T: How much less... ?

S: 2 cents.

Write this number story on the board.

$$(3 \times 66.66) + 0.02 = 200$$

Worksheets N5* and ** are available for individual work.



Name ______________ N 5 *

Complete.

$\frac{1}{2} \times 10 =$ 5

$\frac{1}{2} \times 14 =$ 7

$\frac{1}{2} \times 12 =$ 6

$\frac{1}{2} \times 18 =$ 9

$\frac{1}{2} \times 50 =$ 25

$\frac{1}{2} \times 52 =$ 26

$\frac{1}{2} \times 30 =$ 15

$\frac{1}{2} \times 36 =$ 18

$\frac{1}{2} \times 20 =$ 10

$\frac{1}{2} \times 26 =$ 13

$\frac{1}{2} \times 40 =$ 20

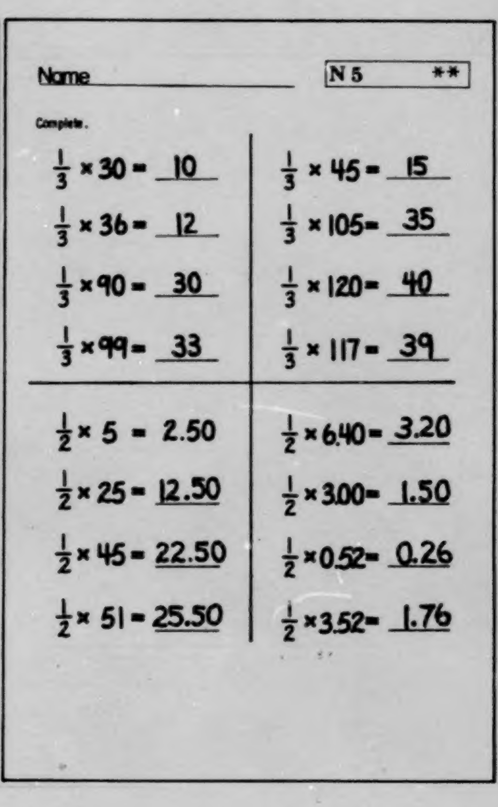
$\frac{1}{2} \times 48 =$ 24

$\frac{1}{2} \times 100 =$ 50

$\frac{1}{2} \times 120 =$ 60

$\frac{1}{2} \times 124 =$ 62

$\frac{1}{2} \times 130 =$ 65



Name ______________ N 5 **

Complete.

$\frac{1}{3} \times 30 =$ 10

$\frac{1}{3} \times 36 =$ 12

$\frac{1}{3} \times 90 =$ 30

$\frac{1}{3} \times 99 =$ 33

$\frac{1}{2} \times 5 =$ 2.50

$\frac{1}{2} \times 25 =$ 12.50

$\frac{1}{2} \times 45 =$ 22.50

$\frac{1}{2} \times 51 =$ 25.50

$\frac{1}{3} \times 45 =$ 15

$\frac{1}{3} \times 105 =$ 35

$\frac{1}{3} \times 120 =$ 40

$\frac{1}{3} \times 117 =$ 39

$\frac{1}{2} \times 6.40 =$ 3.20

$\frac{1}{2} \times 3.00 =$ 1.50

$\frac{1}{2} \times 0.52 =$ 0.26

$\frac{1}{2} \times 3.52 =$ 1.76

N6 An Arrow Road with 10x and +1

CAPSULE LESSON SUMMARY

Find the ending numbers of all possible roads which start at 1 and which have exactly two 10x arrows and nine +1 arrows.

MATERIALS

Teacher: Minicomputer set; ⑩-checkers; colored chalk
Student: Paper; colored pencils

NOTE: Before the lesson begins, magnetize each ⑩-checker by sticking a piece of magnetic material to the back.

DESCRIPTION OF LESSON

Exercise 1: 10x on the Minicomputer

Begin the lesson with mental arithmetic. We suggest these calculations.

4 × 10 (40)
10 × 4 (40)
8 × 10 (80)
10 × 8 (80)

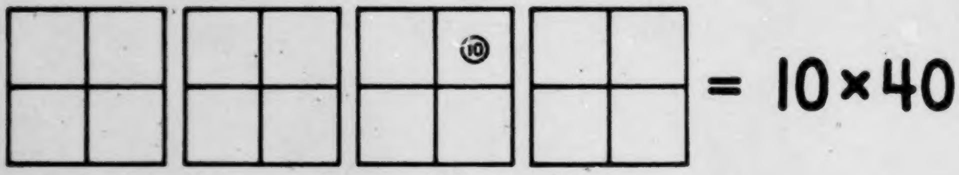
10 × 5 (50)
10 × 0 (0)
10 × 10 (100)
10 × 11 (110)

T: How do you know that 10 x 11 = 110?

S: 10 x 10 = 100, so 11 x 10 = 100 + 10 = 110.

S: I just put a "0" after the "11".

Put a ⑩-checker on the 40-square of the Minicomputer and write the 10x calculation to one side of the Minicomputer.

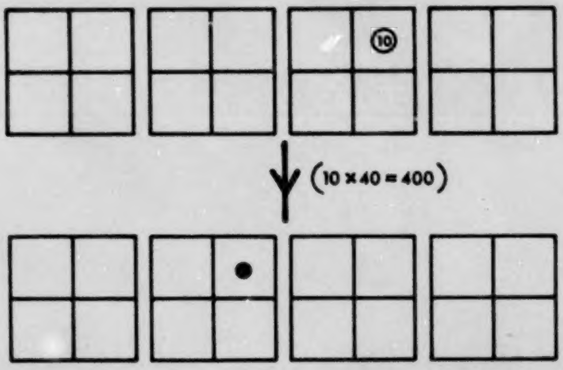


N6

T: What number is 10 × 40?

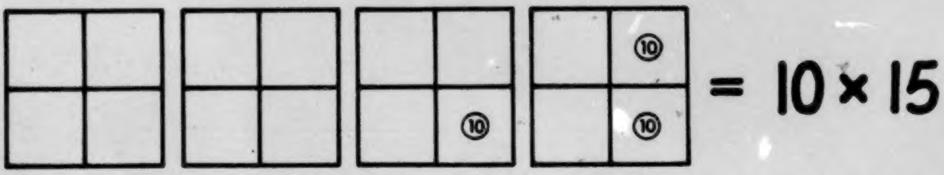
S: 400.

Complete the number story on the board and make the play yourself.



Erase the board.

Put 10 × 15 on the Minicomputer and write the calculation to one side.



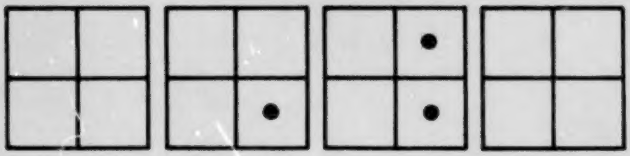
T: What number is 10 × 15? (150)
How do you know?

S: I just put a "0" after the "15"; 10 × 15 = 150.

S: I know that 10 × 10 = 100 and 10 × 5 = 50, so
10 × 15 = 100 + 50 = 150.

S: I imagined the plays. There would be regular checkers on the 100, 40, and 10-squares.

Ask a student to make the plays. Complete the number story to the side of the Minicomputer.



= 10 × 15 = 150

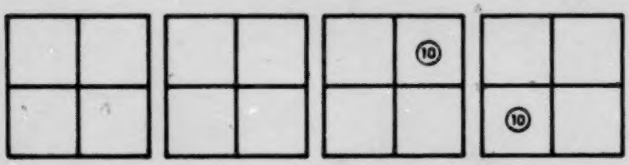
T: Let's record this information in a "10 ×" table.

Begin a table on the board.

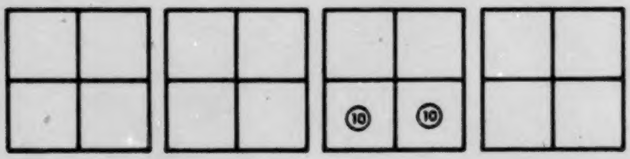
Start	10×
15	150

T: The number in the column on the left is the number we are multiplying by 10. The number in the column on the right is 10 × the number on the left.

Continue this activity with these calculations.



= 10 × 42 = 420



= 10 × 30 = 300

N6

Your table should look like this.

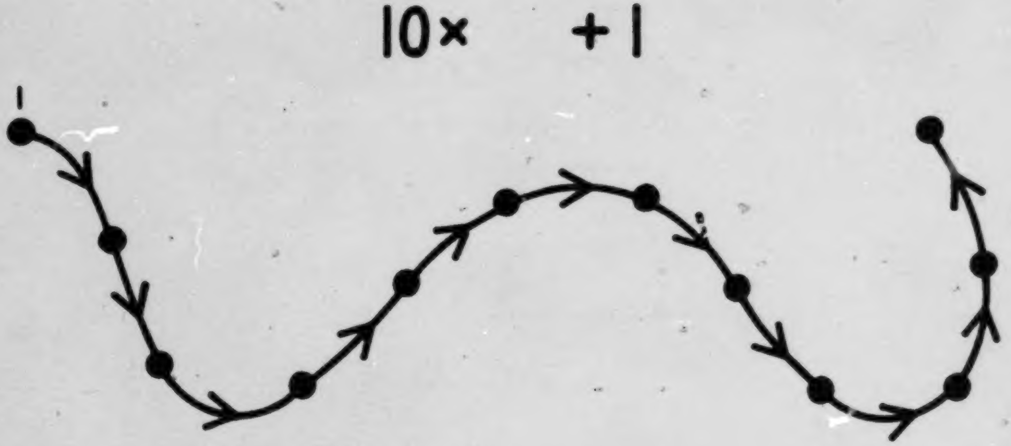
Start	10×
15	150
42	420
30	300

Continue this activity with two and three digit numbers suggested by your class. When you have six or more pairs of numbers in your table, ask the students if they see any patterns. Encourage the class to notice that all the numbers listed in the column on the right end in "0" and if you erase the "0" on the far right of each numeral on the right, the result is the same numeral as on the left.

Erase the board before going on to Exercise 2.

Exercise 2: A 10× and +1 Arrow Road

Draw this arrow picture on the board.

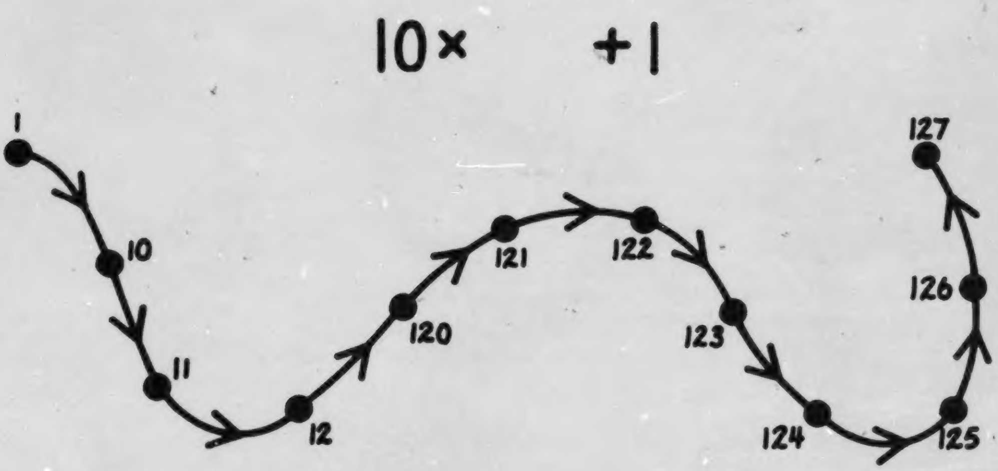


T: What number is the ending dot of this arrow road?
Whisper your answer to me.

Allow several students to whisper their answer to you before asking a student to answer aloud.

S: The ending dot is for 127.

Invite students to label the dots on the board.



T: The starting number of this arrow road is 1 and the ending number is 127. How many 10× arrows are there in this arrow road? (Two)
How many + 1 arrows... ? (Nine)
Would it be possible to draw a different arrow road which also starts at 1 and has two 10× arrows and nine + 1 arrows? (Yes)

Encourage the students to describe at least one arrow road with two 10× arrows and nine + 1 arrows. Ask if the ending number of this road is 127 and allow the students to explain their answers.

Ask the students to draw a different red-blue road on their papers. The road should start at 1 and have exactly two 10× arrows and nine + 1 arrows.

You may need to help individual students get started. When a student completes a road different from the road on the board, record the ending number of this road on the board. Encourage those who finish quickly to build another red-blue road starting at 1 with the largest (or smallest) possible ending number.

Soon you should have a list of about fifteen to eighteen possible ending numbers on the board. There are fifty-five possible ending numbers[*]; a few of which are listed here.

424
127
109
208
280
631
1,000
226
415
811
334
550
820
406
721
190
361
901

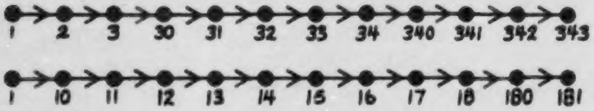
T: What is the largest number in our list of ending numbers? Would it be possible to have an ending number larger than this one? What is the largest possible ending number?

S: 1,000.

T: In what order should we draw the red and the blue arrows if we want to make the ending number as large as possible?

---

[*] For your information, the possible ending numbers are 1,000 and any three-digit number in which the sum of the digits is 10. For example, 343 (3 + 4 + 3 = 10) is a possible ending number, but 185 (1 + 8 + 5 ≠ 10) is not.



(all nine +1 arrows have been used -- cannot get to 185)

S: We should draw the nine blue arrows first and then the two 10× arrows.

If necessary, tell your class that the largest possible ending number is 1,000.

T: What is the smallest number in our list of ending numbers?
Would it be possible to have an ending number smaller than this one?
What is the smallest possible ending number?

S: 109.

T: In what order should we draw the red and blue arrows if we want to make the ending number as small as possible?

S: We should draw the two red arrows first and then the nine blue arrows.

If necessary, tell your class that the smallest possible ending number is 109.

Perhaps a student will notice that the sum of the digits of each number except 1,000 is 10. You may wish to point this out yourself if none of the students mentions it.

N7

N7 Subtraction Problems #2

CAPSULE LESSON SUMMARY

Compare 7 - 3 to 3 - 7, 8 - 5 to 5 - 8, and so on. Calculate "68 - 29" by subtracting "8 - 9" and "60 - 20".

$$\begin{array}{r} 68 \\ \underline{-29} \\ 4\hat{1} \end{array} = 40 + \hat{1} = 39$$

MATERIALS

Teacher: Minicomputer set
Student: Paper; pencil; Worksheets N7*, **, ***, ****

DESCRIPTION OF LESSON

Begin this lesson with a few minutes of mental arithmetic.

T: What number is 7 - 3? (4)
What number is 3 - 7? ($\hat{4}$)
How do you know?

S: 3 - 3 = 0 and 0 - 4 = $\hat{4}$.

Demonstrate "7 - 3" on the number line by starting at 7 and moving to the left three marks. Demonstrate "3 - 7" on the number line by starting at 3 and moving to the left seven marks.

T: What number is 8 - 5? (3)
5 - 8 = . . . ? ($\hat{3}$)
4 - 9 = . . . ? ($\hat{5}$)
9 - 4 = . . . ? (5)

T: Do you see any patterns?

S: The answers are the same except one number is positive and the other is negative.[*]

---

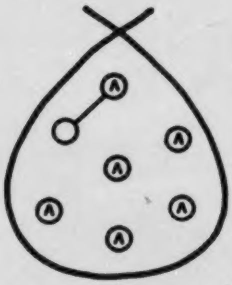
[*] This response is not exactly correct, but it is a likely student response and the intention is correct.

T: What number is 6 - 1? (5)

What number is 1 - 6? ($\hat{5}$)

Could a "peanut picture" help us calculate "1 - 6"? (Yes)

Draw a bag with one regular peanut and six magic peanuts on the board.



Conclude that $1 + \hat{6} = \hat{5}$, so $1 - 6 = \hat{5}$.

T: What number is 12 - 5? (7)

What number is 5 - 12? ($\hat{7}$)

Continue with these calculations:

15 − 9 (6)	9 − 8 (1)	6 − 14 ($\hat{8}$)
9 − 15 ($\hat{6}$)	8 − 9 ($\hat{1}$)	14 − 6 (8)

Write this problem on the board.

$$\begin{array}{r} 68 \\ -\,29 \\ \hline \end{array}$$

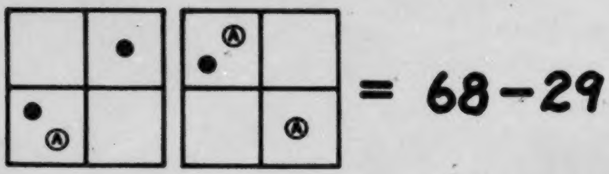
T: What number is 68 - 29? Write your answer on your paper.

Look at several of the students' answers.

Display two Minicomputer boards.

T: How could we calculate "68 - 29" on the Minicomputer?

S: First, put 68 on the Minicomputer with regular checkers and $\widehat{29}$ on with negative checkers.

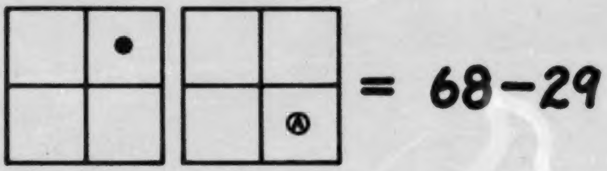


T: What should we do now?

S: We should take off the checkers on the 40-square and on the 8-square.

T: Why?

S: Because there is one regular and one negative checker on each of those squares.



S: We need to make some backward plays.

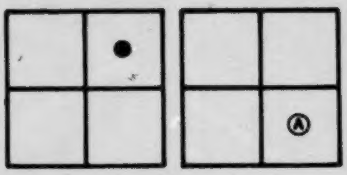
T: I think we can calculate the answer without making any more plays. What number do you see on the tens' board?

S: 40.

T: What number do you see on the ones' board?

S: $\hat{1}$.

Continue the number story.

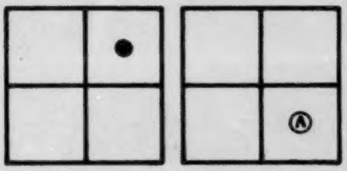


$$= 68 - 29 = 40 + \hat{1}$$

T: What number is 40 + $\hat{1}$? (39)

So what number is 68 - 29? (39)

Complete the number story.



$$= 68 - 29 = 40 + \hat{1} = 39$$

Point to the ones' column of the subtraction problem on the board.

T: What number is 8 - 9?

S: $\hat{1}$.

Write "$\hat{1}$" below the ones' column and point to the tens' column.

T: What number is 6 tens - 2 tens?

S: 40.

Write "4" below the tens' column.

$$\begin{array}{r} 68 \\ -\,29 \\ \hline 4\hat{1} \end{array}$$

N7

T (point to "$4\hat{1}$"): Have you ever seen a number like this before? (No)

This number is 4 tens, or 40, plus $\hat{1}$.

$$\begin{array}{r@{}l} 68 & \\ -29 & \\ \hline 4\hat{1} & = 40 + \hat{1} \end{array}$$

T: What number is 40 + $\hat{1}$?

S: 39.

$$\begin{array}{r@{}l} 68 & \\ -29 & \\ \hline 4\hat{1} & = 40 + \hat{1} = 39 \end{array}$$

Erase the board and remove the checkers from the Minicomputer. Repeat this activity with these problems. (The answers are in the boxes.)

$$\begin{array}{r} 75 \\ -47 \\ \hline \boxed{28} \end{array} \qquad \begin{array}{r} 70 \\ -35 \\ \hline \boxed{35} \end{array} \qquad \begin{array}{r} 49 \\ -51 \\ \hline \boxed{\hat{2}} \end{array}$$

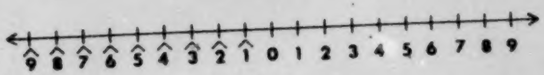
Worksheets N7*, **, ***, and **** are available for individual work.

Optional Activity

If you have some extra time at the end of this lesson, you may wish to play The Number Line Game. Choose a non-integer decimal between 0 and 10 to be your secret number.

N7 *

Name ______________


$\hat{9}$ $\hat{8}$ $\hat{7}$ $\hat{6}$ $\hat{5}$ $\hat{4}$ $\hat{3}$ $\hat{2}$ $\hat{1}$ 0 1 2 3 4 5 6 7 8 9

Complete.

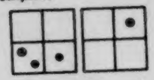
$\begin{array}{r}5\\-2\\\hline 3\end{array}$ $\begin{array}{r}2\\-5\\\hline \hat{3}\end{array}$ $\begin{array}{r}7\\-6\\\hline 1\end{array}$ $\begin{array}{r}6\\-7\\\hline \hat{1}\end{array}$ $\begin{array}{r}4\\-5\\\hline \hat{1}\end{array}$ $\begin{array}{r}5\\-4\\\hline 1\end{array}$ $\begin{array}{r}3\\-6\\\hline \hat{3}\end{array}$ $\begin{array}{r}6\\-3\\\hline 3\end{array}$

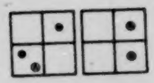
$\begin{array}{r}8\\-1\\\hline 7\end{array}$ $\begin{array}{r}1\\-8\\\hline \hat{7}\end{array}$ $\begin{array}{r}0\\-4\\\hline \hat{4}\end{array}$ $\begin{array}{r}4\\-0\\\hline 4\end{array}$ $\begin{array}{r}9\\-7\\\hline 2\end{array}$ $\begin{array}{r}7\\-9\\\hline \hat{2}\end{array}$ $\begin{array}{r}8\\-2\\\hline 6\end{array}$ $\begin{array}{r}2\\-8\\\hline \hat{6}\end{array}$

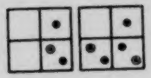
N7 **

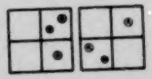
Name ______________

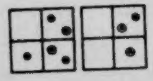
Complete.

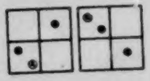

$\begin{array}{r}30\\-24\\\hline 6\end{array}$


$\begin{array}{r}60\\-25\\\hline 35\end{array}$


$\begin{array}{r}57\\-13\\\hline 44\end{array}$


$\begin{array}{r}42\\-56\\\hline \hat{14}\end{array}$

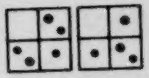

$\begin{array}{r}74\\-55\\\hline 19\end{array}$

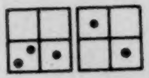

$\begin{array}{r}69\\-28\\\hline 41\end{array}$

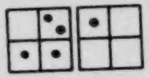
N7 ***

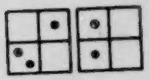
Name ______________

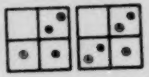
Complete.

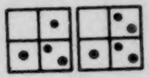

$\begin{array}{r}73\\-65\\\hline 8\end{array}$


$\begin{array}{r}38\\-21\\\hline 17\end{array}$


$\begin{array}{r}70\\-48\\\hline 22\end{array}$

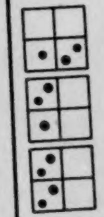

$\begin{array}{r}62\\-28\\\hline 34\end{array}$

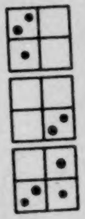

$\begin{array}{r}56\\-67\\\hline \hat{11}\end{array}$

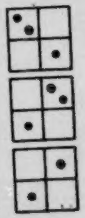

$\begin{array}{r}55\\-37\\\hline 18\end{array}$

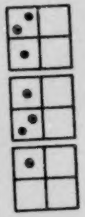
N7 ****

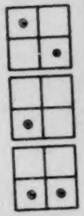
Name ______________


$\begin{array}{r}523\\-433\\\hline 90\end{array}$


$\begin{array}{r}345\\-249\\\hline 96\end{array}$


$\begin{array}{r}691\\-288\\\hline 403\end{array}$


$\begin{array}{r}310\\-232\\\hline 78\end{array}$


$\begin{array}{r}400\\-215\\\hline 185\end{array}$

N8 Roads #1

CAPSULE LESSON SUMMARY

Build arrow roads using 2×, + 1, and - 1 arrows.

MATERIALS

Teacher: Colored chalk

Student : Worksheets N8*, **, ***, and ****; colored pencils

DESCRIPTION OF LESSON

T: What number is 2 × 7? (14)

2 × 7 = 14, so what number is 2 × 17? (34)

How did you calculate "2 × 17"?

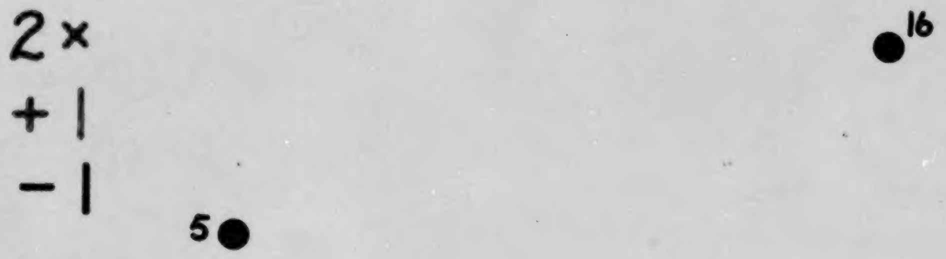
Encourage several students to explain their methods of calculating the answer. Continue this activity with these problems.

2 × 8	(16)	2 × 9	(18)
2 × 18	(36)	2 × 19	(38)
2 × 28	(56)	2 × 109	(218)
		2 × 1,009	(2,018)

Distribute Worksheet N8* to the students.

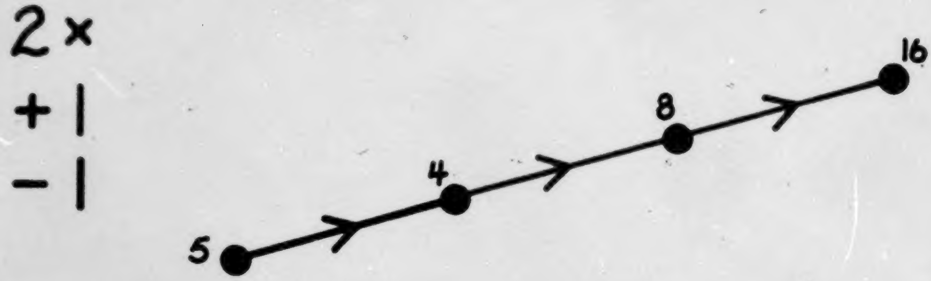
T: Today we are going to build roads using only 2×, + 1, and - 1 arrows. On the first worksheet, build a road between 5 and 16, but try to make your road as short as possible.

While the students are working independently, draw this picture on the board.



Walk around and observe the students' work. As students finish their roads, count the number of arrows in each road. Challenge the students to find even shorter roads by telling the class how many arrows are in the shortest road any student has built up till that time. When most of the students have built a road between 5 and 16, ask a student with the shortest road (of those drawn by your students) to draw this road on the board. Discuss this road very briefly. If this road has more than three arrows, you may wish to tell your students that there is an even shorter road between 5 and 16 and that perhaps they could work on this problem again some other day.

The shortest possible road from 5 to 16 has three arrows. This is for your information only.



Repeat this activity with Worksheets N8** and ***. Encourage the students to find the shortest possible roads, but remind them that their primary objective is to build a road between each pair of numbers. You might challenge them to use ten or fewer arrows in each road.

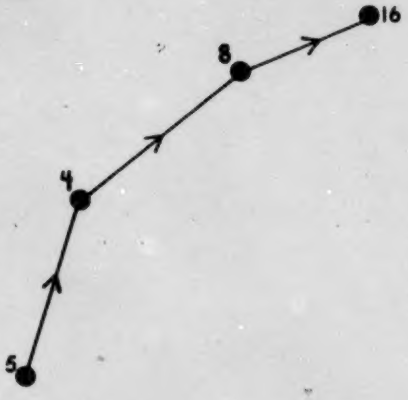
Students who finish Worksheet N8*** quickly should continue with Worksheet N8****.

The shortest possible road between each pair of numbers is shown on page 68. There are many other ways to build a road between each pair of numbers.

Name ______________ **N 8 ***

2x +1 -1

Build a red-blue-green road between 5 and 16. Try to use as few arrows as possible.

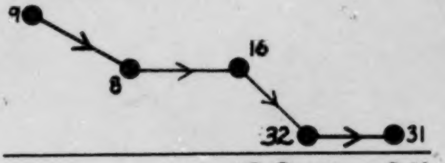


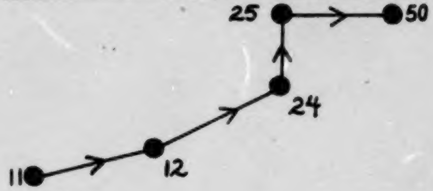
Other solutions are possible; however, this is a shortest road.

Name ______________ **N 8 ****

2x +1 -1

Build a red-blue-green road between these pairs of numbers. Try to use as few arrows as possible.



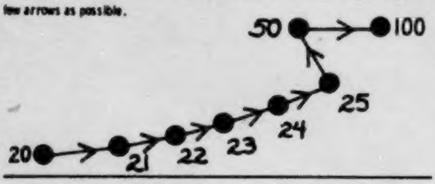

11
12
24
25
50

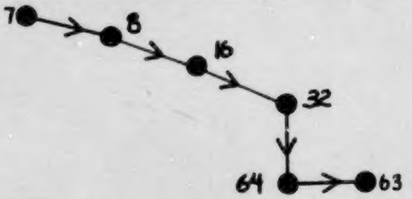
Other solutions are possible; however, these are shortest roads.

Name ______________ **N 8 *****

2x +1 -1

Build a red-blue-green road between these pairs of numbers. Try to use as few arrows as possible.


20
21
22
23
24
25
50
100

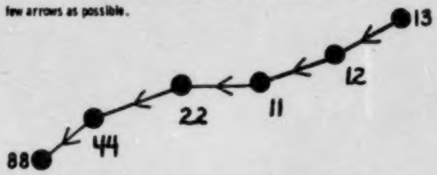

7
8
16
32
64
63

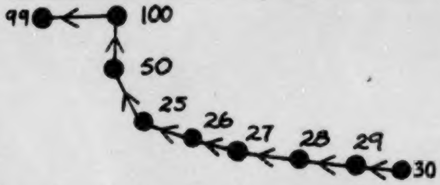
Other solutions are possible; however, these are shortest roads.

Name ______________ **N 8 ******

2x +1 -1

Build a red-blue-green road between these pairs of numbers. Try to use as few arrows as possible.


13
12
11
22
44
88


30
29
28
27
26
25
50
100
99

Other solutions are possible; however, these are shortest roads.

N9 Hand-Calculator Games #1

CAPSULE LESSON SUMMARY

Introduce the symbol "÷" and do several division calculations on the hand-calculator. Using only the keys $\boxed{2}$, $\boxed{8}$, $\boxed{+}$, $\boxed{-}$, $\boxed{\times}$, $\boxed{\div}$, and $\boxed{=}$, put various numbers on the display of a hand-calculator with exactly five presses of keys. Lifting the restriction on the number of presses, put some odd numbers on the display using the same limit on keys.

MATERIALS

Teacher: None
Student: Hand-calculator*

DESCRIPTION OF LESSON

Each student should have a hand-calculator. Ask the students to listen very carefully and to do exactly what you tell them.

Exercise 1

T: Display 100 on your hand-calculator.

Draw this symbol on the board.

÷

T: Look for the button with this symbol on it. This symbol is read "divided by." Press $\boxed{\div}\boxed{2}\boxed{=}$.
What number is on the display?

S: 50.

Students who do not have 50 on their displays should press $\boxed{C}$ and then put 50 on the display.

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*See Role and Use of Hand-Calculators on page 14 of the General Introduction.

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T: Press $\boxed{\div}\,\boxed{2}\,\boxed{=}$.
What number is on the display?

S: 25.

T: What does the calculator do when we press $\boxed{\div}\,\boxed{2}\,\boxed{=}$?

S: It divides the number by 2.

S: It finds one-half of the number.

T: Dividing by 2 is the same as taking one-half of a number.

Ask the students to press $\boxed{C}$ and then to display 80.

T: What number is $\frac{1}{2} \times 80$? (40)
Press $\boxed{\div}\,\boxed{2}\,\boxed{=}$.
What number is on the display?

S: 40.

T: What number is $\frac{1}{2} \times 40$? (20)
Press $\boxed{\div}\,\boxed{2}\,\boxed{=}$.
What number is on the display?

S: 20.

Continue this halving sequence.

$\boxed{\div}\,\boxed{2}\,\boxed{=}$ (10)

$\boxed{\div}\,\boxed{2}\,\boxed{=}$ (5)

T: If we continue dividing by 2, what number will we get next? ($2\frac{1}{2}$)
Press $\boxed{\div}$ $\boxed{2}$ $\boxed{=}$.
What number is on the display? (2.5)
What is another name for 2.5? ("$2\frac{1}{2}$")
What does 2.5 dollars mean?

S: 2 dollars and 50 cents.

T: One-half of $5 is $2.50.
What is one-half of $2.50? ($1.25)
Press $\boxed{\div}$ $\boxed{2}$ $\boxed{=}$.
What number is on the display?

S: 1.25.

Ask the students to press $\boxed{C}$ and then to display 60.

T: What number is 60 ÷ 2? (30)
Press $\boxed{\div}$ $\boxed{2}$ $\boxed{=}$.
What number is on the display?

S: 30.

T: What number is 30 ÷ 3? (10)
Press $\boxed{\div}$ $\boxed{3}$ $\boxed{=}$.
What number is on the display?

S: 10.

T: What does the calculator do when we press $\boxed{\div}$ $\boxed{3}$ $\boxed{=}$?

S: It divides the number by 3.

S: It finds one-third of the number.

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T: Dividing by 3 is the same as finding one-third of a number.

Ask two or three students to put 38 on their calculators and to do exactly what you tell them. The other students should turn their calculators off.

T: What number is on the display? (38)
Cover the display.
Press $\boxed{+}$ $\boxed{3}$ $\boxed{+}$ $\boxed{3}$ $\boxed{\div}$ $\boxed{2}$ $\boxed{=}$.

When you are telling the students which buttons to press, you should pause after each numeral key so the students will have an opportunity to do each calculation mentally.

T: Don't look. What number should be on the display? (22)

Allow several students to answer before asking the students using the calculators to look at their displays. If any calculator has a number other than 22 on the display, ask the student with that calculator to press $\boxed{C}$ and then to put 22 on the display.

Continue this activity with the following or similar sequences.

$\boxed{+}$ $\boxed{1}$ $\boxed{0}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$ (52)

$\boxed{+}$ $\boxed{5}$ $\boxed{+}$ $\boxed{5}$ $\boxed{\div}$ $\boxed{2}$ $\boxed{=}$ (31)

$\boxed{+}$ $\boxed{1}$ $\boxed{0}$ $\boxed{=}$ $\boxed{=}$ (51)

$\boxed{+}$ $\boxed{1}$ $\boxed{\div}$ $\boxed{2}$ $\boxed{=}$ (26)

Exercise 2

T: Today we are going to play a new game using the hand-calculator. In this game you may only use a few of the keys on the hand-calculator.

Write these symbols on the board.

$\boxed{2}\ \boxed{8}\ \boxed{+}\ \boxed{-}\ \boxed{\times}\ \boxed{\div}\ \boxed{=}$

T: You may choose any of these keys (point to the list), and you may press a key more than once if you wish, but you must press exactly five times. What numbers can you put on the display following these rules?

All the students should use their calculators. Be patient! The students may have difficulty restricting their use of the calculators to fit the rules of the game. As necessary, remind the class that they may only use the keys you have listed on the board and that they must press exactly five times.

As students find numbers which can be put on the calculator with these restrictions, record on the board the sequence of keys that were pressed to get each number. For example,

$\boxed{8}\ \boxed{2}\ \boxed{+}\ \boxed{8}\ \boxed{=}$	90
$\boxed{8}\ \boxed{\div}\ \boxed{2}\ \boxed{2}\ \boxed{=}$	0.3636363
$\boxed{8}\ \boxed{8}\ \boxed{\times}\ \boxed{8}\ \boxed{=}$	704
$\boxed{8}\ \boxed{2}\ \boxed{\div}\ \boxed{8}\ \boxed{=}$	10.25
$\boxed{8}\ \boxed{\times}\ \boxed{8}\ \boxed{=}\ \boxed{=}$	512
$\boxed{+}\ \boxed{=}\ \boxed{=}\ \boxed{+}\ \boxed{\times}$	0

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When decimal numbers are suggested, ask where (i.e. between which two integers) each number is on the number line. For example, 0.3636363 is between 0 and 1, and 10.25 is between 10 and 11 on the number line.

When your class has suggested several numbers, encourage the students to find as large a number as they can to put on their hand-calculators following these rules. Continue recording on the board sequences of keys that were pressed, but ask students to read the resulting numbers. When a new sequence is suggested, determine whether the new number is the largest number found thus far. Several sequences which result in large numbers are given below. Do not insist that your students find the largest possible number.

$\boxed{8}\ \boxed{\times}\ \boxed{=}\ \boxed{=}\ \boxed{=}$	4,096
$\boxed{8}\ \boxed{2}\ \boxed{\times}\ \boxed{=}\ \boxed{=}$	551,368
$\boxed{8}\ \boxed{8}\ \boxed{8}\ \boxed{8}\ \boxed{8}$	88,888
$\boxed{8}\ \boxed{8}\ \boxed{\times}\ \boxed{=}\ \boxed{=}$	681,472
$\boxed{8}\ \boxed{8}\ \boxed{8}\ \boxed{\times}\ \boxed{=}$	788,544

Erase everything on the board except the list of keys which may be used.

T: Now we are going to change the rules and play a different game. These are still the only keys you may use, but you may press as many times as you wish. Which odd numbers can you put on the calculator using only these keys?

Record on the board any sequences which result in odd numbers. For example,

$$\boxed{2}\ \boxed{\div}\ \boxed{2}\ \boxed{=}\qquad 1$$

$$\boxed{8}\ \boxed{+}\ \boxed{8}\ \boxed{+}\ \boxed{8}\ \boxed{\div}\ \boxed{8}\ \boxed{=}\qquad 3$$

After one or two odd numbers have been recorded, you can ask for specific odd numbers including 5, 11, 7, and 9. Record any sequence which results in an odd number, even if it is not the particular number you are asking for at that moment.

N 10

N10 Return Arrows #1

CAPSULE LESSON SUMMARY

Ask a student to select a number and to keep it secret. Then give that student a sequence of operations, one at a time, to perform. Record the sequence with an arrow picture. Ask the student to reveal the end result and then, as a class, use return arrows to discover the secret number.

MATERIALS

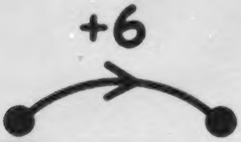
Teacher: Colored chalk
Student: Paper; colored pencils

DESCRIPTION OF LESSON

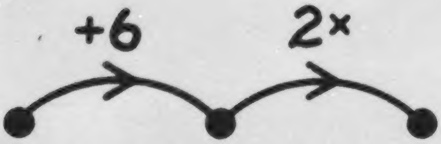
Ask one of your better calculators (Andy, in the following dialogue) to choose any whole number between 0 and 50 and to write that number on a piece of paper. Ask that student to follow your instructions very carefully.

T: Don't tell us what your number is, Andy. Add 6 to your number. You may use scrap paper if you want.

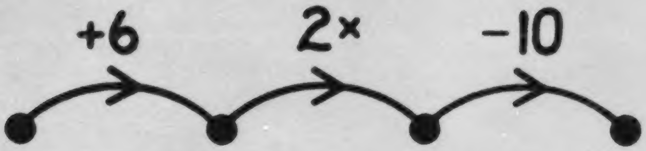
While the student is working, draw this picture on the board.



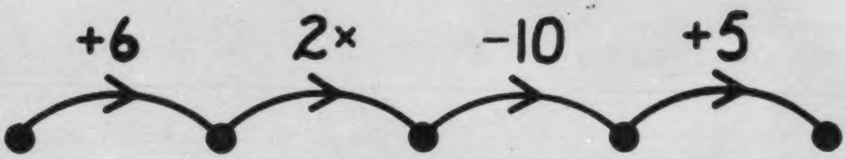
T: This dot (point to the starting dot) is for the number Andy chose and wrote on his paper.
This dot (point to the ending dot) is for the number which is 6 more than Andy's number. Are you ready to continue, Andy? (Yes)
Multiply your new number by 2.



T: Now subtract 10.



T: Now add 5.



T: What is your last number, Andy?

Label the ending dot with the number the student gives. Suppose the student's ending number is 49.

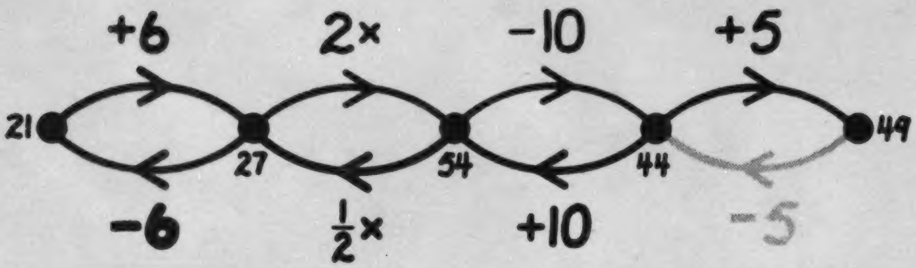
T: Do you think we could figure out what Andy's secret number is? (Yes) How?

S: We could draw the return arrows.

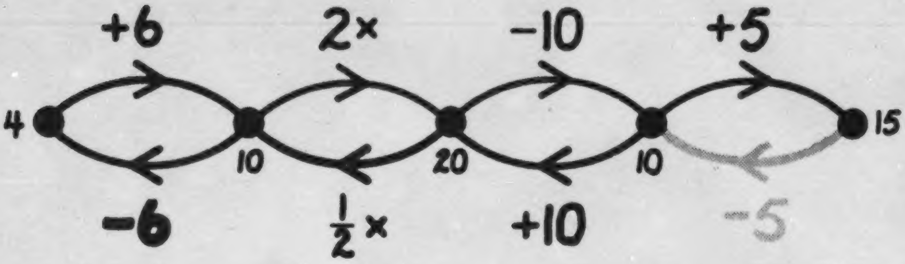
Draw and label the return arrows and the dots with your class's assistance.

N 10

Conclude that Andy's secret number is 21. Check to see if this is the number that Andy wrote on his paper.



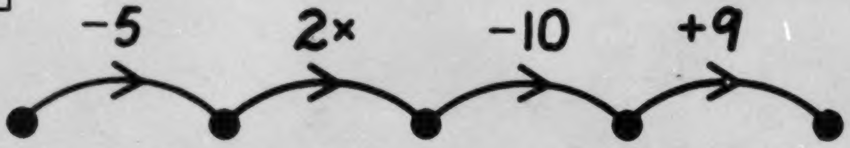
NOTE: Depending upon Andy's choice of starting numbers, two dots in the picture could be for the same number. For example, if 4 were the starting number, two dots in the picture would be for the number 10.



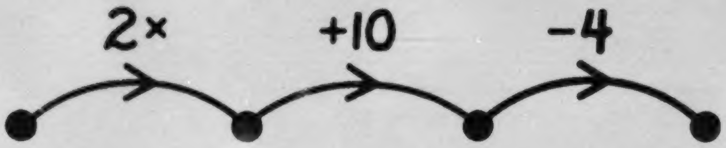
In this lesson, we are not concerned about there possibly being two dots for the same number, because the object is for the arrow pictures to show the sequences of calculations.

Erase the board and repeat this activity with these sequences of calculations. In each case let a student choose the starting number and reveal the ending number.

#1

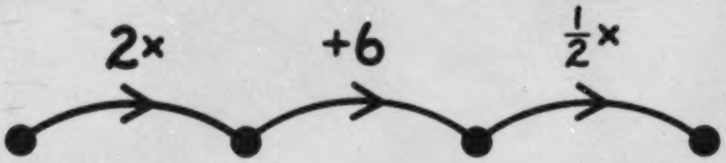


#2



With each of these sequences, ask all the students, except the student who chose the secret number, to copy the arrow picture on their papers as you draw it on the board. When the ending dot has been labeled, ask the students to draw the return arrows and to label the dots in their pictures. Look at the students' papers before collectively completing the picture on the board.

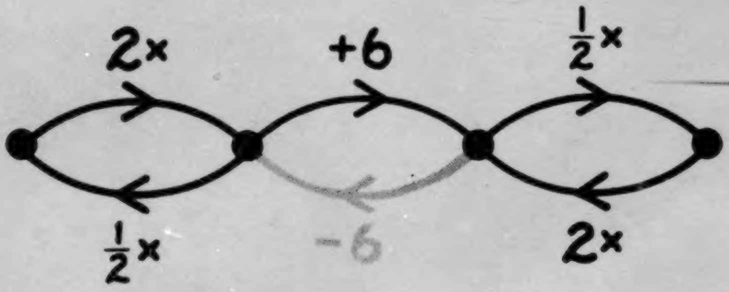
#3



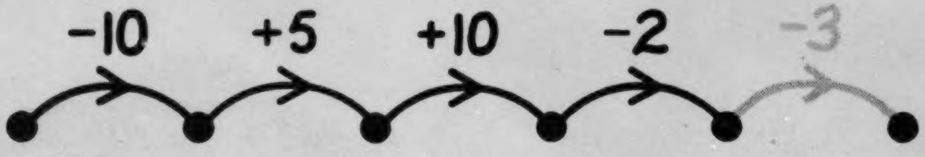
When the ending dot has been labeled, ask the students to whisper the secret number to you before allowing a student to answer aloud. Encourage the students to explain how they calculated the secret number.

N 10

Draw the return arrows and label the dots with the students' help. Both 2 x arrows should be the same color, and both $\frac{1}{2}$ x arrows should be the same color.



#4



When the ending dot has been labeled, ask the students to predict the secret number. They should quickly discover that the starting dot and the ending dot are for the same number.

T: Do you think that is always true -- no matter what is the starting number?

Assign various numbers to the starting dot and find the ending numbers with the class.

T: Why do you think the starting and ending numbers are always the same?

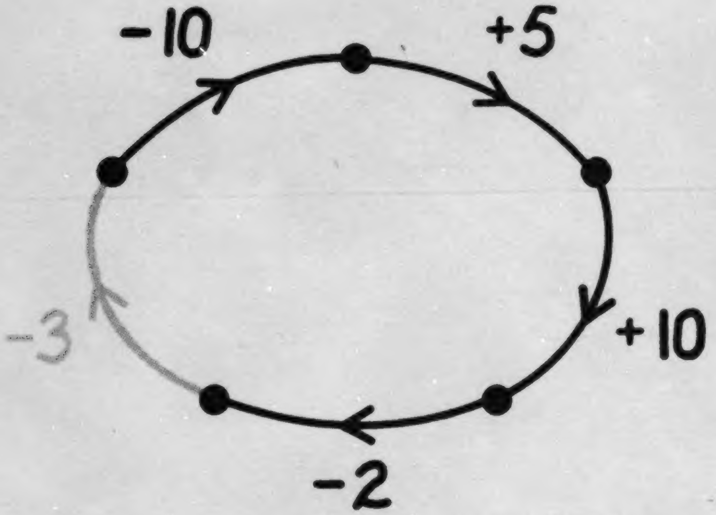
S: In the road we subtract 10, but we also add 10. We add 5, but we subtract 2 and 3.

T: How could we draw this arrow picture to show that the starting number and the ending number are the same?

S: Draw it like a circle.

S: Make a loop of arrows.

Redraw the picture.



NOTE: You can involve more students in these activities by asking two students to each choose a secret number and to follow your instructions as you tell them which calculations to make. In this case, both students will have ending numbers, and if these numbers are different, your class will have two secret numbers to calculate. Avoid writing two numbers near the same dot. This can be done by considering each ending number in turn and erasing any previous labels, by drawing each arrow road twice and writing one of the ending numbers by each of the ending dots, or by writing the appropriate numbers at some distance from the dots.

If there is time remaining, ask each student to think of a number. Then give them a sequence of operations, one at a time, to perform as you draw the corresponding arrow pictures. Call on students to tell their ending numbers and then quickly respond with their starting numbers. Ask the class to try to explain how you can be figuring their starting numbers so quickly. We suggest that you use one or more of these sequences:

- +5, -8, +1, +12 which is equivalent to +10;
- ×2, +1, -4, +3 which is equivalent to ×2;
- +8, -10, +1, +1, ÷2 which is equivalent to ÷2.

N11 Subtraction Problems #3

CAPSULE LESSON SUMMARY

Pose sequences of subtraction calculations that have the same answer and ask students for others. Notice how such sequences can be generated.

MATERIALS

Teacher: None

Student: Worksheets N11*, **, ***, and ****; paper; pencil

DESCRIPTION OF LESSON

Exercise 1

T: What number is 12 - 5?

S: 7.

Write "12 - 5 = 7" on the board.

T: What number is 13 - 6?

S: 7.

Write "13 - 6 = 7" on the board.

Continue this activity until you have this list of number stories on the board.

12 - 5 = 7

13 - 6 = 7

14 - 7 = 7

15 - 8 = 7

T: What patterns do you see?

Let students comment.

T: What are some other subtraction problems with 7 as the answer?

Encourage the students to contribute several problems. Record correct number stories in your list on the board.

T: What number is 16 - 9?

S: 7.

Start another list on the board.

16 - 9 = 7
26 - 19

T: What number is 26 - 19? How do you know?

S: It's still 7 -- you just added 10 to both numbers.

S: I counted from 19 to 26, and the answer is 7.

T: What patterns do you see with these two number stories?

S: If you add 10 to 16, you get 26. If you add 10 to 9, you get 19.

S: The answer to both problems is 7.

Record the answer and pose the next problem.

16 - 9 = 7
26 - 19 = 7
36 - 29

S: 36 - 29 = 7. The answer is the same because you made both numbers 10 bigger.

Continue this activity until you have these number stories on the board.

16 - 9 = 7
26 - 19 = 7
36 - 29 = 7
46 - 39 = 7
56 - 49 = 7

T: What is another subtraction problem with 7 as the answer?

Let students make suggestions and record them on the board. A sample dialogue is given.

S: 66 - 59 = ?

T: How do you know that problem has the same answer as these others?

S: I added 10 to 56 and also to 49.

NOTE: Students may or may not use a preceding problem to find a new problem with the same answer. Accept all valid approaches.

S: 1,056 - 1,049 = 7.

T: How do you know that problem has the same answer as the others?

S: I added 1,000 to 56 and also to 49.

Continue this activity until approximately ten number stories are in your list. If calculations with very large numbers are suggested (for example, "1,000,066 - 1,000,059"), ask a student to write the number story on the board. If only large numbers are suggested, encourage the students to offer number stories with numbers less than 100.

Erase the board before going on to Exercise 2.

Exercise 2

Write this problem on the board.

$$\begin{array}{r} 28 \\ -15 \\ \hline \end{array}$$

T: What number is 28 - 15? (13)
How do you know?

S: 20 - 10 = 10 and 8 - 5 = 3, so 28 - 15 = 13.

S: 28 - 10 = 18 and 18 - 5 = 13.

Record the answer and pose another problem.

$$\begin{array}{r} 28 \\ -15 \\ \hline 13 \end{array} \qquad \begin{array}{r} 29 \\ -16 \\ \hline \end{array}$$

T: What number is 29 - 16? (13)
How did I change the problem?

S: You added 1 to 28 and 1 to 15.

Continue this activity until you have these calculations on the board.

$$\begin{array}{r} 28 \\ -15 \\ \hline 13 \end{array} \qquad \begin{array}{r} 29 \\ -16 \\ \hline 13 \end{array} \qquad \begin{array}{r} 30 \\ -17 \\ \hline 13 \end{array} \qquad \begin{array}{r} 32 \\ -19 \\ \hline 13 \end{array} \qquad \begin{array}{r} 52 \\ -39 \\ \hline 13 \end{array}$$

T: How do you know that 52 - 39 = 13?

S: You added 20 to each of the numbers 32 and 19 so the answer is the same as 32 - 19.

T: Let's solve this problem a different way just to check that 52 - 39 = 13. What do you suggest?

Your students know many ways to solve this problem. Perhaps they will suggest one of the following methods.

$$\begin{array}{r} 53 \\ -40 \\ \hline 13 \end{array} \quad \text{or} \quad \begin{array}{r} 52 \\ -39 \\ \hline 2\hat{7} \end{array} = 20 + \hat{7} = 13 \quad \text{or} \quad \begin{array}{r} 52 \\ -30 \\ \hline 22 \\ -9 \\ \hline 13 \end{array} \quad \text{or} \quad \begin{array}{r} 52 \\ -40 \\ \hline 12 \\ +1 \\ \hline 13 \end{array}$$

Encourage the students to suggest other subtraction problems with 13 as the answer. Record these problems on the board as they are suggested and ask the students to explain how they know that the answer is 13.

Erase the board before going onto Exercise 3.

Exercise 3

Write this problem on the board.

$$\begin{array}{r} 65 \\ -\,40 \\ \hline \end{array}$$

T: What number is 65 - 40? (25)
How do you know?

Encourage the students to explain their calculations.

Record the answer and pose another problem.

$$\begin{array}{r} 65 \\ -\,40 \\ \hline 25 \end{array} \qquad \begin{array}{r} 63 \\ -\,\square \\ \hline 25 \end{array}$$

T: 63 minus what number is 25?

S: 38.

T: How do you know?

S: The answers are the same and 65 - 2 = 63, so I subtracted 2 from 40.

$$\begin{array}{r} 65 \\ -40 \\ \hline 25 \end{array} \qquad \begin{array}{r} 63 \\ -\boxed{38} \\ \hline 25 \end{array} \qquad \begin{array}{r} 60 \\ -\boxed{} \\ \hline 25 \end{array}$$

S: 60 is 3 less than 63, so the missing number is 35 because 38 - 3 = 35.

$$\begin{array}{r} 65 \\ -40 \\ \hline 25 \end{array} \qquad \begin{array}{r} 63 \\ -\boxed{38} \\ \hline 25 \end{array} \qquad \begin{array}{r} 60 \\ -\boxed{35} \\ \hline 25 \end{array} \qquad \begin{array}{r} 71 \\ -\boxed{} \\ \hline 25 \end{array}$$

S: 71 is 11 more than 60, so 71 - 46 = 25.

$$\begin{array}{r} 65 \\ -40 \\ \hline 25 \end{array} \qquad \begin{array}{r} 63 \\ -\boxed{38} \\ \hline 25 \end{array} \qquad \begin{array}{r} 60 \\ -\boxed{35} \\ \hline 25 \end{array} \qquad \begin{array}{r} 71 \\ -\boxed{46} \\ \hline 25 \end{array} \qquad \begin{array}{r} \boxed{} \\ -39 \\ \hline 25 \end{array}$$

T: What number minus 39 is 25?

S: 64.

T: How do you know?

S (referring to "63 - 38 = 25"): 38 + 1 = 39, so I added 1 to 63.

N 11

Worksheets N11*, **, ***, and **** are available for individual work. Students who finish quickly should be encouraged to write many subtraction problems with 78 as the answer.

Name ______________________ N 11 *

Complete.

$8 - 5 = \boxed{3}$

$9 - 6 = \boxed{3}$

$10 - \boxed{7} = 3$

$\boxed{11} - 8 = 3$

$10 - 5 = \boxed{5}$

$12 - 7 = \boxed{5}$

$14 - \boxed{9} = 5$

$\boxed{16} - 11 = 5$

$7 - 8 = \boxed{\uparrow}$

$8 - 8 = \boxed{0}$

$\boxed{9} - 8 = 1$

$10 - \boxed{8} = 2$

$20 - 10 = \boxed{10}$

$30 - \boxed{10} = 20$

$\boxed{40} - 10 = 30$

$50 - \boxed{10} = 40$

Name ______________________ N 11 **

Complete.

$$\begin{array}{r}18\\-10\\\hline 8\end{array}\quad\begin{array}{r}19\\-11\\\hline 8\end{array}\quad\begin{array}{r}20\\-\boxed{12}\\\hline 8\end{array}\quad\begin{array}{r}\boxed{21}\\-13\\\hline 8\end{array}$$

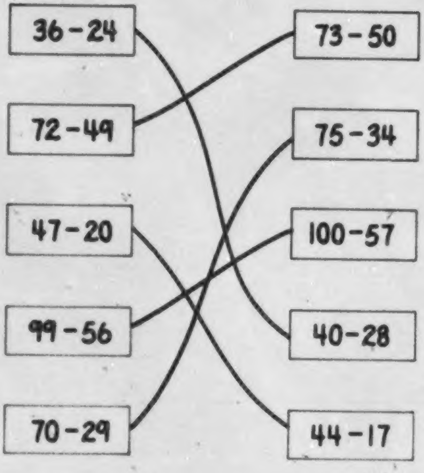
$$\begin{array}{r}42\\-21\\\hline 21\end{array}\quad\begin{array}{r}41\\-\boxed{21}\\\hline 20\end{array}\quad\begin{array}{r}40\\-21\\\hline 19\end{array}\quad\begin{array}{r}\boxed{39}\\-21\\\hline 18\end{array}$$

$$\begin{array}{r}14\\-7\\\hline 7\end{array}\quad\begin{array}{r}16\\-\boxed{9}\\\hline 7\end{array}\quad\begin{array}{r}\boxed{17}\\-10\\\hline 7\end{array}\quad\begin{array}{r}20\\-13\\\hline 7\end{array}$$

$$\begin{array}{r}13\\-8\\\hline 5\end{array}\quad\begin{array}{r}23\\-18\\\hline 5\end{array}\quad\begin{array}{r}33\\-28\\\hline 5\end{array}\quad\begin{array}{r}43\\-38\\\hline 5\end{array}$$

Name ______________________ N 11 ***

Pair names for the same number.



36 − 24

72 − 49

47 − 20

99 − 56

70 − 29

73 − 50

75 − 34

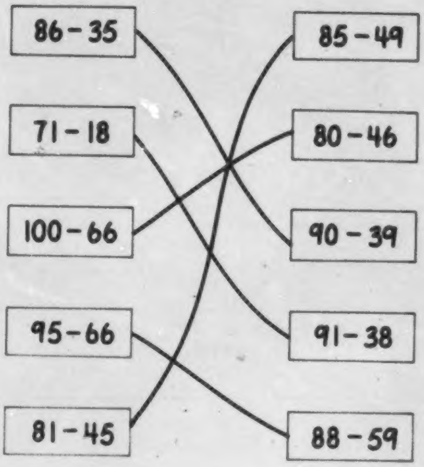
100 − 57

40 − 28

44 − 17

Name ______________________ N 11 ****

Pair names for the same number.



86 − 35

71 − 18

100 − 66

95 − 66

81 − 45

85 − 49

80 − 46

90 − 39

91 − 38

88 − 59

N 12

N12 Division Problems #1

CAPSULE LESSON SUMMARY

Evenly divide a quantity of fruit between two classes.

MATERIALS

Teacher: Colored chalk
Student: Paper; pencil; Worksheets N12*, **, ***, and ****

DESCRIPTION OF LESSON

Begin the lesson with a short mental arithmetic activity involving the function $\frac{1}{2}x$.

$\frac{1}{2} \times 16$ (8)	$\frac{1}{2} \times 24$ (12)	$\frac{1}{2} \times 32$ (16)
$\frac{1}{2} \times 20$ (10)	$\frac{1}{2} \times 30$ (15)	$\frac{1}{2} \times 38$ (19)

Ask several students to explain how they calculated "$\frac{1}{2} \times 38$".

Tell the following or similar stories to your class. Choose other teachers' classes in your school as well as your own to star in them. Substitute those teachers' names for "Ms. Briggs", "Mr. Greene", and "Ms. Jackson" in the lesson description.

T: Let's pretend that I have brought 536 cherries to school with me today. Since there are so many cherries I would like to keep half of them for our class and give half of the cherries to Ms. Briggs' class. I have brought two bags and I want to put half of the cherries in each bag.

Is $\frac{1}{2} \times 536$ more or less than 100? (More)
Is $\frac{1}{2} \times 536$ more or less than 200? (More)
How can you be sure that $\frac{1}{2} \times 536$ is more than 200?

S: 2 x 200 = 400 and 536 is more than 400.

T: Is $\frac{1}{2}$ x 536 more or less than 300? (Less)
How do you know?

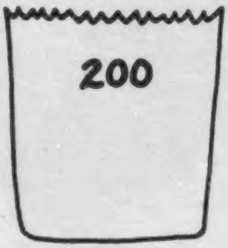
S: 2 x 300 = 600 and 536 is less than 600.

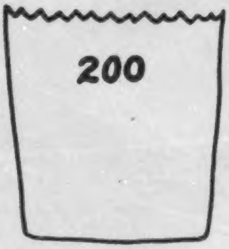
T: We know that $\frac{1}{2}$ x 536 is between 200 and 300. Therefore, each class should get at least 200 cherries. Let's begin by putting 200 cherries in each of the bags.

Draw two bags on the board and write "200" on each.

Ms. Schweitzer
(Write your name here.)

Ms. Briggs
(Write the other teacher's name here.)





T: Now there are 200 cherries in each of these bags. How many cherries have we distributed altogether?

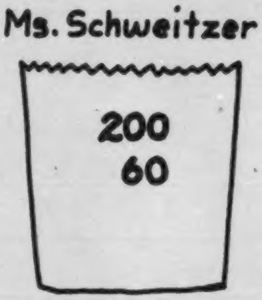
S: 400.

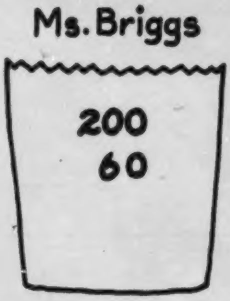
T: How many cherries do we still need to divide between the two classes? (136)
How could we do this?

Perhaps a student will suggest putting 60 more in each.

S: We could put 60 more cherries in each of these bags because 2 x 60 = 120.

T: 120 is less than 136, so let's put 60 more cherries in each bag.


Ms. Schweitzer
200
60


Ms. Briggs
200
60

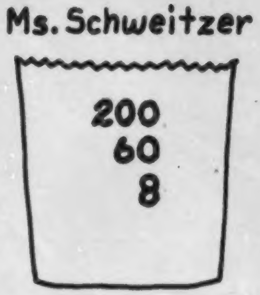
T: How many cherries are in each bag now? (260)

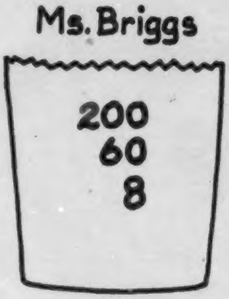
How many cherries have been distributed altogether? (520)

How many cherries do we still need to share between our class and Ms. Briggs' class? (16)

What number is $\frac{1}{2} \times 16$?

S: We can put 8 more cherries in each bag.


Ms. Schweitzer
200
60
8


Ms. Briggs
200
60
8

T: How many cherries will each class receive? (268)

Are there any cherries left over? (No)

How could we be sure that $\frac{1}{2} \times 536 = 268$?

S: We could calculate "2 x 268" and see if the answer is 536.

Ask the students to calculate "268 + 268" on their papers. Conclude that $\frac{1}{2} \times 536 = 268$.

T: What number stories can we write about this problem?

Record number stories on the board as they are suggested. If the division number story is not suggested, suggest it yourself. Several appropriate number stories are listed here.

$2 \times 268 = 536$ $268 + 268 = 536$

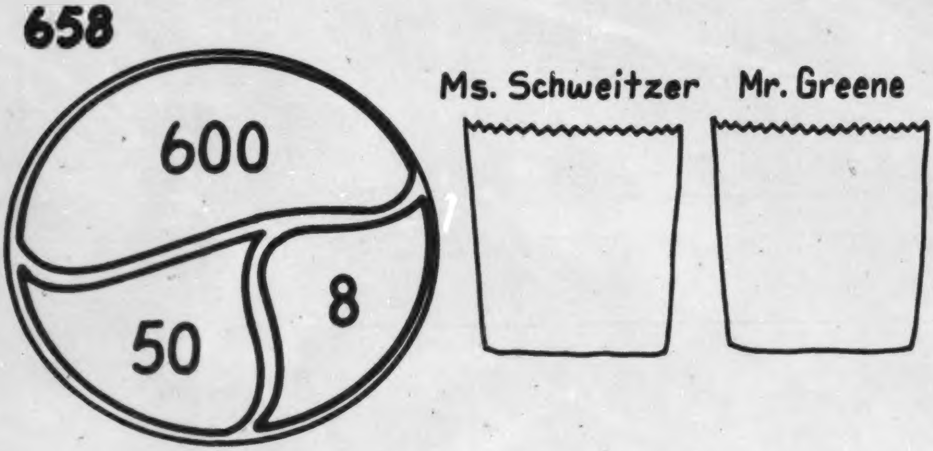
$\frac{1}{2} \times 536 = 268$ $536 \div 2 = 268$

$\frac{1}{2} \times 536 = 200 + 60 + 8$

Erase the board before continuing with the next story.

T: Let's pretend that I have 658 peaches to share with Mr. Greene's class. We will keep half of the peaches for our class, and give half of them to Mr. Greene's class. How many peaches will each class receive? What number is $\frac{1}{2} \times 658$? Try to solve this problem by yourself before we solve it together.

Draw this picture on the board while the students are working.

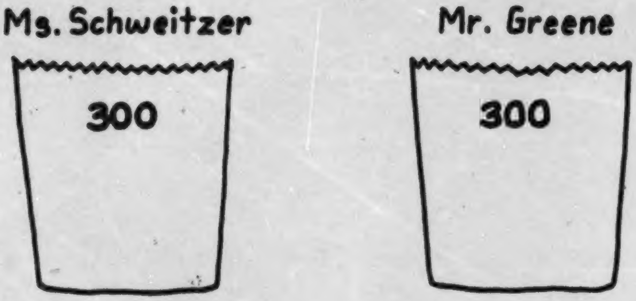


N12

Before continuing with the collective lesson, look at several of the students' papers.

T: There are 658 peaches inside the big string. Inside the blue string are 600 peaches, inside the red string are 50 peaches, and inside the green string are 8 peaches. Let's divide the peaches in each of the strings between the two classes. The blue string has 600 peaches in it. What number is $\frac{1}{2}$ x 600? (300)
Each class will get 300 of the peaches in the blue string?

Write "300" on the bag for each teacher.



T: There are 50 peaches in the red string. How many of these peaches should each class receive?

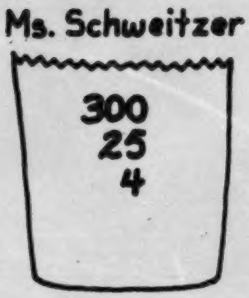
S: 25.

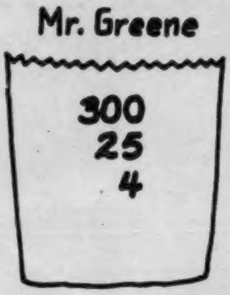
T: $\frac{1}{2}$ x 50 = 25.

Write "25" on the bag for each teacher.

T: In the green string there are only 8 peaches. How many of these peaches should we give to each class?

S: 4.


Ms. Schweitzer
300
25
4


Mr. Greene
300
25
4

T: How many peaches will each class receive altogether?

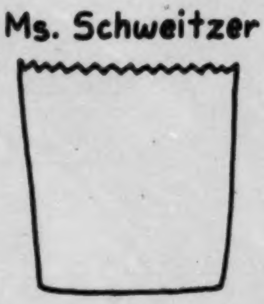
S: 329.

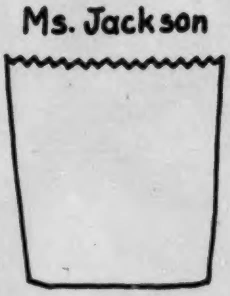
Ask the students to check if this answer is correct. Conclude that $\frac{1}{2} \times 658 = 329$.

Erase the board before continuing with the next story.

T: Tomorrow I am going to bring 375 apples to school. I have promised to give half of the apples to Ms. Jackson's class. How many apples will be left for our class?

Encourage the students to try to solve this problem on their own while you draw the picture on the board.

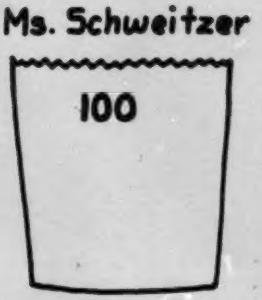

Ms. Schweitzer

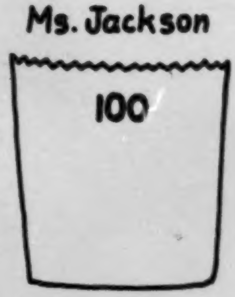


T: Would we be able to give each class at least 100 apples? (Yes)
2 x 100 = 200, so each class will receive at least 100 apples.
Could we give each class 200 apples? (No)
Why not?

S: 2 x 200 = 400 and 375 is less than 400.

T: Let's begin by giving each class 100 apples.


Ms. Schweitzer
100

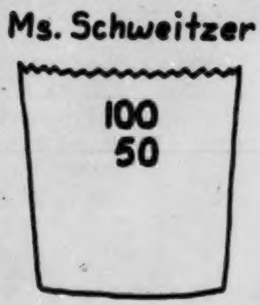

Ms. Jackson
100

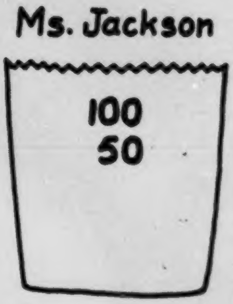
T: How many apples do we still need to distribute?

S: 175.

T: What do you suggest we do now?

S: We can give each class 50 more apples because 2 x 50 = 100 and 100 is less than 175.


Ms. Schweitzer
100
50


Ms. Jackson
100
50

T: How many apples have been distributed? (300)
How many apples do we still need to divide between the two classes? (75)

S: $\frac{1}{2} \times 70 = 35$, so let's give each class 35 more apples.

Write "35" on the bag for each teacher.

T: How many apples do we still need to share between our class and Ms. Jackson's class?

S: 5.

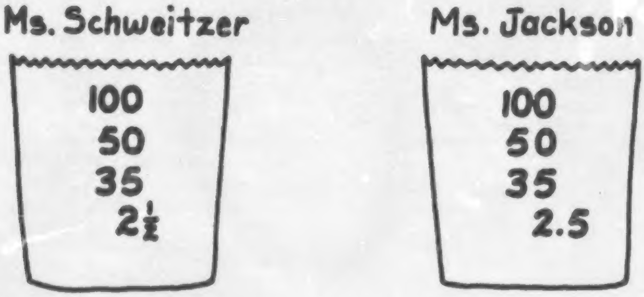
T: What do you suggest?

S: We give Ms. Jackson's class 2 apples and we keep the other 3.

S: We need to cut one of the apples in half.

S: Each class should get $2\frac{1}{2}$ more apples because $\frac{1}{2} \times 5 = 2\frac{1}{2}$.

Invite a student to write "$2\frac{1}{2}$" on the board. Ask if anyone knows a different way to write this number. If no one suggests "2.5" or "2.50", suggest "2.5" yourself. Emphasize that "2.5" and "$2\frac{1}{2}$" are different ways to write the same number. Write "2.5" on one of the bags and "$2\frac{1}{2}$" on the other bag.



N 12

T: How many apples will each class receive?

Write these calculations on the board and invite two students to solve them.

$$\begin{array}{r} 100 \\ 50 \\ 35 \\ +\ 2\frac{1}{2} \\ \hline 187\frac{1}{2} \end{array} \qquad \begin{array}{r} 100 \\ 50 \\ 35 \\ +\ 2.5 \\ \hline 187.5 \end{array}$$

T: Are $187\frac{1}{2}$ and 187.5 the same number? (Yes)

Each class will receive $187\frac{1}{2}$ apples. Is $\frac{1}{2} \times 375 = 187\frac{1}{2}$?

Check this answer on your paper.

Write these calculations on the board while the students are working.

$$\begin{array}{r} 187\frac{1}{2} \\ +187\frac{1}{2} \\ \hline \end{array} \qquad \begin{array}{r} 187.5 \\ +187.5 \\ \hline \end{array}$$

T: What number is $\frac{1}{2} + \frac{1}{2}$? (1)

We write "1" above the ones' column.

$$\begin{array}{r} 18\overset{1}{7}\frac{1}{2} \\ +\ 187\frac{1}{2} \\ \hline \end{array} \qquad \begin{array}{r} 187.5 \\ +187.5 \\ \hline \end{array}$$

Invite a student to complete the calculation on the board. Ask another student to complete the addition problem on the right.

$$\begin{array}{r} \shortmid\ \shortmid\ \shortmid\ \ \\ 187\frac{1}{2} \\ +187\frac{1}{2} \\ \hline 375 \end{array} \qquad \begin{array}{r} \shortmid\ \shortmid\ \shortmid\ \ \ \ \ \\ 187.5 \\ +187.5 \\ \hline 375.0 \end{array}$$

Conclude that $2 \times 187\frac{1}{2} = 375$. Ask the students to suggest other number stories about this problem. Several of the appropriate number stories are listed below. If the division number story is not suggested, suggest it yourself.

$$2 \times 187.5 = 375 \qquad 375 \div 2 = 187\frac{1}{2}$$

$$\frac{1}{2} \times 375 = 187\frac{1}{2} \qquad 375 - 187.5 = 187.5$$

Worksheets N12*, **, ***, and **** are available for individual work.

Name ____________ **N 12 ***

Complete.

$\frac{1}{2} \times 20 =$ 10

$\frac{1}{2} \times 24 =$ 12

$\frac{1}{2} \times 40 =$ 20

$\frac{1}{2} \times 46 =$ 23

$60 \div 2 =$ 30

$68 \div 2 =$ 34

$80 \div 2 =$ 40

$82 \div 2 =$ 41

$\frac{1}{2} \times 50 =$ 25

$\frac{1}{2} \times 52 =$ 26

$\frac{1}{2} \times 54 =$ 27

$\frac{1}{2} \times 56 =$ 28

$30 \div 2 =$ 15

$32 \div 2 =$ 16

$34 \div 2 =$ 17

$36 \div 2 =$ 18

Name ____________ **N 12 ****

Complete.

$4 \div 2 =$ 2

$5 \div 2 =$ 2.5

$6 \div 2 =$ 3

$7 \div 2 =$ 3.5

$\frac{1}{2} \times 600 =$ 300

$\frac{1}{2} \times 640 =$ 320

$\frac{1}{2} \times 642 =$ 321

$\frac{1}{2} \times 650 =$ 325

$\frac{1}{2} \times 90 =$ 45

$\frac{1}{2} \times 94 =$ 47

$\frac{1}{2} \times 96 =$ 48

$\frac{1}{2} \times 95 =$ 47.5

$100 \div 2 =$ 50

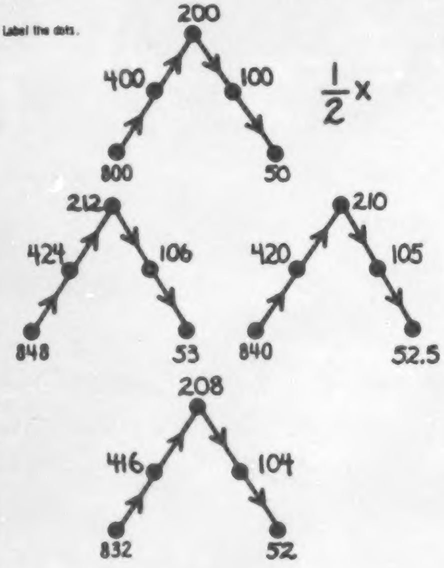
$130 \div 2 =$ 65

$138 \div 2 =$ 69

$139 \div 2 =$ 69.5

Name ____________ **N 12 *****

Label the dots.



Name ____________ **N 12 ******

Complete.

$\frac{1}{2} \times 500 =$ 250

$\frac{1}{2} \times 510 =$ 255

$\frac{1}{2} \times 300 =$ 150

$\frac{1}{2} \times 380 =$ 190

$900 \div 2 =$ 450

$948 \div 2 =$ 474

$916 \div 2 =$ 458

$974 \div 2 =$ 487

$\frac{1}{2} \times 1{,}000 =$ 500

$\frac{1}{2} \times 700 =$ 350

$\frac{1}{2} \times 36 =$ 18

$\frac{1}{2} \times 1{,}736 =$ 868

$1{,}800 \div 2 =$ 900

$1{,}798 \div 2 =$ 899

$1{,}776 \div 2 =$ 888

$1{,}777 \div 2 =$ 888.5

N13 Subtraction Problems #4

CAPSULE LESSON SUMMARY

Through a series of estimations compute "$\frac{1}{3} \times 100$". Decide how many of Mr. Chipper's watermelons are missing if one day he has 363 and the next only 295. Subtract one number from another by finding and doing an easier calculation with the same answer.

MATERIALS

Teacher: Colored chalk
Student: Paper; pencil

DESCRIPTION OF LESSON

Exercise 1: One-third of 100

T: What number is $\frac{1}{2} \times 10$? How do you know?

S: $5 + 5 = 10$, so $\frac{1}{2} \times 10 = 5$.

T: Since $\frac{1}{2} \times 10 = 5$, what number is $10 \div 2$?

S: 5.

T: $\frac{1}{2} \times 10$ is the same as $10 \div 2$.
Who could write number stories on the board to show this?

The students could suggest these two number stories and may suggest others.

$$\frac{1}{2} \times 10 = 5$$
$$10 \div 2 = 5$$

N 13

Write this number story on the board and explain that it is another way to show that 10 ÷ 2 = 5.

$$\begin{array}{r} 5 \\ 2\overline{)10} \end{array}$$

Continue this activity with these calculations.

$\frac{1}{5} \times 20$ (4) $100 \div 2$ (50) $\frac{1}{4} \times 100$ (25)

$20 \div 5$ (4) $\frac{1}{2} \times 100$ (50) $100 \div 4$ (25)

T: What number is 100 ÷ 3?

Allow several students to whisper their answers to you before you guide the students in calculating " 100 ÷ 3". A possible dialogue is given here.

S: I think $\frac{1}{3} \times 100 = 25$.

T: How could we check to see if this is true?

S: We could calculate "3 x 25".

$$\begin{array}{r} 25 \\ 25 \\ \underline{+\,25} \end{array}$$

T: What number is 3 x 25?

Invite a student to do the addition at the board.

T: 3 x 25 = 75. Is $\frac{1}{3} \times 100$ larger or smaller than 25?

S: Larger.

Record the inequality on the board.

$$100 \div 3 > 25$$

S: I think 100 ÷ 3 = 36.

S: We could add "36 + 36 + 36".

Invite a student to do the addition at the board.

$$\begin{array}{r} {}^{1} \\ 36 \\ 36 \\ +36 \\ \hline 108 \end{array}$$

T: Is 36 too large or too small?

S: Too large.

$$\begin{array}{l} 100 \div 3 > 25 \\ 100 \div 3 < 36 \end{array}$$

T: We know that $\frac{1}{3} \times 100$ is between 25 and 36.

S: Let's try 30.

T: 3 x 30 = . . . ?

S: 90.

N 13

S: $\frac{1}{3} \times 100$ is more than 30.

Replace "25" with "30".

$$100 \div 3 > 30$$
$$100 \div 3 < 36$$

The students most likely will suggest 33, then 34, and conclude that $100 \div 3$ is between 33 and 34. Perhaps someone will suggest $33\frac{1}{2}$. Write the appropriate addition problem on the board and ask for a volunteer to solve it. You may need to remind the students that $\frac{1}{2} + \frac{1}{2} = 1$.

$$\begin{array}{r} 1\,1\phantom{\tfrac{1}{2}} \\ 33\tfrac{1}{2} \\ 33\tfrac{1}{2} \\ 33\tfrac{1}{2} \\ \hline 100\tfrac{1}{2} \end{array}$$

Conclude that $33\frac{1}{2}$ is not correct.

S: I think $100 \div 3 = 33\frac{1}{3}$.

Write the appropriate problem on the board and ask for a volunteer to solve it. You may need to remind the students that $\frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$.

$$\begin{array}{r} 1\,1\phantom{\tfrac{1}{3}} \\ 33\tfrac{1}{3} \\ 33\tfrac{1}{3} \\ 33\tfrac{1}{3} \\ \hline 100\phantom{\tfrac{1}{3}} \end{array}$$

Conclude that $100 \div 3 = 33\frac{1}{3}$. Erase the board before going on to Exercise 2.

Exercise 2: Subtraction Problem

Tell the following or a similar story to your class.

T: There is a man named Mr. Chipper who grows watermelons. Everyday Mr. Chipper counts his watermelons. Why would he do this?

Allow the students to express their opinions.

T: Yesterday Mr. Chipper counted 363 watermelons but today counted only 295. What could have happened to the other watermelons?

The students very likely will suggest that some watermelons were stolen.

T: Mr. Chipper is very upset because some of his watermelons are missing. He thinks that they may have been stolen, so he decides to go to the sheriff's office. Mr. Chipper tells the sheriff, "Yesterday I had 363 watermelons and now I have only 295. I think someone stole some of my watermelons." The sheriff pulls out the official forms and begins to complete them; he asks Mr. Chipper how many watermelons were stolen.

Mr. Chipper thinks a long time before he tells the sheriff that he can't count the missing watermelons until they are found. The sheriff says he can't fill out the forms until he knows how many watermelons are missing. Can you help them figure this out?

Let the students work on this problem independently for a few minutes. Write this problem on the board.

$$\begin{array}{r} 363 \\ - \\ \hline 295 \end{array}$$

N 13

T: After a long time the sheriff writes this problem on a piece of paper. "You had 363 watermelons, then someone took away some of the watermelons and now you have 295 watermelons left," explains the sheriff.

Mr. Chipper says "I think they took about 100 watermelons." The sheriff writes this problem on the paper.

$$\begin{array}{r} 363 \\ - \\ \hline 295 \end{array} \qquad\qquad \begin{array}{r} 363 \\ -100 \\ \hline \end{array}$$

T: What number is 363 - 100?

S: 263.

Record that 363 - 100 = 263.

T: Are more or less than 100 watermelons missing?

S: Less.

T: If 100 watermelons were taken, Mr. Chipper would have only 263 watermelons left. So we know that less than 100 watermelons are missing.

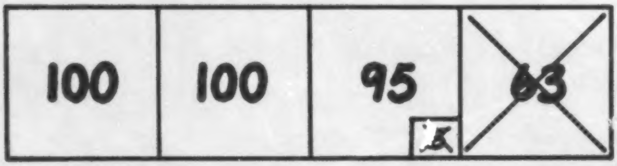
Draw this picture on the board.

100	100	100	63

T: The sheriff finally draws this picture after he and Mr. Chipper have tried many other numbers that do not work. The sheriff says, "This is a picture of the 363 watermelons that you had yesterday. Today you have only 295 watermelons."

How could we show in this picture that Mr. Chipper has only 295 watermelons left?

S: We could cross out the 63 watermelons and then 5 more watermelons from one of the 100's.



T: Does this picture tell us how many watermelons are missing? How?

S: 63 + 5 = 68, so 68 watermelons were taken.

Record "68" in the space for the missing number.

$$\begin{array}{r} 363 \\ -\ 68 \\ \hline 295 \end{array}$$

T: We want to check if 363 - 68 = 295.
Can you suggest an easy way to calculate "363 - 68"?

N 13

Perhaps your students will suggest one of these methods.

$$\begin{array}{r} 363 \\ -\ 60 \\ \hline 303 \\ -\ \ \ 8 \\ \hline 295 \end{array} \quad \text{or} \quad \begin{array}{r} 363 \\ -\ 63 \\ \hline 300 \\ -\ \ \ 5 \\ \hline 295 \end{array} \quad \text{or} \quad \begin{array}{r} 363 \\ -\ 68 \\ \hline 30\hat{5} \end{array} = 300 + \hat{5} = 295$$

Erase the board before going on to Exercise 3.

NOTE: Exercise 3 is very important for the development of the subtraction algorithm. If you do not have time for it today, be sure to do the exercise with your class before the day you do Lesson N17.

Exercise 3: Subtraction Problems with the Same Answer

Write this problem on the board.

$$\begin{array}{r} 52 \\ -19 \\ \hline \end{array}$$

T: What number is 52 - 19? Write the answer on your paper.

Look at many of the students' papers before asking a student to answer aloud. Ask several students to explain how they calculated the answer. If no one suggests adding 1 to each of the numbers in this subtraction problem, suggest this yourself.

$$\begin{array}{r} 52 \\ -19 \\ \hline \end{array} \qquad \begin{array}{r} 53 \\ -20 \\ \hline \end{array}$$

T: Why would we want to calculate "53 - 20" instead of "52 - 19"?

S: Both problems have the same answer, but "53 - 20" is an easier problem to solve.

T: What number is 53 - 20? (33)
So what number is 52 - 19? (33)
On your paper write many subtraction problems with 33 as the answer.

Look at the students' work. Encourage the students to write subtraction problems with large and small numbers. Invite students who have written correct subtraction problems to write them on the board. Emphasize that when both numbers are increased or decreased by the same amount, the answer is the same as before.

Erase the board and write this problem.

$$534 - 97$$

T: What number is 534 - 97? Write the answer on your paper.

Look at many of the students' papers before asking a student to answer aloud. Ask several students to explain how they calculated the answer. If no one suggests adding 3 to each of the numbers in this subtraction problem, suggest this yourself. Indicate that you intend to add 3 to each number by writing a small "3" above the ones' digit of each number.

$$53\overset{3}{4} - 9\overset{3}{7}$$

T: If we add 3 to each of these numbers, what will the new subtraction problem be?

S: "537 - 100".

$$53\overset{3}{4} - 9\overset{3}{7} = 537 - 100$$

T: What number is 537 - 100?

S: 437.

$$53\overset{3}{4} - 9\overset{3}{7} = 537 - 100 = 437$$

Emphasize that both problems have the same answer but that the second problem is easier to solve.

Use a similar approach to do these calculations.

$$37\overset{1}{2} - 4\overset{1}{9} = 373 - 50 = 323$$

$$13\overset{2}{2} - 1\overset{2}{8} = 134 - 20 = 114$$

$$1{,}55\overset{1}{5} - 99\overset{1}{9} = 1{,}556 - 1{,}000 = 556$$

Write these problems on the board and ask the students to solve as many of them as they can.

$$73 - 18 \qquad 255 - 98 \qquad 1{,}463 - 997$$

$$95 - 29 \qquad 865 - 199 \qquad 2{,}551 - 995$$

Look at the students' work. After 5-10 minutes solve these problems collectively. A sample dialogue is given here.

T: Nicole found an easy way to calculate "73 - 18". She added 2 to each of the numbers.

$$7\overset{2}{3} - 1\overset{2}{8} = 75 - 20$$

T: What number is 75 - 20?

S: 55.

$$7\overset{2}{3} - 1\overset{2}{8} = 75 - 20 = 55$$

T: The answer to the first problem is 55.

Frankie has found a way to make the second problem easier. He added 1 to each number.

$$9\overset{1}{5} - 2\overset{1}{9} = 96 - 30$$

N 13

T: What number is 96 - 30?

S: 66.

Continue this activity until all the problems have been solved.

$$7\overset{2}{3} - 1\overset{2}{8} = 75 - 20 = 55$$

$$9\overset{1}{5} - 2\overset{1}{9} = 96 - 30 = 66$$

$$25\overset{2}{5} - 9\overset{2}{8} = 257 - 100 = 157$$

$$86\overset{1}{5} - 19\overset{1}{9} = 866 - 200 = 666$$

$$1{,}46\overset{3}{3} - 99\overset{3}{7} = 1{,}466 - 1{,}000 = 466$$

$$2{,}55\overset{5}{1} - 99\overset{5}{5} = 2{,}556 - 1{,}000 = 1{,}556$$

N14 Roads #2

CAPSULE LESSON SUMMARY

Build a road from one number to another using only 10 x, +1, and -1 arrows.

MATERIALS

Teacher: Colored chalk
Student: Paper; colored pencils; Worksheets N14*, **, ***, and ****

DESCRIPTION OF LESSON

Begin this lesson with a short mental arithmetic activity involving the function 10 x.

10 x 4 (40)	10 x 7 (70)	10 x 16 (160)
10 x 5 (50)	10 x 10 (100)	10 x 0 (0)
10 x 2 (20)	10 x 12 (120)	10 x 1 (10)

T: What number is 10 x 20?

S: 200.

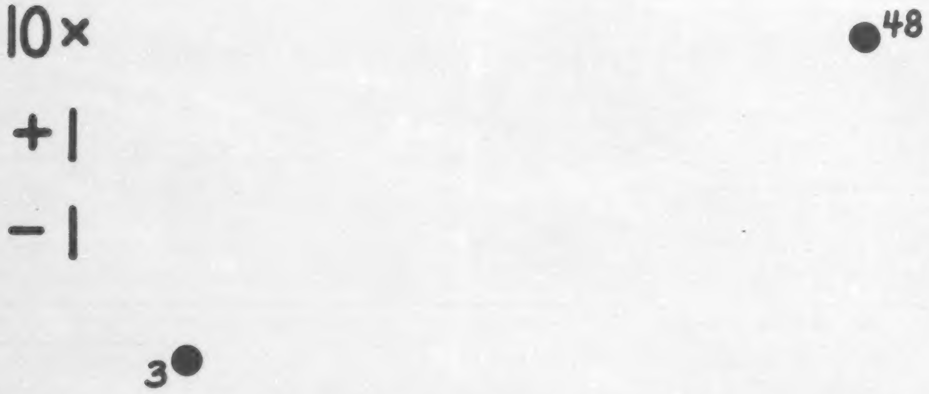
Write "10 x 20 = 200" on the board.

Continue this activity with these calculations.

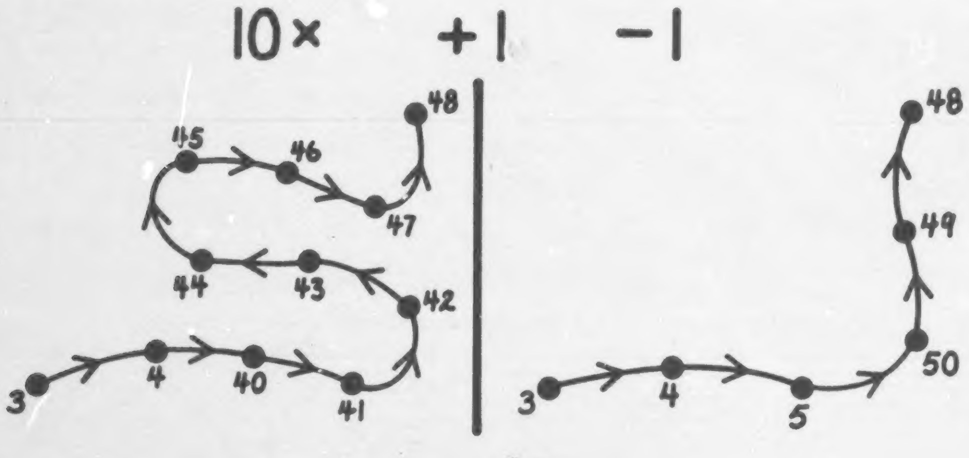
10 x 23
10 x 200
10 x 230
10 x 203

Ask the students to suggest several numbers which are multiples of 10. As each number is suggested, ask the students to tell you the appropriate 10 x number story. For example, if 670 is suggested, the corresponding number story would be "10 x 67 = 670".

Erase the number stories and then put this information on the board.



Ask the students to copy what you have drawn on the board and to build a road between 3 and 48. The problem is solved when one road has been built, but encourage the students to try to build a shortest possible road. As students finish their roads, tell the class how many arrows are in the shortest road you have seen up till that time. When most of the students have built a road between 3 and 48, ask students with the following arrow roads to draw them on the board. You may need to draw one of these roads yourself if none of your students has built it.



Briefly discuss these roads with your class. Ask which multiple of 10 is closest to 48. (50)

Distribute copies of Worksheet N14* and ask the students to build a road between each pair of numbers, trying to make their roads as short as possible. After a few minutes ask a student who has built a road between 2 and 53 with less than ten arrows to draw that road on the board.

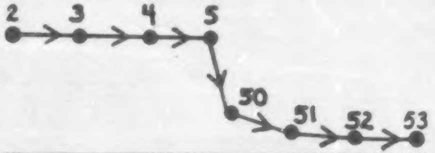
Repeat this activity with the other worksheets. Encourage the students to find shortest possible roads, but remind them that they have solved a problem when they build a road between the two numbers, no matter how long the road.

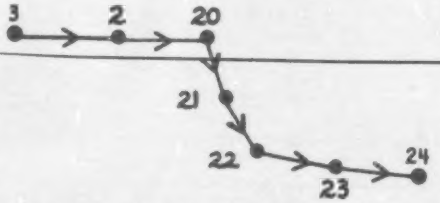
There are many other ways to build a road between each pair of numbers on the worksheets, although there is only one shortest road in each case.

Name ____________ **N 14 ***

10× +1 −1

Build a red-blue-green road between each pair of numbers.
Try to use as few arrows as possible.


2
3
4
5
50
51
52
53

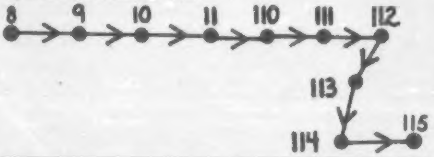

3
2
20
21
22
23
24

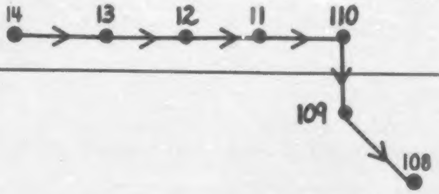
Other solutions are possible; however, these are shortest roads.

Name ____________ **N 14 ****

10× +1 −1

Build a red-blue-green road between each pair of numbers.
Try to use as few arrows as possible.


8
9
10
11
110
111
112
113
114
115

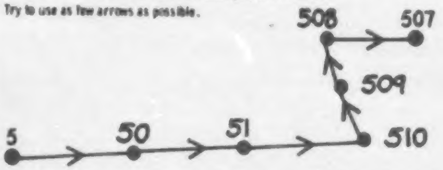

14
13
12
11
110
109
108

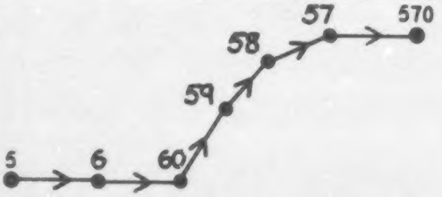
Other solutions are possible; however, these are shortest roads.

Name ____________ **N 14 *****

10× +1 −1

Build a red-blue-green road between each pair of numbers.
Try to use as few arrows as possible.


5
50
51
510
509
508
507

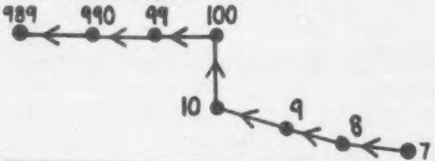

5
6
60
59
58
57
570

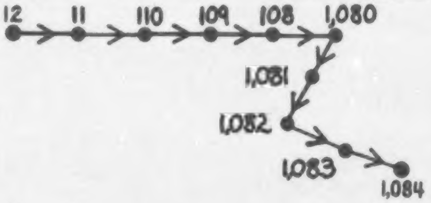
Other solutions are possible; however, these are shortest roads.

Name ____________ **N 14 ******

10× +1 −1

Build a red-blue-green road between each pair of numbers.
Try to use as few arrows as possible.


7
8
9
10
100
99
990
989


12
11
110
109
108
1,080
1,081
1,082
1,083
1,084

Other solutions are possible; however, these are shortest roads.

N15 Minicomputer Golf #1

CAPSULE LESSON SUMMARY

Explore the effect of moving various checkers in a configuration on the Minicomputer — after a move, is there a larger, a smaller, or the same number on the Minicomputer? Introduce a version of the game "Minicomputer Golf", in which checkers are moved from a starting configuration in order to reach a specific goal.

MATERIALS

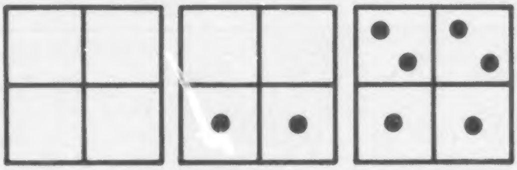
Teacher: Minicomputer set; colored chalk
Student: None

DESCRIPTION OF LESSON

Exercise 1

T: I am going to put a number on the Minicomputer. See if you can figure out which number it is.

Gradually put this configuration on the Minicomputer, starting with the checkers on the squares of largest value. Pause frequently so your students can do the mental calculations.



Let the students whisper the number to you before letting one of them answer aloud.

S: 57.

Invite several students to explain how they knew this number was 57.

N 15

Write these words on the board.

More
Same
Less

T: I am going to move one of these checkers to another square. Tell me if the new number is more than, less than, or the same as the number on the Minicomputer now.

Move a checker from the 2-square to the 1-square.

Point to each of the words on the board in turn and ask the students to hold up their hands when you point to the word which describes the new number. The students should indicate that the new number is less than before.

T: How much less . . . ?

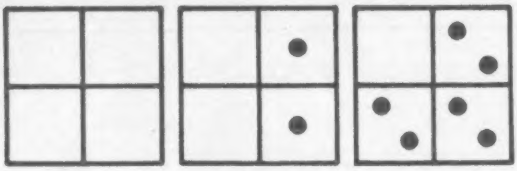
S: 1 less.

Repeat this activity several times as suggested below. Do not return checkers to their original positions. Each move will start from a new number on the Minicomputer.

Move a checker:

- from the 4-square to the 1-square (3 less)
- from the 20-square to the 40-square (20 more)
- from the 10-square to the 2-square (8 less)
- from the 1-square to the 10-square (9 more)
- from the 8-square to the 4-square (4 less)
- from the 8-square to the 2-square (6 less)

Put this configuration on the Minicomputer.



T: Who can move exactly one checker and make the number 2 more than it is now?

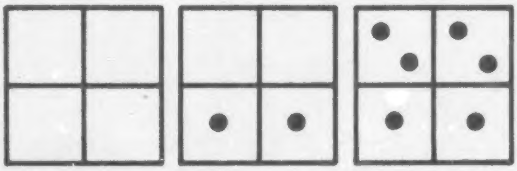
A student moves a checker from the 2-square to the 4-square.

Continue this activity by asking for volunteers to make these changes. Again, do not return checkers to their original positions; otherwise, some changes may be impossible.

- 9 more (from the 1-square to the 10-square)
- 19 more (from the 1-square to the 20-square)
- 10 less (from the 20-square to the 10-square)
- 3 less (from the 4-square to the 1-square)
- 30 less (from the 40-square to the 10-square)
- 6 more (from the 4-square to the 10-square or from the 2-square to the 8-square)
- 99 more (from the 1-square to the 100-square)

Exercise 2: Minicomputer Golf

Put this configuration on the Minicomputer.



T: What number is this?

S: 57.

T: Today we are going to play a game called "Minicomputer Golf."

Draw and label a dot for 57.

T: Our goal is to reach 200 by moving the checkers.

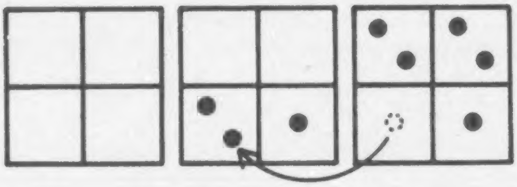
Draw and label a dot for 200.

T: Do we need to make the number on the Minicomputer larger or smaller?

S: Larger.

Invite a student to move exactly one checker from any square to any other square of the Minicomputer. When the checker has been moved, ask the student how much larger or smaller is the new number. If the student is unable to tell you, replace the checker in its previous position and ask the student to make another move. Continue in this way until the goal is reached. The move which reaches the goal is the winning move. We describe a sample game.

The first volunteer moves a checker from the 2-square to the 20-square.



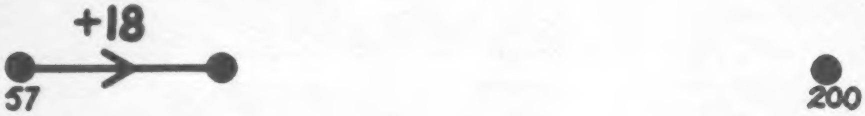
T: Is the new number larger or smaller than the number before?

S: Larger.

T: How much larger . . . ?

S: 18 larger.

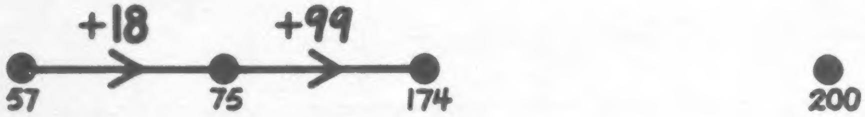
A +18 arrow starting at 57 records the increase.



T: What number is 57 + 18? (75)

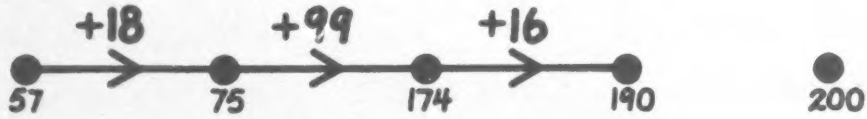
Some students might look at the Minicomputer to calculate the number (75); others might do the addition mentally. If necessary, write the addition problem on the board and solve it collectively. Label the dot with the answer.

The next volunteer moves a checker from the 1-square to the 100-square and tells the class that the number is now 99 larger. The class calculates and finds that 75 + 99 = 174, and so decides that the new number is 174.



T: Do we need to make the number on the Minicomputer larger or smaller? (Larger)
How much larger . . . ? (26)

A student moves a checker from the 4-square to the 20-square.

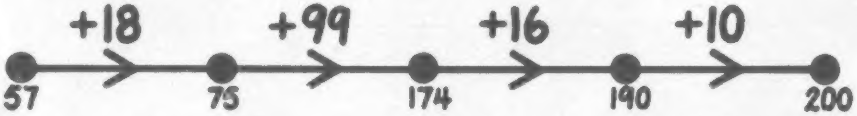


N 15

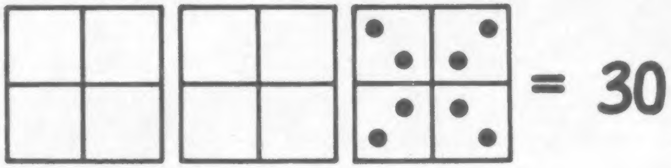
T: What is the number on the Minicomputer now? (190)
What is the distance from 190 to 200?

S: 10; we need to make the number 10 more.

A student moves a checker from the 10-square to the 20-square and the goal is reached.



Play the game a second time but start with this configuration . . .



. . . and set the goal at 200.

If your class wishes to play other games of Minicomputer Golf, we suggest that you start with any number represented by eight positive checkers on the Minicomputer and choose any multiple of 100 up to 1,000 as your goal. Of course there are other choices, but they may be too difficult for your students or they may require calculations which slow the game down to the point of becoming boring.

N16 Subtraction and Composition #2

CAPSULE LESSON SUMMARY

Find the composite of several subtraction arrows. Build arrow roads to record ways of subtracting "650 - 298".

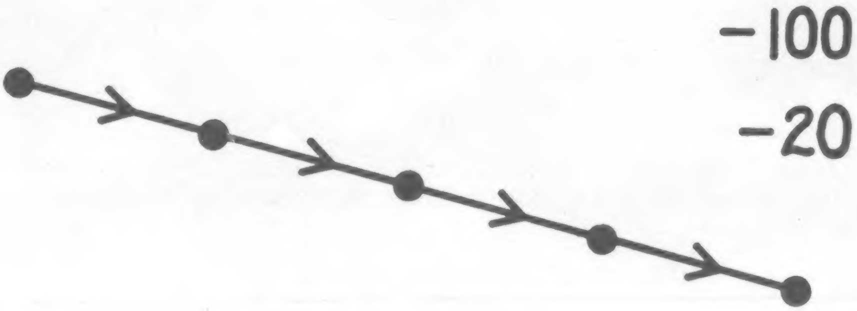
MATERIALS

Teacher: Colored chalk
Student: Paper; colored pencils

DESCRIPTION OF LESSON

Exercise 1

Draw this arrow picture on the board.



T: Point to the smallest number in this arrow picture.

The students should point to the ending dot of the road.

T: How do you know that this is the smallest number?

S: Because when we subtract we get smaller numbers.*

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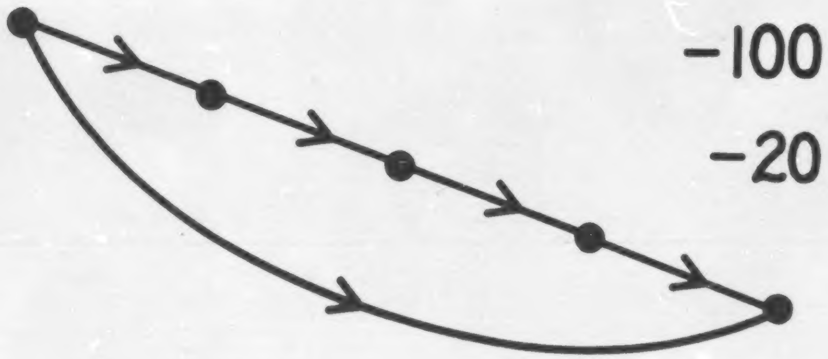
* This is true if a positive number is subtracted.

N 16

Draw a green arrow from the starting dot to the ending dot.



T: What could this green arrow be for? How do you know?

S: There are three - 100 arrows and one - 20 arrow, so the green arrow is for - 320.

Write "- 320" in green near the arrow picture. Label the starting dot "575".

T: If this dot were for 575, what would the ending number be? (255)

Let a student explain the answer by pointing to the dots in the arrow picture.

S: These numbers would be 575, 475, 375, 275, and 255.

Erase "575".

T: If the starting dot were for 837, what number would be at the ending dot? How do you know?

S: 517, because 800 - 300 = 500 and 37 - 20 = 17.

Repeat this activity with 915, 306, and 1,200 as starting numbers of the road. Erase the board before going on to Exercise 2.

Exercise 2

In the beginning of this exercise, do not tell the students what the arrows are for.

T: Draw an arrow road on your paper with several blue arrows (not more than ten) and exactly one red arrow. When you finish, draw a green arrow from your starting dot to your ending dot.

Help individual students if necessary. When many of the students have finished this task, instruct the students to write on their papers "- 100" in blue and "- 30" in red.

T: Each of your blue arrows is for - 100 and the red arrow is for - 30. Since you have drawn many different arrow pictures, your green arrows will be for different things also. Who can tell us what his or her green arrow could be for?

S: My green arrow is for -630 because I drew six blue arrows and one red arrow. 6 x 100 = 600 and 600 + 30 = 630.

T: Sandra's green arrow is for - 630, so she should write "- 630" on her paper in green.
Whose green arrow is for something different than Sandra's?

S: My green arrow is for -1,030. I drew ten blue arrows and one red arrow. 10 x 100 = 1,000 and 1,000 + 30 = 1,030.

Ask all the students to label their green arrows. Let students who are having difficulty seek the help of the class by describing their arrow roads. Continue this activity until all the students have labeled their green arrows.

T: Choose any number you like to be your starting number. Write that number near your starting dot and then label all the other dots.

Look at the students' papers. Undoubtedly some of your students will have some negative numbers in their arrow pictures. Students who finish quickly may be able to help other students who need assistance in labeling their dots.

T: Write a number story about your green arrow on your paper.

Let several students write their number stories on the board. For example,

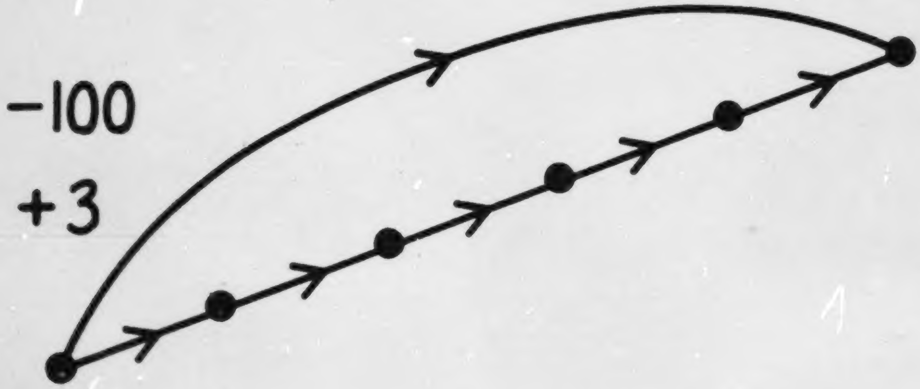
$$1{,}000 - 630 = 370 \qquad 400 - 630 = \widehat{230}$$

$$350 - 230 = 120 \qquad 1{,}050 - 530 = 520$$

Erase the board before going on to Exercise 3.

Exercise 3

Draw this arrow picture on the board.



T: What could this green arrow be for? How do you know?

S: There are four - 100 arrows and one + 3 arrow. 4 x 100 = 400 and 400 - 3 = 397, so the green arrow is for - 397.

Write "- 397" in green near the arrow picture.

T (point to the starting dot): If this dot were for 1,000, what number would be at the ending dot? (603)

Let a student explain the answer by pointing to the dots in the arrow picture.

Repeat this activity with 500, 750, and 1,500 as starting numbers.

Erase the board before going on to Exercise 4.

Exercise 4

Write this problem on the board.

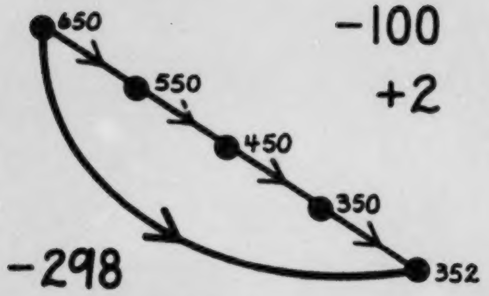
650 - 298 = ?

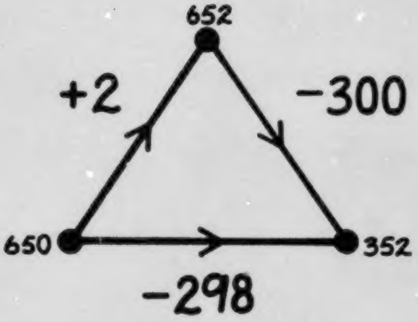
T: Solve this problem using any method you like and then draw an arrow picture to show how you calculated the answer.

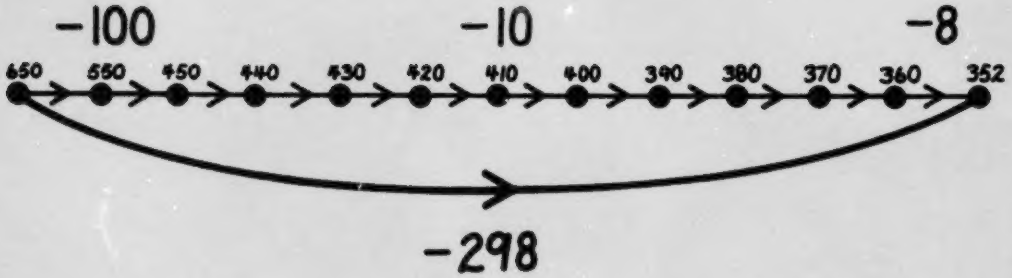
Some students may need help drawing an arrow picture. If these students are able to explain to you how they calculated the answer, describe for them the corresponding arrow picture. Suggest to students who are unable to calculate "650 - 298" that they start at 650 and draw two - 100 arrows, nine - 10 arrows, and one - 8 arrow.

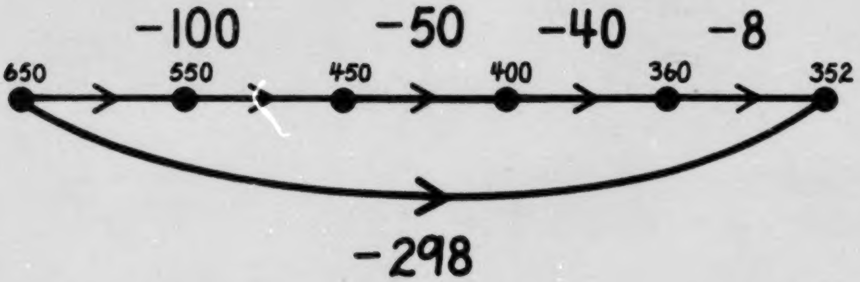
Ask students who finish quickly to draw their arrow pictures on the board. Try to get at least three different arrow pictures. A few of the many possibilities are shown on the following page.

N 16


650
-100
550
+2
450
350
-298
352


652
+2
-300
650
352
-298


-100
-10
-8
650
550
450
440
430
420
410
400
390
380
370
360
352
-298


-100
-50
-40
-8
650
550
450
400
360
352
-298

Discuss each arrow picture briefly and conclude that 650 - 298 = 352. Emphasize that there are many ways to solve this problem.

N17 Subtraction Algorithm #1

CAPSULE LESSON SUMMARY

Introduce one of two standard subtraction algorithms.

MATERIALS

Teacher: Colored chalk

Student: Paper; pencil; colored pencils; SUBTRACTION PROBLEMS booklet

DESCRIPTION OF LESSON

Exercise 1

T: What number is 2 + 4? (6)

2 + 4 = 6, so what number is 20 + 40? (60)

What number is 120 + 40? (160)

123 + 40 = . . . ? (163)

I have a friend named "Nick" who would write the problem "123 + 40" like this.

On the board write "123" and a red "4" above the "2".

$$1\overset{4}{2}3$$

T: Why would he write a red "4" above the "2" in "123"?

S: Because 40 has 4 tens, and if you add 4 tens to 123 it is the same as adding 40 to 123.

T: 4 tens + 2 tens is . . . ? (6 tens)

$$1\overset{4}{2}3 = 163$$

N 17

T: What number is 236 + 50?

On the board write "236" and a red "5" above the "3".

$$2\overset{5}{3}6$$

S: 236 + 50 = 286.

T: 5 tens + 3 tens is . . . ? (8 tens)

$$2\overset{5}{3}6 = 286$$

T: What number is 352 + 50?

On the board write "352" and a red "5" above the "5".

$$3\overset{5}{5}2$$

S: 2 fiftys make another hundred.

S: 10 tens is the same as 1 hundred, so 352 + 50 = 402.

$$3\overset{5}{5}2 = 402$$

T: This is a problem I saw on Nick's paper. Can you solve it? Write the answer on a piece of paper.

$$\overset{1}{4}1\overset{3}{6}$$

Look at many of the students papers before asking a student to answer aloud. Ask several students to explain how they calculated the answer (519).

Continue this activity with these problems. Answers are in boxes.

$$5\overset{3}{6}5 = \boxed{595} \qquad 4\overset{5}{4}\overset{2}{6} = \boxed{498}$$

$$1{,}2\overset{5}{3}8 = \boxed{1{,}288} \qquad 2{,}\overset{2}{4}5\overset{3}{3} = \boxed{2{,}656}$$

$$3{,}\overset{1}{4}\overset{6}{0}7 = \boxed{3{,}567} \qquad 2{,}\overset{1}{5}\overset{8}{1}\overset{4}{4} = \boxed{2{,}698}$$

T: What number is 475 + 10?

On the board write "475" and a red "10" above the "5".

$$47\overset{10}{5}$$

S: 475 + 10 = 485.

T: Could Nick have written this problem a different way?

S: Nick could have written a "1" above the "7" in "475" instead.

Record the equivalent expression and the answer.

$$47\overset{10}{5} = 4\overset{1}{7}5 = 485$$

T: What about this problem?

$$6\overset{10}{7}8$$

S: 10 tens is the same as 1 hundred, so Nick could have written a "1" above the "6" in "678" instead.

$$6\overset{10}{7}8 = \overset{1}{6}78$$

T: What number is 678 + 100?

S: 778.

$$6\overset{10}{7}8 = \overset{1}{6}78 = 778$$

Erase the board before going on to Exercise 2.

Exercise 2

Write this problem on the board and ask the students to solve it on their papers.

$$\begin{array}{r} 986 \\ -352 \\ \hline \end{array}$$

Look at the students' papers and then invite someone to do the subtraction calculation at the board.

$$\begin{array}{r} 986 \\ -352 \\ \hline 634 \end{array}$$

T: Nick would solve this problem the same way we did.

Pose another problem.

$$\begin{array}{r} 986 \\ -352 \\ \hline 634 \end{array} \qquad \begin{array}{r} 872 \\ -428 \\ \hline \end{array}$$

T: Is this an easier or a harder problem? Why?

Let students comment and then ask the class to solve the problem individually. Many students may solve this problem using one of the methods shown below.

$$\begin{array}{r} 872 \\ -428 \\ \hline 45\hat{6} \end{array} = 450 + \hat{6} = 444 \qquad \text{or} \qquad \begin{array}{r} 872 \\ -400 \\ \hline 472 \\ -20 \\ \hline 452 \\ -8 \\ \hline 444 \end{array}$$

$$\begin{array}{r} 87\overset{2}{2} \\ -42\overset{2}{8} \\ \hline \end{array} = \begin{array}{r} 874 \\ -430 \\ \hline 444 \end{array}$$

Ask a student who added 2 to both numbers to explain this method to the class. If necessary, show this approach yourself.

T: Nick solved this problem in another way. He didn't add 2 to both numbers; he added 10 to both numbers.

Pause for a moment to allow students to consider how that might help.

T: This is what he wrote.

$$\begin{array}{r} 87\overset{10}{2} \\ -\,4\overset{1}{2}8 \\ \hline \end{array}$$

T: Did Nick add 10 to each of these numbers? (Yes)

Emphasize that 10 ones is the same as 1 ten.

T: Then Nick thought, "10 + 2 = 12 and 12 - 8 = 4", and he wrote a "4" below the ones' column.

$$\begin{array}{r} 87\overset{10}{2} \\ -\,4\overset{1}{2}8 \\ \hline 4 \end{array}$$

T: Nick continued, "2 + 1 = 3 and 7 - 3 = 4", and he wrote a "4" below the tens' column.

$$\begin{array}{r} 87\overset{10}{2} \\ -\,4\overset{1}{2}8 \\ \hline 44 \end{array}$$

T: "8 - 4 = 4", so he wrote a "4" below the hundreds' column.

$$\begin{array}{r} 87\overset{10}{2} \\ -\,4\overset{1}{2}8 \\ \hline 444 \end{array}$$

Conclude that 872 - 428 = 444.

Ask the students to solve this problem using Nick's method of adding 10 to each of the numbers.

$$\begin{array}{r} 8,745 \\ -\,5,337 \\ \hline \end{array}$$

After a few minutes, allow a student to explain how Nick would solve this problem.

$$\begin{array}{r} 8,74\overset{10}{5} \\ -\,5,3\overset{1}{3}7 \\ \hline 3,408 \end{array}$$

Let the students solve these problems on their own.

$$\begin{array}{r} 473 \\ -\,158 \\ \hline \end{array} \qquad \begin{array}{r} 860 \\ -\,729 \\ \hline \end{array} \qquad \begin{array}{r} 5,983 \\ -\,2,657 \\ \hline \end{array}$$

NOTE: Some students may have difficulty knowing which numbers they should add and which they should subtract. One suggestion is to encircle the numbers to be added together as shown below.

$$\begin{array}{r} 47\enclose{circle}{\overset{10}{3}} \\ -\,1\enclose{circle}{\overset{1}{5}}8 \\ \hline \end{array}$$

N17

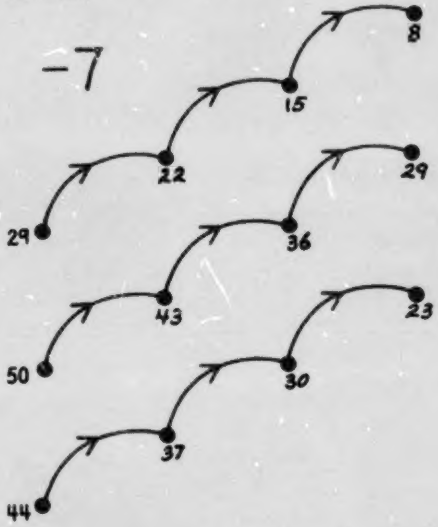
The worksheets for the entire sequence of lessons (N17, N21, N25, and N32) involving the subtraction algorithm for CSMP Mathematics for the Upper Primary Grades Part IV are contained in a sixteen-page booklet, SUBTRACTION PROBLEMS. The students should begin on page 2 today and progress at their own pace. Allow 5 to 10 minutes today for independent work in this problem book. After the lesson, collect the booklets, check them, and store them for future use in the remaining lessons on the subtraction algorithm.

Subtraction Problems

Subtract.

$\begin{array}{r}56\\-21\\\hline 35\end{array}$	$\begin{array}{r}45\\-13\\\hline 32\end{array}$	$\begin{array}{r}57\\-34\\\hline 23\end{array}$
$\begin{array}{r}73\\-33\\\hline 40\end{array}$	$\begin{array}{r}87\\-52\\\hline 35\end{array}$	$\begin{array}{r}69\\-51\\\hline 18\end{array}$
$\begin{array}{r}159\\-30\\\hline 129\end{array}$	$\begin{array}{r}268\\-166\\\hline 102\end{array}$	$\begin{array}{r}100\\-20\\\hline 80\end{array}$

Label the dots.



Subtract.

$\begin{array}{r}78\\-32\\\hline 46\end{array}$	$\begin{array}{r}108\\-91\\\hline 17\end{array}$	$\begin{array}{r}96\\-50\\\hline 46\end{array}$
$\begin{array}{r}648\\-325\\\hline 323\end{array}$	$\begin{array}{r}127\\-117\\\hline 10\end{array}$	$\begin{array}{r}354\\-152\\\hline 202\end{array}$
$\begin{array}{r}1{,}463\\-212\\\hline 1251\end{array}$	$\begin{array}{r}607\\-405\\\hline 202\end{array}$	$\begin{array}{r}1{,}738\\-637\\\hline 1{,}101\end{array}$

Complete.

12 − 8 = 4 22 − 8 = 14 32 − 8 = 24	16 − 7 = 9 26 − 7 = 19 36 − 7 = 29
14 − 6 = 8 24 − 6 = 18 34 − 6 = 28	15 − 9 = 6 25 − 9 = 16 35 − 9 = 26
13 − 9 = 4 23 − 9 = 14 33 − 9 = 24	14 − 8 = 6 24 − 8 = 16 34 − 8 = 26

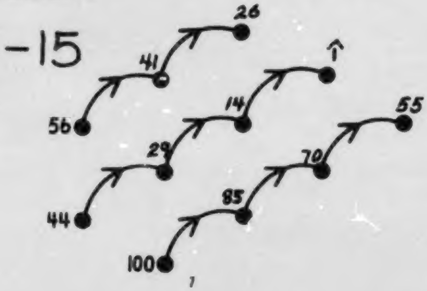
Subtract.

$\begin{array}{r} 4\overset{10}{3} \\ -14 \\ \hline 29 \end{array}$	$\begin{array}{r} 7\overset{10}{2} \\ -44 \\ \hline 28 \end{array}$	$\begin{array}{r} 8\overset{10}{1} \\ -23 \\ \hline 58 \end{array}$
$\begin{array}{r} 60 \\ -25 \\ \hline 35 \end{array}$	$\begin{array}{r} 87 \\ -77 \\ \hline 10 \end{array}$	$\begin{array}{r} 84 \\ -27 \\ \hline 57 \end{array}$
$\begin{array}{r} 56 \\ -48 \\ \hline 8 \end{array}$	$\begin{array}{r} 80 \\ -59 \\ \hline 21 \end{array}$	$\begin{array}{r} 101 \\ -50 \\ \hline 51 \end{array}$

Subtract.

$\begin{array}{r} 50 \\ -26 \\ \hline 24 \end{array}$	$\begin{array}{r} 41 \\ -24 \\ \hline 17 \end{array}$	$\begin{array}{r} 43 \\ -38 \\ \hline 5 \end{array}$
$\begin{array}{r} 426 \\ -312 \\ \hline 114 \end{array}$	$\begin{array}{r} 371 \\ -222 \\ \hline 149 \end{array}$	$\begin{array}{r} 356 \\ -40 \\ \hline 316 \end{array}$

Label the dots.



Complete.

35 − 45 = $\widehat{10}$

263 − 121 = 142

352 − 38 = 314

200 − 99 = 101

460 − 217 = 243

100 − 25 = 75

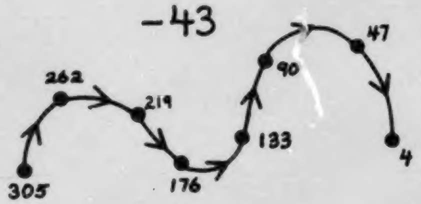
Subtract.

$\begin{array}{r} 174 \\ -65 \\ \hline 109 \end{array}$	$\begin{array}{r} 365 \\ -28 \\ \hline 337 \end{array}$	$\begin{array}{r} 259 \\ -132 \\ \hline 127 \end{array}$
$\begin{array}{r} 6{,}152 \\ -2{,}048 \\ \hline 4{,}104 \end{array}$	$\begin{array}{r} 7{,}933 \\ -5{,}617 \\ \hline 2{,}316 \end{array}$	$\begin{array}{r} 5{,}472 \\ -306 \\ \hline 5{,}166 \end{array}$
$\begin{array}{r} 706 \\ -524 \\ \hline 182 \end{array}$	$\begin{array}{r} 572 \\ -47 \\ \hline 525 \end{array}$	$\begin{array}{r} 629 \\ -598 \\ \hline 31 \end{array}$

Subtract.

615 - 475 140	707 - 332 375	462 - 155 307	518 - 104 414
153 - 63 90	663 - 247 416	538 - 367 171	984 - 576 408

Label the dots.

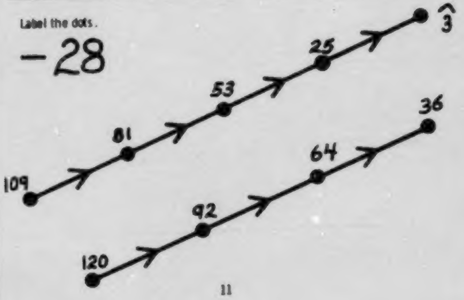


-43
305
262
219
176
133
90
47
4

Subtract.

1,451 - 339 1,112	2,603 - 1,063 1,540	3,580 - 436 3,144
5,069 - 2,245 2,824	1,719 - 359 1,360	18,217 - 609 17,608

Label the dots.



-28
109
81
53
25
3
120
92
64
36

Subtract.

653 - 206 447	827 - 754 73	147 - 75 72
873 - 618 255	416 - 284 132	496 - 369 127
5,829 - 390 5,439	1,028 - 414 614	2,873 - 2,618 255
2,648 - 1,908 740	4,627 - 96 4,531	7,164 - 5,008 2,156

Complete.

636 - 386 = 250

503 - 261 = 242

885 - 467 = 418

1,387 - 956 = 431

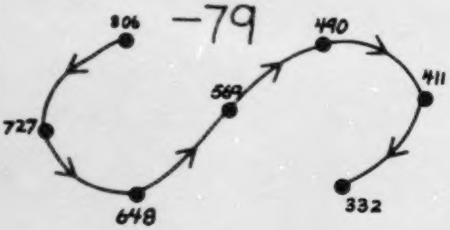
4,586 - 2,177 = 2,409

2,163 - 660 = 1,503

Subtract.

$\begin{array}{r} ^{10\;10} \\ 500 \\ -174 \\ \hline 326 \end{array}$	$\begin{array}{r} 153 \\ -69 \\ \hline 84 \end{array}$	$\begin{array}{r} 876 \\ -485 \\ \hline 391 \end{array}$
$\begin{array}{r} 156 \\ -59 \\ \hline 97 \end{array}$	$\begin{array}{r} 1{,}860 \\ -75 \\ \hline 1{,}785 \end{array}$	$\begin{array}{r} 928 \\ -589 \\ \hline 339 \end{array}$

Label the dots.



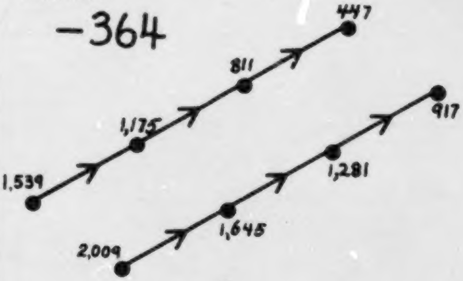
−79

Subtract.

$\begin{array}{r} 3{,}456 \\ -1{,}852 \\ \hline 1{,}604 \end{array}$	$\begin{array}{r} 6{,}072 \\ -429 \\ \hline 5{,}643 \end{array}$	$\begin{array}{r} 5{,}329 \\ -1{,}890 \\ \hline 3{,}439 \end{array}$
$\begin{array}{r} 5{,}900 \\ -358 \\ \hline 5{,}342 \end{array}$	$\begin{array}{r} 1{,}397 \\ -538 \\ \hline 859 \end{array}$	$\begin{array}{r} ^{10\;10\;10} \\ 1{,}000 \\ -679 \\ \hline 321 \end{array}$

Label the dots.

−364



Subtract.

$\begin{array}{r} 9{,}024 \\ -3{,}567 \\ \hline 5{,}457 \end{array}$	$\begin{array}{r} 8{,}204 \\ -7{,}999 \\ \hline 205 \end{array}$	$\begin{array}{r} 7{,}001 \\ -643 \\ \hline 6{,}358 \end{array}$
$\begin{array}{r} 3{,}760 \\ -584 \\ \hline 3{,}176 \end{array}$	$\begin{array}{r} 5{,}000 \\ -2{,}027 \\ \hline 2{,}973 \end{array}$	$\begin{array}{r} 2{,}546 \\ -1{,}494 \\ \hline 1{,}052 \end{array}$
$\begin{array}{r} 3{,}253 \\ -1{,}355 \\ \hline 1{,}898 \end{array}$	$\begin{array}{r} 9{,}086 \\ -2{,}194 \\ \hline 6{,}892 \end{array}$	$\begin{array}{r} 4{,}800 \\ -999 \\ \hline 3{,}801 \end{array}$
$\begin{array}{r} 22{,}270 \\ -4{,}631 \\ \hline 17{,}639 \end{array}$	$\begin{array}{r} 14{,}200 \\ -8{,}663 \\ \hline 5{,}537 \end{array}$	$\begin{array}{r} 10{,}000 \\ -9{,}876 \\ \hline 124 \end{array}$

N18 Division Problems #2

CAPSULE LESSON SUMMARY

Evenly divide 108 books among three classes. Do a similar activity with 192 ballons.

MATERIALS

Teacher: None
Student: Paper; pencil; Worksheets N18*, **, ***

DESCRIPTION OF LESSON

Begin this lesson with a short mental arithmetic activity involving the functions $3\times$ and $\frac{1}{3}\times$.

3×10 (30) 3×12 (36) $\frac{1}{3} \times 60$ (20)

$\frac{1}{3} \times 30$ (10) $\frac{1}{3} \times 36$ (12) $\frac{1}{3} \times 69$ (23)

Occasionally ask students to explain how they did a calculation.

Tell the following or similar stories to your class. Choose other teachers' classes as well as your own to star in them. Substitute those teachers' names for "Ms. Briggs", "Ms. Bollinger", "Ms. Powers", "Mr. Hickel", and "Mr. Kenfeld" in the lesson description.

T: Let's pretend that someone donated 108 books to our school. We would like to keep one-third of them for our class, to give one-third of the books to Ms. Briggs' class, and to give another one-third to Ms. Bollinger's class.

N 18

Draw a chart on the board.

Ms. Schweitzer (Write your name here.)	**Ms. Briggs** (Write two other teachers' names here.)	**Ms. Bollinger**

T: How many books should we give to each of these three classes? Remember that we want to give one-third of the books to each class. Write the answer on your paper.

Walk around the room and look at many of the students' answers. Do not tell them yet whether or not their answers are correct.

T: Will each classroom receive at least one hundred books? How do you know?

S: No, because if we give one hundred of the books to Ms. Bollingers' class, there will be only eight books left for our class and Ms. Briggs' class.

T: Could we give fifty books to each class? (No) Why not?

S: 3 × 50 = 150 and 150 is more than 108.

T: Could we give forty books to each class?

S: No, because 3 × 40 = 120 and 120 is more than 108.

T: Could we give thirty books to each class?

S: We could do that.

Record "30" in each column of the chart.

Ms. Schweitzer	Ms. Briggs	Ms. Bollinger
30	30	30

T: After we give thirty books to each class, how many books will we have left to share? How do you know?

S: Eighteen. 90 + 10 = 100, so 90 + 18 = 108.

T: What number is $\frac{1}{3} \times 18$?

S: 6.

Write "6" in each column.

T: How many books should we give to each class?

S: Thirty-six.

T: What number stories can we write about this problem?

Record number stories on the board as they are suggested. For example,

$$36 + 36 + 36 = 108$$

$$3 \times 36 = 108$$

$$36 \times 3 = 108$$

$$\frac{1}{3} \times 108 = 36$$

$$108 \div 3 = 36$$

N 18

T: This is another way to record the division story.

$$\begin{array}{r} 36 \\ 3\overline{)108} \end{array}$$

T: 108 divided by 3 equals 36.

Erase the board before continuing with the next story.

T: There are 192 balloons left over from the school carnival. Suppose we want to divide them equally among the three first grade classes (or choose another grade level).

(Write the names of their teachers here.)

Ms. Powers	Mr. Hickel	Mr. Kenfeld

T: We are sharing 192 balloons. See if you can figure out how many balloons each class should receive. Write the answer on your paper.

Look quickly at the students' work.

T: How shall we begin to share the 192 balloons?

Follow the suggestions of your students. A sample dialogue follows.

S: Give fifty balloons to each class.

T: Is it possible to give fifty balloons to each class? (Yes)

Write "50" in each column of the chart.

T: What number is 3 × 50?

S: 150.

T: How many balloons do we still have to distribute?

S: Forty-two.

T: What shall we do now?

S: Give ten more balloons to each class.

T: $10 \times 3 = 30$, so that's possible to do.

Write "10" in each column of the chart.

T: How many balloons are left now?

S: Twelve.

T: What number is $\frac{1}{3} \times 12$?

S: 4.

Write "4" in each column.

Ms. Powers	Mr. Hickel	Mr. Kenfeld
50	50	50
10	10	10
4	4	4

N18

T: How many balloons did we give to each class?

S: Sixty-four.

T: How can we check that $\frac{1}{3} \times 192$ really is 64?

S: Calculate "64 + 64 + 64".

Ask the students to calculate "3 × 64" on their papers. Conclude that $\frac{1}{3} \times 192 = 64$ and so each class receives 64 balloons. Ask the students to write number stories about this situation on their papers. Invite several students each to record a number story on the board. A few of the possible number stories are shown here.

$$192 \div 3 = 64$$

$$64 \times 3 = 192$$

$$\frac{1}{3} \times 192 = 64$$

$$3 \times (50 + 10 + 4) = 192$$

$$\begin{array}{r} 64 \\ 3\overline{)192} \end{array}$$

$$\begin{array}{r} 64 \\ \times\ 3 \\ \hline 192 \end{array}$$

$$\begin{array}{r} 64 \\ 64 \\ +\ 64 \\ \hline 192 \end{array}$$

Worksheets N18*, **, and *** are available for individual work.

Name ____________________ **N 18** *

Share 26 dollars fairly between Dick and Linda.

For Dick	For Linda
10	10
3	3
13	13

Write a number story about this sharing. 13 + 13 = 26

Share 27 pencils fairly among Andrea, Sheila, and Rob.

For Andrea	For Sheila	For Rob
5	5	5
2	2	2
2	2	2
9	9	9

Write a number story about this sharing. 27 ÷ 3 = 9

Other ways to show the sharings and other number stories are possible.

Share 34 pennies fairly between Pat and Gary.

For Pat	For Gary
15	15
2	2
17	17

Write a number story about this sharing. $\frac{1}{2} \times 34 = 17$

Share 54 dimes fairly among Bill, Stanley, and Lisa.

For Bill	For Stanley	For Lisa
10	10	10
5	5	5
3	3	3
18	18	18

Write a number story about this sharing. 3 × 18 = 54

Other ways to show the sharings and other number stories are possible.

Name ____________________ **N 18** **

Share 114 pictures between Arthur and Maria.

For Arthur	For Maria
50	50
7	7
57	57

Write a number story about this sharing. 114 ÷ 2 = 57

Share 81 candies among Nora, Brad, and Mark.

For Nora	For Brad	For Mark
20	20	20
7	7	7
27	27	27

Write a number story about this sharing. 27 + 27 + 27 = 81

Other ways to show the sharings and other number stories are possible.

Share 186 stamps fairly between Andy and Pam.

For Andy	For Pam
50	50
40	40
3	3
93	93

Write a number story about this sharing. 2 × 93 = 186

Share 129 marbles fairly among John, Ann, and Cathy.

For John	For Ann	For Cathy
40	40	40
3	3	3
43	43	43

Write a number story about this sharing. 129 ÷ 3 = 43

Other ways to show the sharings and other number stories are possible.

Name ____________________ **N 18 *****

Share 483 jacks fairly among Paula, Stacey, and Joy.

For Paula	For Stacey	For Joy
100	100	100
60	60	60
1	1	1
161	161	161

Write a number story for this sharing. 3×161=483

Share 735 cards among Wally, James, and Amy.

For Wally	For James	For Amy
200	200	200
40	40	40
5	5	5
245	245	245

Write a number story for this sharing. 735÷3=245

Other ways to show the sharings and other number stories are possible.

Share 819 vegetable seeds among Mike, Ellen, and Eric.

For Mike	For Ellen	For Eric
250	250	250
20	20	20
3	3	3
273	273	273

Write a number story for this sharing. 273+273+273=819

Share 1,932 books among Sandra, Leo, and Christy.

For Sandra	For Leo	For Christy
500	500	500
100	100	100
40	40	40
4	4	4
644	644	644

Write a number story for this sharing. $\frac{1}{3}$ × 1,932=644

Other ways to show the sharings and other number stories are possible.

N19 Decimals on the Number Line

CAPSULE LESSON SUMMARY

Multiply non-integer decimals by 10 on the Minicomputer. List the corresponding number stories and look for patterns. Locate decimals between 0 and 1 on a number line using money as a support.

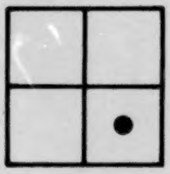
MATERIALS

Teacher: Minicomputer set; colored chalk; ⑩-checkers
Student: Worksheet N19*, **, and ***; pencil

DESCRIPTION OF LESSON

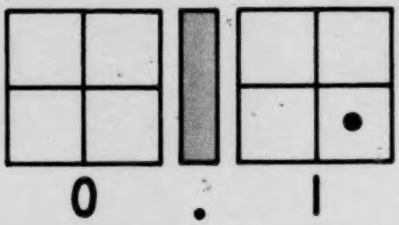
Exercise 1

Display one Minicomputer board and put a checker on the 1-square.



T: This checker is for $1. How could you show one dime on the Minicomputer?

Students should suggest putting up another board to the right and drawing a green bar between the boards. Follow this suggestion. Then ask a student to put one dime on the Minicomputer and to write the number below the Minicomputer.

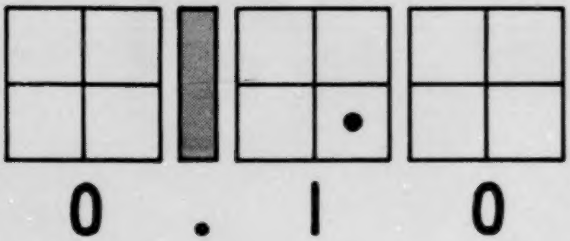


N 19

Add another new board on the right.

T: When I put another Minicomputer board on the right, the number is not changed. It is still one dime, but does this suggest another way to write this number?

S: You can write "0.10".

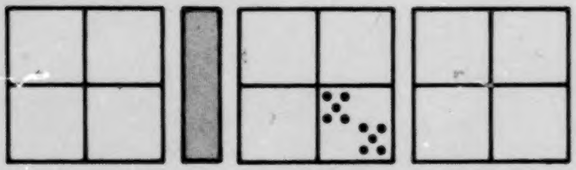


Record this information on the board.

$$0.1 = 0.10$$

T: The number on the Minicomputer is one dime.

Put nine more checkers on the 0.10-square and write the corresponding calculation to the right of the Minicomputer.



$= 10 \times 0.10$

T: Now how many dimes are on the Minicomputer?

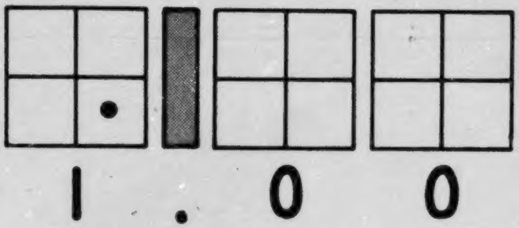
S: Ten.

T: How much money is this?

S: $1.

T: Ten dimes is the same as $1.

Ask students to make plays on the Minicomputer until the standard configuration for 1.00 is obtained. Ask a student to write the number below the Minicomputer.



= 10 × 0.10 = 1.00

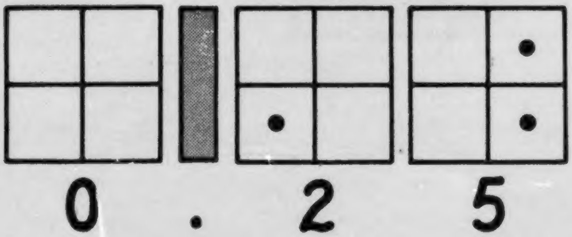
T: Do you know any other ways to write this number?

If necessary, suggest "1" and "1.0" yourself.* Begin a list of number stories on the board.

$$10 \times 0.10 = 1.00 = 1.0 = 1$$

Erase the board and remove the checker from the Minicomputer.

Ask a student to put a quarter on the Minicomputer and to write the number below the Minicomputer.



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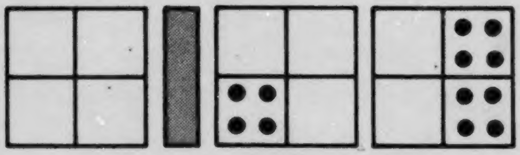
* There is an infinite number of expressions, including "01" and "001.000", which can be used to represent this number.

N 19

T: One quarter is on the Minicomputer.

Put three more checkers on the 0.20-square, on the 0.04-square, and on the 0.01-square.



T: Now how many quarters are on the Minicomputer?

S: Four.

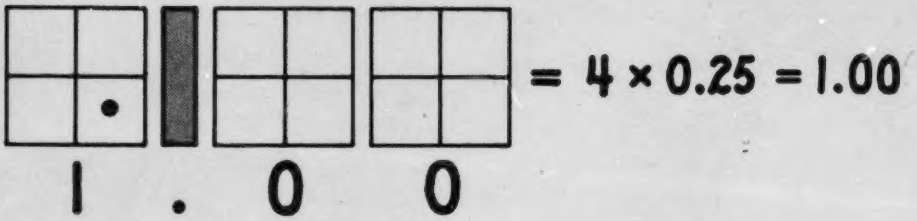
Write the calculation "4 × 0.25" to the right of the Minicomputer.

T: How much money is this?

S: \$1.

T: Four quarters is the same as \$1.

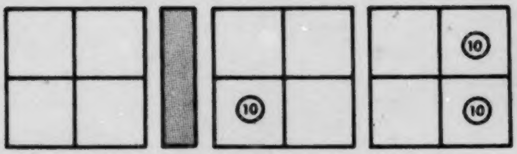
Ask students to make plays on the Minicomputer until the standard configuration for 1.00 is obtained. Ask a student to write the number below the Minicomputer.



T: How much money is ten quarters?

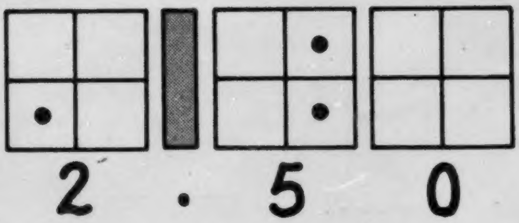
Write the calculation "10 × 0.25" and let students whisper their answers to you.

Put 10 × 0.25 on the Minicomputer with ⑩-checkers.



= 10 × 0.25

Ask for a volunteer to make plays with the ⑩-checkers and to write the number below the Minicomputer.



= 10 × 0.25 = 2.50

T: How much money is ten quarters? ($2.50)
10 × 0.25 = 2.50.[*] What is another name for 2.50?

S: 2.5.

Add this number story to the list forming on the board.

$$10 \times 0.10 = 1.00 = 1.0 = 1$$
$$10 \times 0.25 = 2.50 = 2.5$$

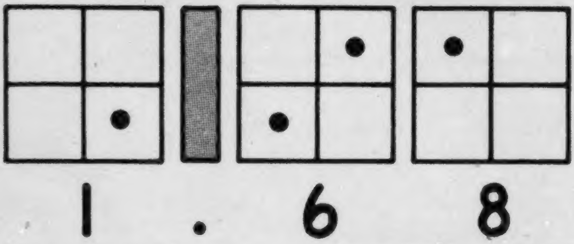
Erase the board except for the list of number stories and remove the checkers from the Minicomputer. Add a fourth Minicomputer board.

---

[*] "10 × 0.25 = 2.50" is read "ten times zero point two five is equal to two point five zero".

N 19

Ask a student to put $1.68 on the Minicomputer and to write the number below the Minicomputer.

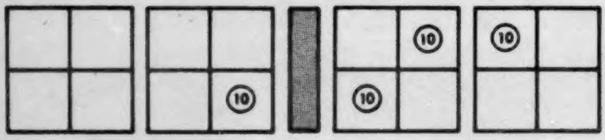


Write this problem on the board.

10 × 1.68

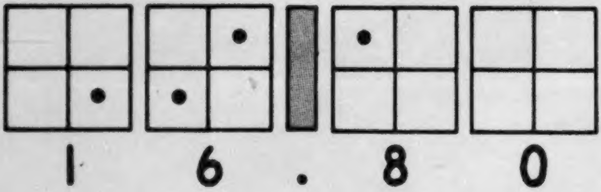
T: What number is 10 × 1.68?

Put 10 × 1.68 on the Minicomputer with ⑩-checkers and allow several students to whisper their answers to you. Write the calculation to the right of the Minicomputer.



= 10 × 1.68

Ask for a volunteer to make plays with the ⑩-checkers and to write the number below the Minicomputer.



= 10 × 1.68 = 16.80

T: How much money is 10 × \$1.68? (\$16.80)
10 × 1.68 = 16.80.
What is another name for 16.80?

S: 16.8.

Add this number story to the list on the board.

$$10 \times 0.10 = 1.00 = 1.0 = 1$$
$$10 \times 0.25 = 2.50 = 2.5$$
$$10 \times 1.68 = 16.80 = 16.8$$

T: Do you notice any patterns we could use when we multiply by 10?

Encourage the students to express themselves freely. Students should compare the positions of the decimal points in the answers to their positions in the problems.

Write this problem on the board.

$$10 \times 0.50$$

T: What number is 10 × 0.50?

S: 5.00.

Complete the number story and pose another problem.

$$10 \times 0.43$$

T: What number is 10 × 0.43?

S: 4.30.

N 19

Complete the number story. You should have this list on the board.

$$10 \times 0.10 = 1.00 = 1.0 = 1$$

$$10 \times 0.25 = 2.50 = 2.5$$

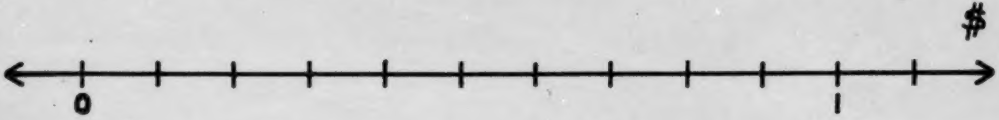
$$10 \times 1.68 = 16.80 = 16.8$$

$$10 \times 0.50 = 5.00 = 5.0 = 5$$

$$10 \times 0.43 = 4.30 = 4.3$$

Exercise 2

Draw this number line on the board. Each segment should be approximately 10 cm in length. Explain to the students that this is a "dollar" number line.



T: How many spaces are there between $0 and $1 (Ten)

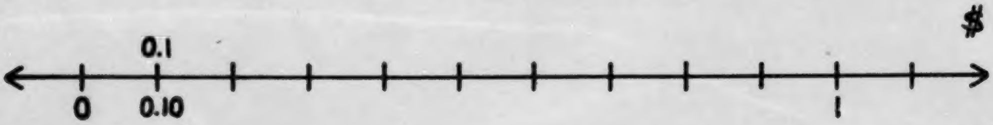
What could these marks be for? (Dimes)

Where is the mark for one dime?

The student should point to the first mark to the right of the mark for 0. Label it "0.10".

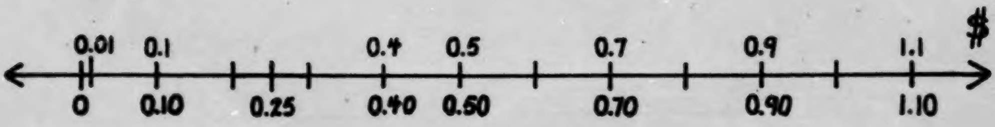
T: What is another name for 0.10?

Accept correct names. When someone suggests "0.1", write it near the same mark.



Ask students to point to and label the marks for these amounts.

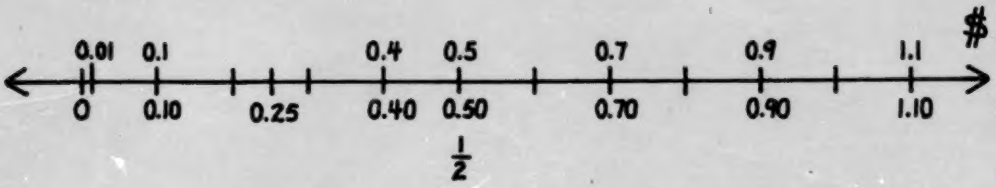
- forty cents
- nine dimes
- seventy cents
- the amount exactly halfway between $0 and $1 on the number line
- $1.10
- one penny
- one quarter


$
0.01
0.1
0.4
0.5
0.7
0.9
1.1
0
0.10
0.25
0.40
0.50
0.70
0.90
1.10

T: What part of a dollar is 50¢?

S: Half a dollar.

T: We can also use "$\frac{1}{2}$" as a label for the mark already labeled "0.5" and "0.50". This mark is halfway between "0" and "1".


$
0.01
0.1
0.4
0.5
0.7
0.9
1.1
0
0.10
0.25
0.40
0.50
0.70
0.90
1.10
1/2

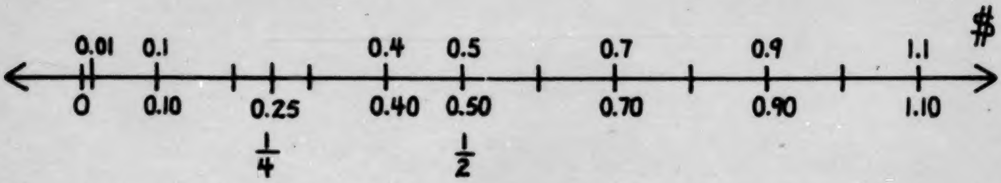
T: What part of a dollar is $0.25?

How many quarters are there in one dollar? (Four)

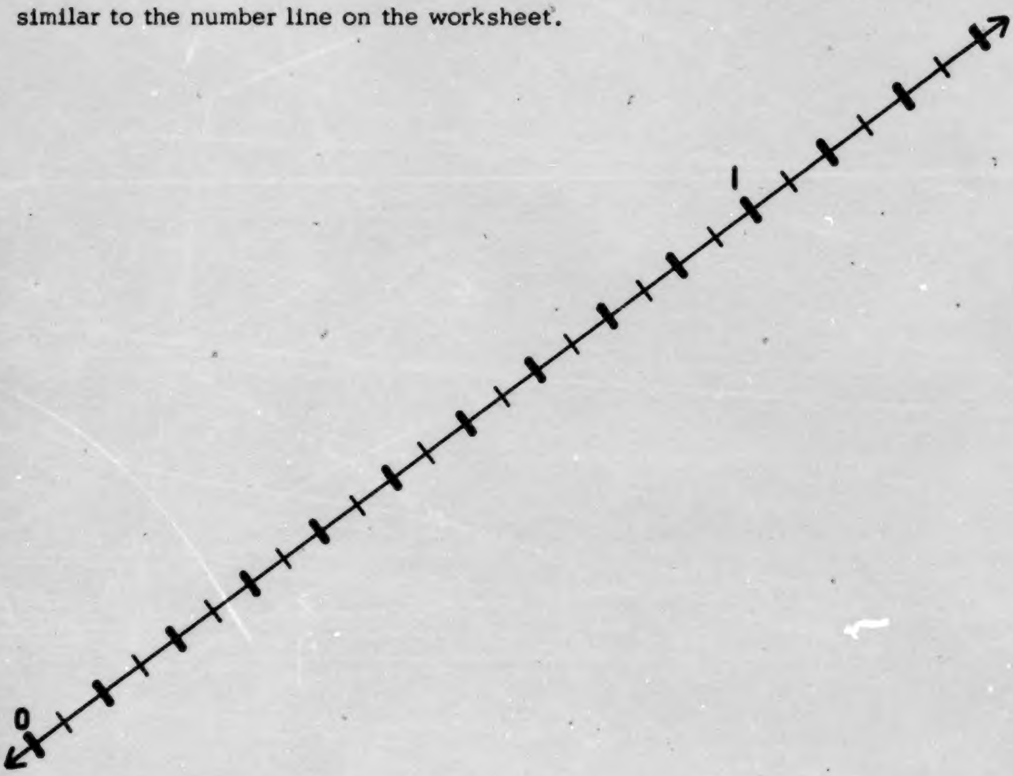
So one quarter is one-fourth of a dollar.

N19

T: We can also use "$\frac{1}{4}$" as a label for the mark already labeled "0.25". This mark is one-fourth of the way from "0" to "1".


$
0.01
0.1
0.4
0.5
0.7
0.9
1.1
0
0.10
0.25
0.40
0.50
0.70
0.90
1.10
1/4
1/2

Distribute copies of Worksheet N19*. Erase the board and draw a number line similar to the number line on the worksheet.

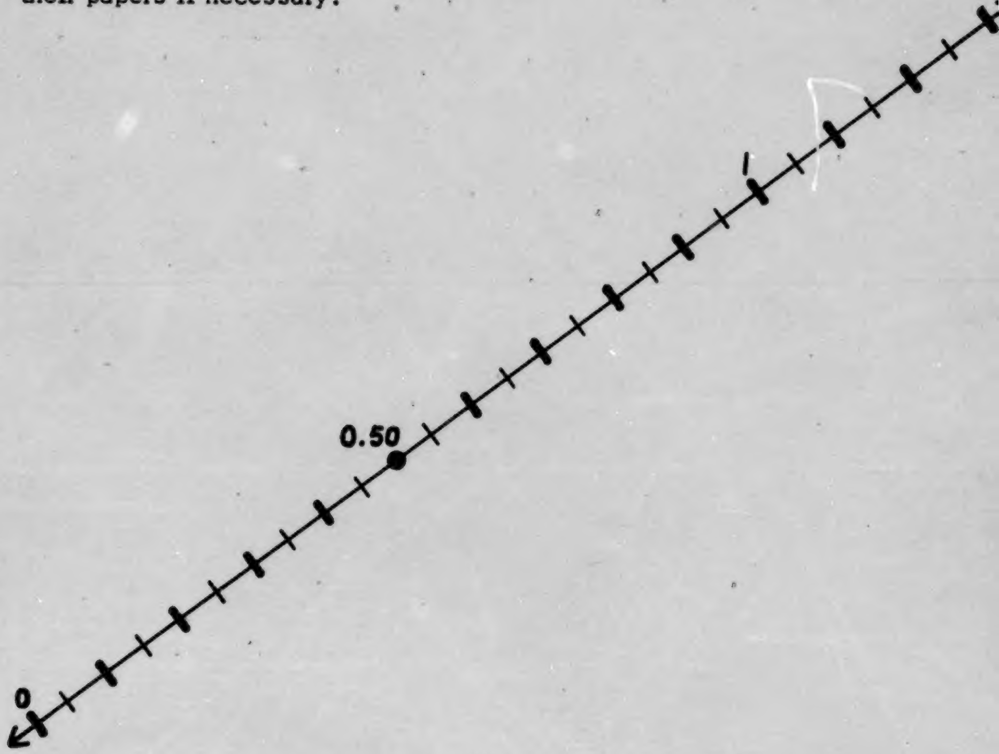


T: What could these heavy marks be for? (Dimes)

What could the light marks be for? (Nickels)

Draw and label a dot for 50¢ on your number line.

Walk around and look at the students' worksheets. Invite a student to draw and label a dot for 50¢ on the number line on the board. Ask students to correc their papers if necessary.

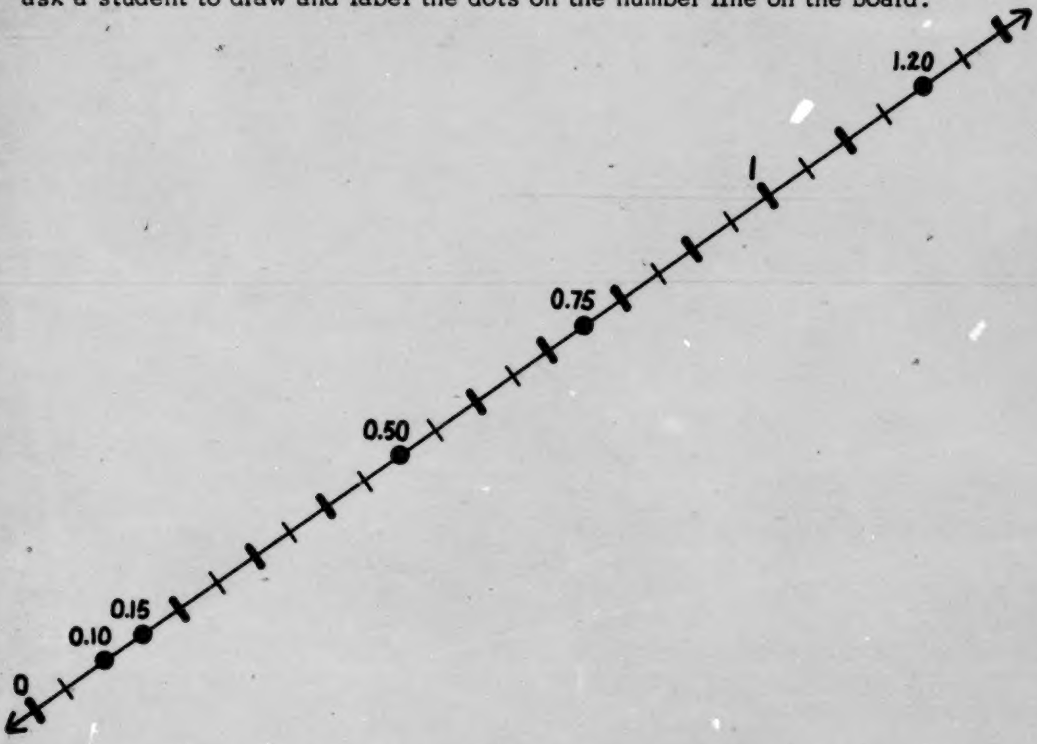


T: How did you know that this is the dot for 50¢ on this number line?

S: The heavy marks are for dimes, so I just counted 10, 20, 30, 40, 50.

N 19

Ask the students to draw and label dots on their number lines for 10¢, 15¢, 75¢, and $1.20. After the students have had some time to draw and label each dot, ask a student to draw and label the dots on the number line on the board.

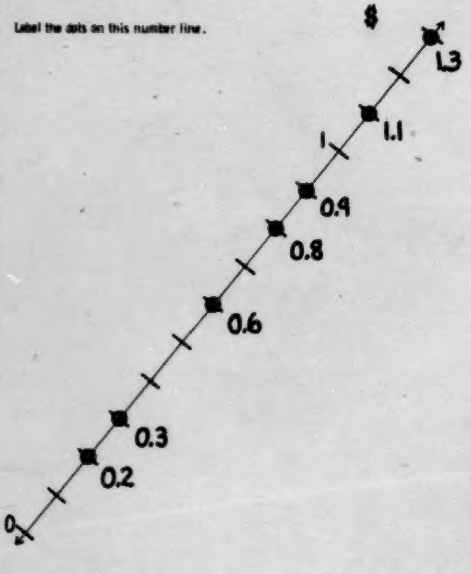


Worksheets N19** and *** are available for individual work.

Name ____________________

N 19 **

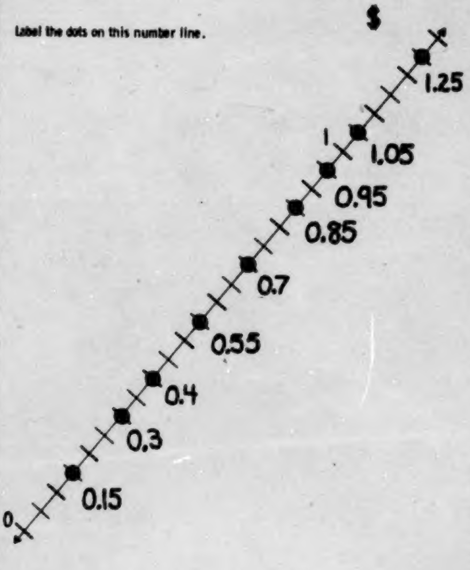
Label the dots on this number line.


$
0
0.2
0.3
0.6
0.8
0.9
1
1.1
1.3

Name ____________________

N 19 ***

Label the dots on this number line.


$
0
0.15
0.3
0.4
0.55
0.7
0.85
0.95
1
1.05
1.25

N 20

N20 Return Arrows #2

CAPSULE LESSON SUMMARY

Perform sequences of calculations on a hand-calculator and predict results before observing them. With a secret number displayed on the calculator, perform a sequence of operations one at a time. Record the sequence of operations in an arrow picture and use the picture to help discover the secret number.

MATERIALS

Teacher: Colored chalk
Student: Paper; pencil; hand-calculator

DESCRIPTION OF LESSON

In Exercise 1 only two or three students should have hand-calculators; in Exercise 2, every student should have one if possible.

Exercise 1

Give calculators to two or three students and ask them to put 10 on their calculators and to do exactly what you tell them.

T: What number is on the display? (10)
Cover the display. Press $\boxed{\times}\boxed{3}\boxed{+}\boxed{6}\boxed{=}$.

When you are telling students which buttons to press, you should pause after each numeral key so the students will have an opportunity to do each calculation mentally.

T: Don't look. What number should be on the display? (36)

Allow several students to answer before asking the students using the calculators to look at their displays. If any calculator has a number other than 36 on the display, ask the student with that calculator to press $\boxed{C}$ and then to put 36 on the display.

T: Cover the display again. Press $\boxed{-}\boxed{1}\boxed{0}\boxed{\div}\boxed{2}\boxed{=}$.
Don't look. What number should be on display? (13)

Allow several students to answer before asking the students using the calculators to look at their displays. If necessary, let the students correct the numbers on the hand-calculators.

Continue this activity with the following sequences.

$\boxed{-}\boxed{2}\boxed{\times}\boxed{3}\boxed{=}$ (33)
$\boxed{\times}\boxed{2}\boxed{+}\boxed{4}\boxed{=}$ (70)
$\boxed{\div}\boxed{2}\boxed{=}$ (35)
$\boxed{+}\boxed{1}\boxed{\div}\boxed{3}\boxed{=}$ (12)

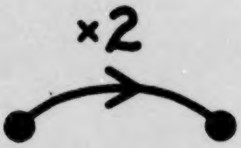
Exercise 2

Distribute calculators to the rest of the class.

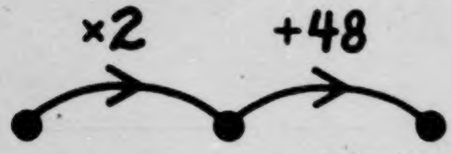
Invite a student who has shown some competence in the use of a hand-calculator to choose a whole number between 50 and 100, to write it on a piece of paper, and then to put it on the display of the calculator. This number should not be revealed to the other students. Ask the student to follow your instructions very carefully.

T: Press $\boxed{\times}\boxed{2}$.

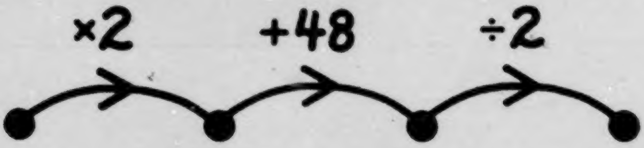
Draw a ×2 arrow on the board.



T: Now press $\boxed{+}\boxed{4}\boxed{8}$ (read: plus forty-eight).

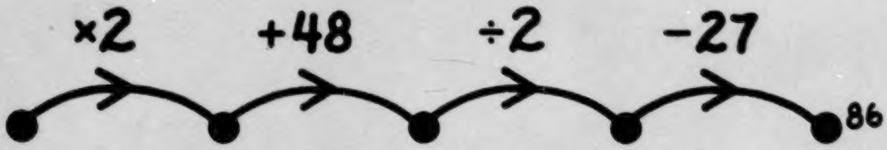


T: Press $\boxed{\div}\boxed{2}$.



T: And press $\boxed{-}\boxed{2}\boxed{7}\boxed{=}$.
What number is on the display?

Label the ending dot with the number given. (Here we assume the student's ending number is 86.)



T: Try to figure out what the secret (starting) number is.
Write it on your paper.

Allow a few minutes for the students to work on this problem. Encourage them to use their hand-calculators and suggest that return arrows might help them to determine the secret number.

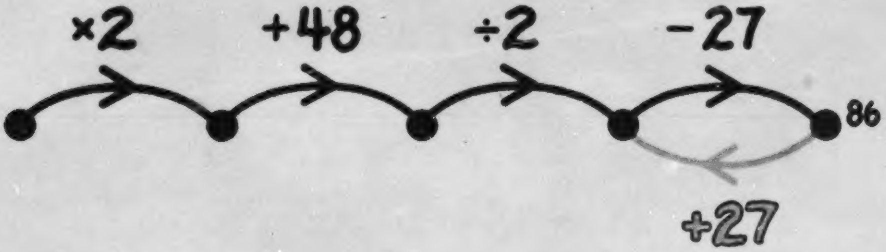
Ask all of the students to put the ending number (86 in this example) on their hand-calculators.

T: Let's label all the dots in this arrow picture with the help of our calculators.

Trace the return arrow for the - 27 arrow.

T: What could this return arrow be for? (+27)
What color could this arrow be?

Draw and label the +27 arrow. The +27 arrow can be any color which hasn't been used already in your arrow picture.



T: Press $\boxed{+}$ $\boxed{2}$ $\boxed{7}$ $\boxed{=}$. What number is on the display?

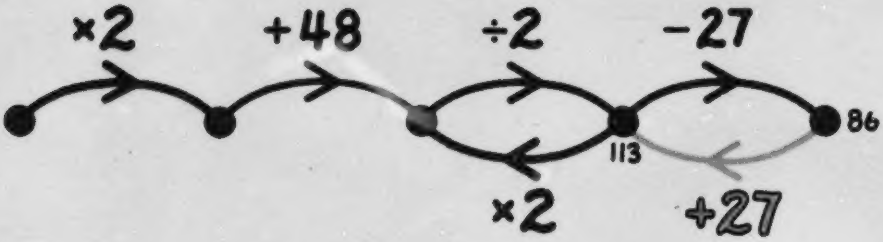
S: 113.

Label the dot for 113. Trace the return arrow for the ÷2 arrow.

T: What could this return arrow be for? (×2)
What color should this arrow be? Why?

N 20

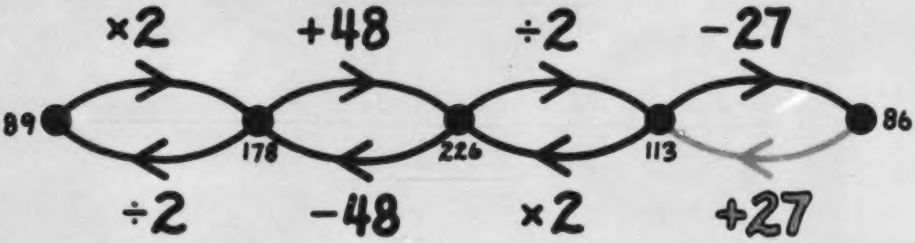
S: It should be green, because the other ×2 arrow is green.



T: Press $\boxed{\times}$ $\boxed{2}$ $\boxed{=}$. What number is on the display?

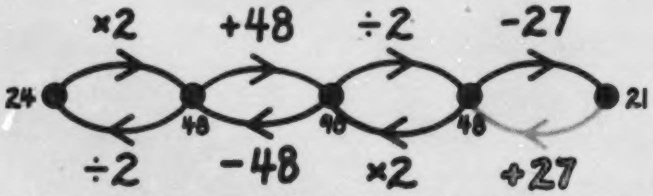
S: 226.

Continue this activity until all the return arrows are drawn and all the dots are labeled.



Check to see if the starting (secret) number is the same as the number written on the student's paper.

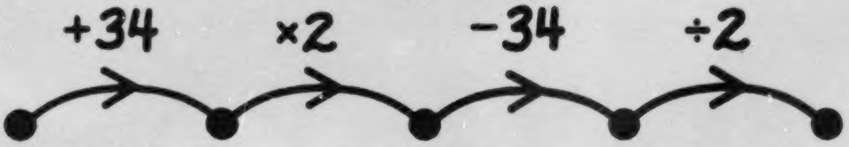
NOTE: Depending on what number is chosen as the starting number, two dots in the picture could be for the same number. For example, if 24 were the starting number, two dots in the picture would be for the number 48.



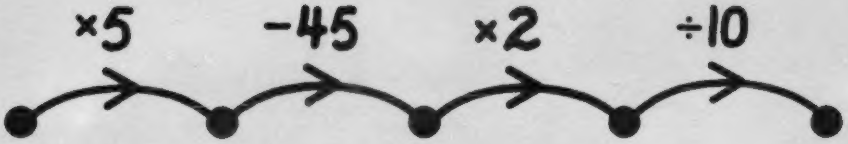
In this lesson we are not concerned about this situation possibly occurring because we want the arrow pictures to show the sequences of calculations.

Erase the board and repeat this activity with these sequences of calculations. Let several students have a turn at choosing the secret number.

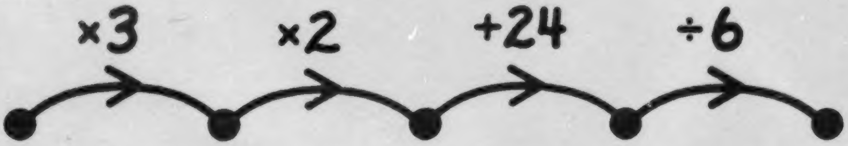
#1



#2



#3



You can involve more students in this activity by asking two students to each choose a secret number and then to follow your instructions as you tell them which buttons to press. In this case, both students will have ending numbers, and, if these numbers are different, your class will have two secret numbers to calculate. Avoid writing two numbers near the same dot. This can be done by considering each ending number in turn and erasing any previous labels, or by drawing each arrow road twice and writing one of the ending numbers by each of the ending dots, or by writing the appropriate numbers at some distance from the dots.

N21 Subtraction Algorithm #2

CAPSULE LESSON SUMMARY

Use the subtraction algorithm introduced in N17 to solve problems such as:

$$\begin{array}{r}362\\ \underline{-135}\end{array}\qquad\begin{array}{r}608\\ \underline{-427}\end{array}\qquad\begin{array}{r}1,342\\ \underline{-67}\end{array}$$

MATERIALS

Teacher: Colored chalk

Student: Paper; pencil; SUBTRACTION PROBLEMS booklet

DESCRIPTION OF LESSON

T: What number is 465 + 30?

S: 495.

Write "465" on the board.

T: How would Nick show that he wants to add 30 to 465?

S: Nick would put a red "3" above the "6".

$$4\overset{3}{6}5$$

T: The red "3" tells us that we want to add 3 tens to the 6 tens in 465. 3 + 6 = 9, so the answer has 9 tens.

$$4\overset{3}{6}5 = 495$$

Erase the board and then write this problem.

$$2,3\overset{10}{4}8$$

T: Nick wants to add 2,348 and 10 tens. Write the answer on your paper.

Look at several of the students papers before asking a student to answer aloud.

S: 2,348 + 100 = 2,448.

T: Could Nick have written this problem a different way?

S: He could have put a red "1" above the "3".

On the board write the equivalent expressions and the answer.

$$2,3\overset{10}{4}8 = 2,\overset{1}{3}48 = 2,448$$

T: 10 tens is the same as 100, so writing "10" above the number of tens and "1" above the number of hundreds are just different ways to show that we are adding 100.

Write these problems on the board and ask the students to solve as many of them as they can. Answers are in boxes.

$57\overset{4}{3} = \boxed{577}$

$1,2\overset{7}{3}8 = \boxed{1,308}$

$\overset{1}{6}\overset{7}{2}\overset{7}{4} = \boxed{801}$

$\overset{2}{4}52 = \boxed{652}$

$2,3\overset{10}{4}0 = \boxed{2,440}$

$4\overset{9}{7}\overset{8}{3} = \boxed{571}$

$5\overset{2}{9}4 = \boxed{614}$

$4\overset{5}{4}\overset{3}{6} = \boxed{499}$

$2,5\overset{10}{7}\overset{4}{6} = \boxed{2,680}$

If necessary, help some of the students individually. When a few students have solved all the problems, ask students to put the answers on the board.

N 21

You may wish to go over a particular problem collectively if several of the students are having difficulty with that problem.

Erase the board and then write this subtraction problem.

$$\begin{array}{r} 362 \\ -135 \\ \hline \end{array}$$

T: How would Nick have solved this subtraction problem?

S: He'd put a "1" above the "3" in the bottom number and a "10" above the "2" in the top number.

$$\begin{array}{r} 36\overset{10}{2} \\ -1\overset{1}{3}5 \\ \hline \end{array}$$

T: Nick would add 10 to each of these numbers.
What is the answer to this problem? Write it on your paper.

Observe the students work. Ask the students who finish quickly to solve this problem using Nick's method.

$$\begin{array}{r} 1{,}396 \\ -209 \\ \hline \end{array}$$

After a few minutes solve the first problem collectively.

T: Nick would think, "10 + 2 = 12 and 12 - 5 = 7", and then he would write a "7" below the ones' column.

$$\begin{array}{r} 36\overset{10}{2} \\ -1\overset{1}{3}5 \\ \hline 7 \end{array}$$

T: Then Nick would think, "1 + 3 = 4 and 6 - 4 = 2", and he would write a "2" below the tens' column.

$$\begin{array}{r} 36\overset{10}{2} \\ -1\overset{1}{3}5 \\ \hline 27 \end{array}$$

T: 3 - 1 = 2, so he would write "2" below the hundreds' column.

$$\begin{array}{r} 36\overset{10}{2} \\ -1\overset{1}{3}5 \\ \hline 227 \end{array}$$

Conclude that 352 - 135 = 227. Ask a student to solve the second problem and to explain each step.

$$\begin{array}{r} 1{,}39\overset{10}{6} \\ -\ 2\overset{1}{0}9 \\ \hline 1{,}187 \end{array}$$

N 21

Write the following subtraction problems on the board, and ask the students to solve as many of them as they can.

$$\begin{array}{r} 71 \\ -36 \\ \hline \end{array} \qquad \begin{array}{r} 150 \\ -34 \\ \hline \end{array} \qquad \begin{array}{r} 926 \\ -418 \\ \hline \end{array} \qquad \begin{array}{r} 608 \\ -427 \\ \hline \end{array}$$

When a few students have solved all four problems, ask students to solve the problems at the board and to explain each step.

$$\begin{array}{r} 7\overset{10}{1} \\ -\overset{1}{3}6 \\ \hline 35 \end{array} \qquad \begin{array}{r} 15\overset{10}{0} \\ -\overset{1}{3}4 \\ \hline 116 \end{array} \qquad \begin{array}{r} 92\overset{10}{6} \\ -4\overset{1}{1}8 \\ \hline 508 \end{array} \qquad \begin{array}{r} 6\overset{10}{0}8 \\ -\overset{1}{4}27 \\ \hline 181 \end{array}$$

Emphasize that in the last problem ("608 - 427"), Nick would not add 10 to each of the numbers because 8 is more than 7. The difficulty lies in the tens' column because 0 tens is smaller than 2 tens; that is why Nick would add 100 to each of the numbers.

Write these subtraction problems on the board and ask the students to solve as many of them as they can.

$$\begin{array}{r} 62 \\ -46 \\ \hline \end{array} \qquad \begin{array}{r} 154 \\ -44 \\ \hline \end{array} \qquad \begin{array}{r} 237 \\ -165 \\ \hline \end{array} \qquad \begin{array}{r} 1{,}342 \\ -\ \ 67 \\ \hline \end{array}$$

When a few students have solved all the problems, ask students to solve the problems at the board and to explain each step.

$$\begin{array}{r} 6\overset{10}{2} \\ -\overset{1}{4}6 \\ \hline 16 \end{array} \qquad \begin{array}{r} 154 \\ -44 \\ \hline 110 \end{array} \qquad \begin{array}{r} 2\overset{10}{3}7 \\ -\overset{1}{1}65 \\ \hline 72 \end{array} \qquad \begin{array}{r} 1{,}3\overset{10}{4}\overset{10}{2} \\ -\ \overset{1}{}\overset{1}{6}7 \\ \hline 1{,}275 \end{array}$$

Distribute the students' copies of the SUBTRACTION PROBLEMS booklet and allow 10 to 15 minutes for individual work. Encourage students to correct any errors they made the previous week. After the lesson, collect the booklets, check them, and have them ready for lessons N25 and N32.

N 22

N22 Composition of Functions #2

CAPSULE LESSON SUMMARY

With the support of the Minicomputer, decide that the composition 2 × followed by 2 × followed by 2 × is 8 ×. Draw the corresponding arrow picture. Do several 8 × calculations using the method suggested by the arrow picture.

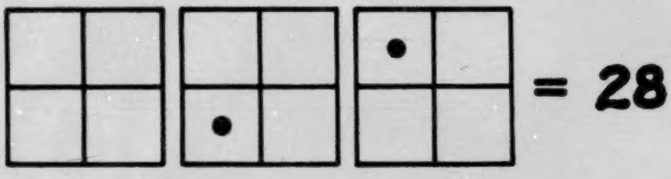
MATERIALS

Teacher: Colored chalk

Student: Worksheets N22*, **, ***, and ****; colored pencils

DESCRIPTION OF LESSON

Ask a student to put 28 on the Minicomputer.



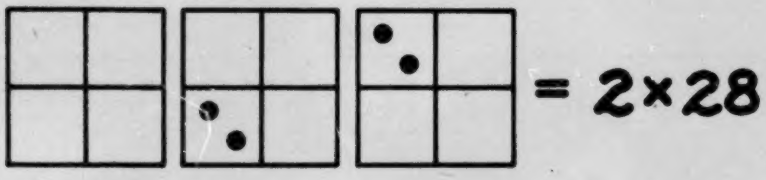
T: What number is $\frac{1}{2} \times 28$? How do you know?

S: $\frac{1}{2} \times 28 = 14$, because $\frac{1}{2} \times 20 = 10$ and $\frac{1}{2} \times 8 = 4$.

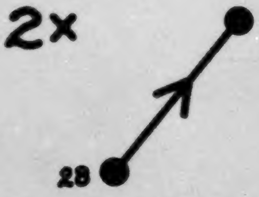
T: What number is $\frac{1}{4} \times 28$? How do you know?

S: 7, because $7 + 7 = 14$ and $14 + 14 = 28$.

Ask a student to double the number on the Minicomputer.

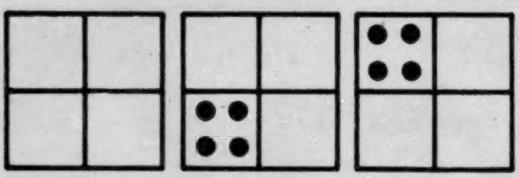


T: We can show this in an arrow picture. We start with 28 and then we double it.

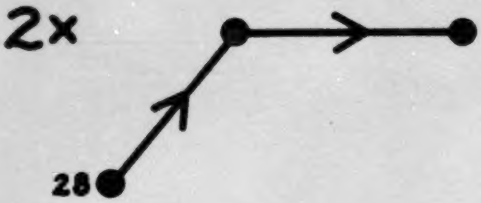


T: The number on the Minicomputer is 2 × 28. How can we double the number now on the Minicomputer?

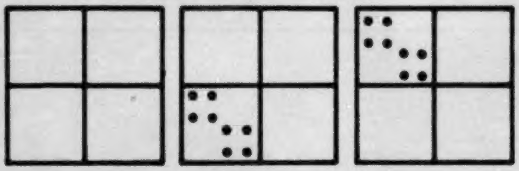
S: Put two more checkers on the 20-square and on the 8-square.



T: There were two checkers on both of these squares and now there are four. I'll draw another 2 × arrow.

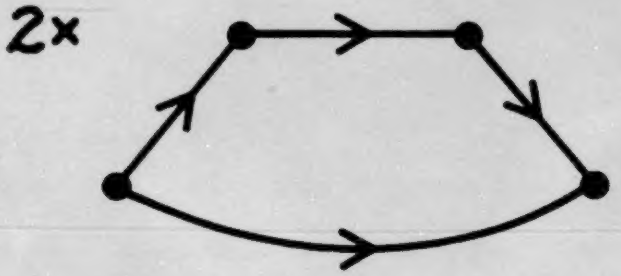


T (point to the ending dot): This is the number on the Minicomputer. We doubled 28 and then we doubled again. I would like to double the number on the Minicomputer again. Who can do this for me?



T: There were four checkers on each of these squares, and now there are eight.

Draw another 2 x arrow in the arrow picture ; then draw a blue arrow from the starting dot of the first arrow to the ending dot of the last arrow.



T: How many times do you see 28 on the Minicomputer? (Eight) What could the blue arrow be for?

S: 8 x.

Label the blue arrow "8 x". Trace the appropriate arrows as you say:

T: 2 x followed by 2 x followed by 2 x is the same as 8 x.

T: What number is on the Minicomputer? Whisper your answer to me.

Let several students whisper the answer before asking several students to tell which calculations they did. A possible dialogue is given here.

S: I figured out the number on the 20-square, and then I figured out the number on the 8-square.

Record the calculations on the board.

$$\begin{array}{r}8 \times 20 = \\ 8 \times 8 = \\ \hline 8 \times 28 =\end{array}$$

T: What number is 8 × 20? (160)

8 × 2 = 16, so 8 × 20 = 160.

What number is 8 × 8? How do you know?

S: 64. 10 × 8 = 80, so 9 × 8 = 80 - 8 = 72 and 8 × 8 = 72 - 8 = 64.

S: 5 × 8 = 40 and 3 × 8 = 24, so 8 × 8 = 40 + 24 = 64.

T: 8 × 20 = 160 and 8 × 8 = 64, so what number is 8 × 28?

How do you know?

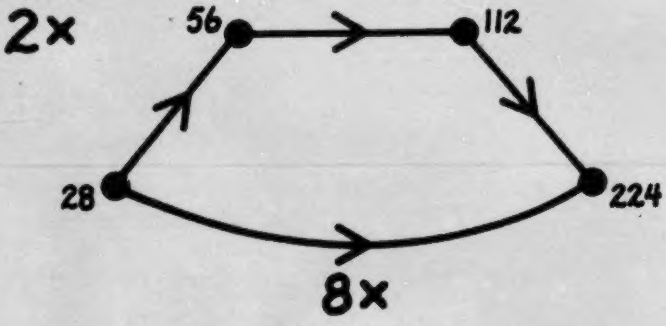
S: 8 × 28 = 224 because 160 + 64 = 224.

$$\begin{array}{r}8 \times 20 = 160 \\ 8 \times 8 = 64 \\ \hline 8 \times 28 = 224\end{array}$$

S: I followed the arrows in the arrow picture.

2 × 28 = 56, 2 × 56 = 112, and 2 × 112 = 224.

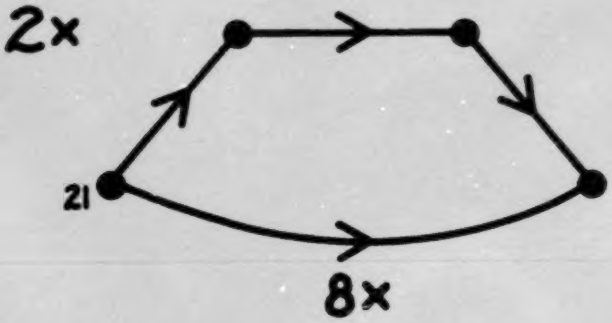
Label the dots in the arrow picture.



N 22

If no one suggests labeling the dots in the arrow picture to find the number on the Minicomputer, suggest this yourself. Conclude that the number on the Minicomputer is 224.

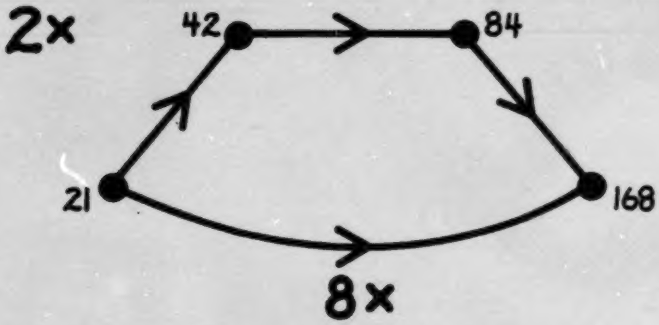
Remove the Minicomputer and erase everything on the board except the arrow picture. Label the starting dot of the red arrow road "21".



T: What number is 8 × 21? How do you know?

S: 2 × 21 = 42, 2 × 42 = 84, and 2 × 84 = 168.

Label the dots in the arrow picture.

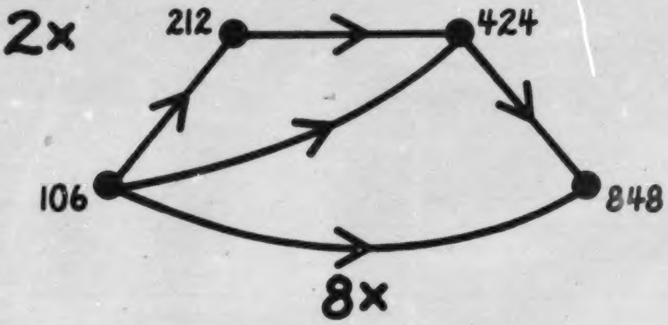


S: 8 × 20 = 160 and 8 × 1 = 8, so 8 × 21 = 160 + 8 = 168.

Erase the labels for the dots and then label the starting dot 106.

T: What number is 8 × 106? How do you know?

Encourage several students to explain how they determined that 8 × 106 = 848. If labeling the dots in the arrow picture is not suggested, suggest this yourself. Draw a green arrow from 106 to 424.



T: What could this green arrow be for?

S: 4 ×.

Label the green arrow "4 ×" and trace the appropriate arrows as you say:

T: A 2 × arrow followed by a 2 × arrow is the same as 4 × arrow. Could we draw another 4 × arrow in this arrow picture? (Yes, from 212 to 848)

Invite a student to trace the arrow.

T: 4 × 212 = 848. A 2 × arrow followed by a 2 × arrow is the same as a 4 × arrow.

Draw this table on the board near the arrow picture.

Start	2×	4×	8×
106			

N 22

T: The number in the first column is the starting number of the red arrow road.

Point to the dot for 106 in the arrow picture.

T: The number in the second column is two times the starting number. What number is 2 × 106?

S: 212.

Trace the red arrow from 106 to 212. Write "212" in the second column.

T: The number in the third column is four times the starting number. What number is 4 × 106?

S: 424.

Trace the green arrow from 106 to 424. Write "424" in the third column.

T: The number in the fourth column is eight times the starting number. What number is 8 × 106?

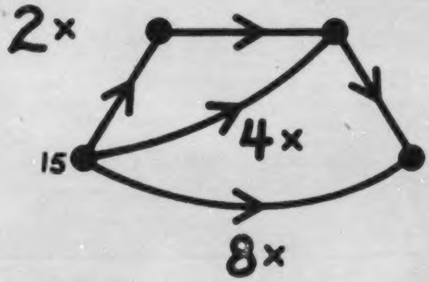
S: 848.

Trace the blue arrow from 106 to 848. Write "848" in the fourth column.

Start	2×	4×	8×
106	212	424	848

Erase the labels for the dots. Label the starting dot "15" and write "15" in the first column of the table.

Start	2×	4×	8×
106	212	424	848
15			



T: What number is 8 × 15? Whisper your answer to me.

Take whispers from several students before filling in the table. Conclude that 8 × 15 = 120.

Start	2×	4×	8×
106	212	424	848
15	30	60	120

Repeat this activity starting at 1,002; at 25; and at 51. Your table should look like this one.

Start	2×	4×	8×
106	212	424	848
15	30	60	120
1,002	2,004	4,008	8,016
25	50	100	200
51	102	204	408

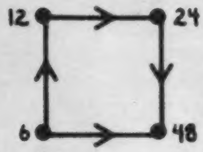
Worksheets N22*, **, ***, and **** are available for individual work.

Name ______________________ N 22 *

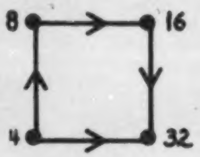
Label the dots and complete the number stories.

2×

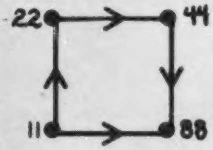
8×



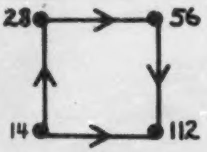
8×6 = 48



8×4 = 32



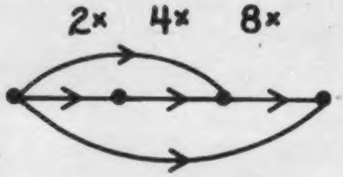
8×11 = 88



8×14 = 112

Name ______________________ N 22 **

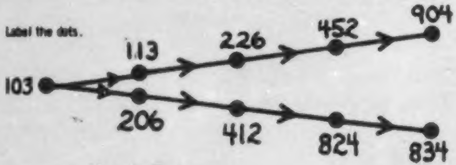
Complete this table.



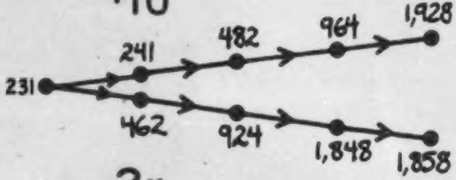
Starting Number	2×	4×	8×
9	18	36	72
17	34	68	136
23	46	92	184
38	76	152	304
47	94	188	376

Name ______________________ N 22 ***

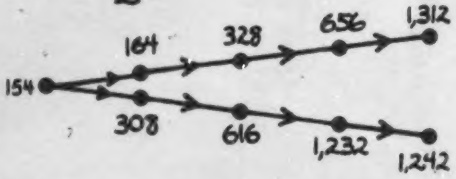
Label the dots.


103
113
226
452
904
206
412
824
834

+10


231
241
482
964
1,928
462
924
1,848
1,858

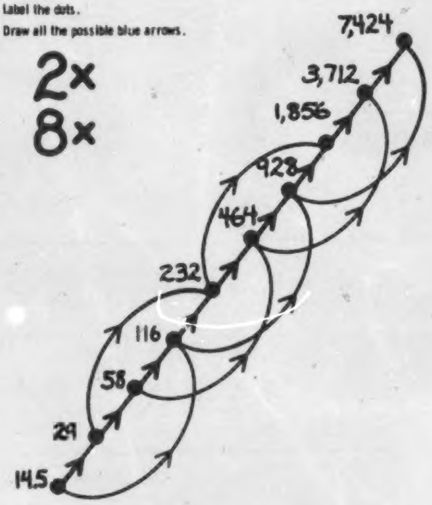
2×


154
164
328
656
1,312
308
616
1,232
1,242

Name ______________________ N 22 ****

Label the dots.
Draw all the possible blue arrows.

2×
8×


14.5
29
58
116
232
464
928
1,856
3,712
7,424

N23 Comparing Prices #1

CAPSULE LESSON SUMMARY

Compare prices of items packaged in different quantities by finding the cost of a certain number of them that can be purchased with either packaging.

MATERIALS

Teacher: Colored chalk
Student: Paper; colored pencils

DESCRIPTION OF LESSON

Exercise 1

Tell the following or a similar story to your class. Choose one of your students to star in this story. Substitute the student's name for "Patty" in the following dialogue.

T: Patty is going to buy a paint brush for each of the students in our school. She goes to two stores to price their paint brushes. The two stores sell exactly the same paint brushes, but one store sells a package of three paint brushes for 40¢ and the other store sells a package of two paint brushes for 24¢.

Record this information on the board.

Store #1 3 brushes for 40¢
Store #2 2 brushes for 24¢

T: Patty needs to know which store offers the better deal. Where should she buy the paint brushes for our school?

N 23

Let the students think about this problem and discuss it collectively. Very likely a student will observe that brushes are cheaper in the second store because each paint brush costs 12¢ there and so three paint brushes cost 36¢. Three paint brushes at the first store cost 40¢.

Perhaps some students will object that in the second store you must buy the paint brushes in packages of two, so you cannot buy exactly three paint brushes there. If necessary, mention this yourself.

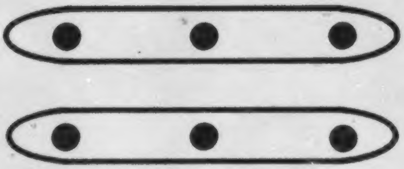
T: Could you buy the same number of paint brushes at each store? (Yes) What is the smallest such number?

S: Six.

T: To get six paint brushes from the first store, how many packages would we buy?

S: Two packages, because there are three brushes in each package.

Draw this picture on the board.

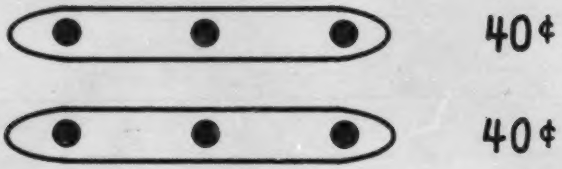


T: How much would it cost to buy six paint brushes from the first store? How do you know?

S: 80¢, because each package costs 40¢ and 2 x 40 = 80.

Record this information on the board.

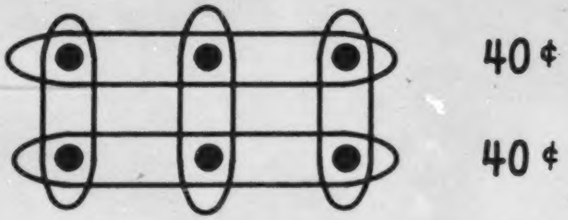
Store #1 3 brushes for 40¢ or 6 brushes for 80¢
Store #2 2 brushes for 24¢



T: To get six paint brushes from the second store, how many packages would we buy?

S: Three packages, because there are only two paint brushes in each package at that store.

Draw blue strings around pairs of dots on the board.



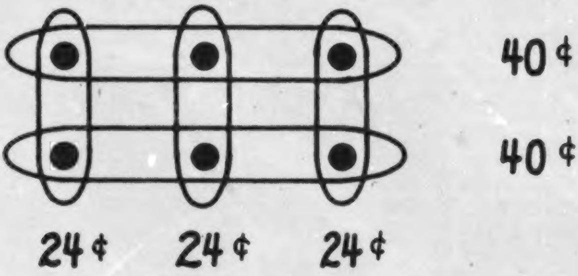
T: How much would it cost to buy six paint brushes from the second store? How do you know?

S: 72¢, because each package costs 24¢ and 24 + 24 + 24 = 72.

N 23

Record this information on the board.

Store #1 3 brushes for 40¢ or 6 brushes for 80¢
Store #2 2 brushes for 24¢ or 6 brushes for 72¢



T: Which store has the best price if we are buying six paint brushes?

S: The second store.

T: How much will we save if we buy six paint brushes at the second store instead of at the first store?

S: 8¢.

T: Patty needs to buy a paint brush for each student in our school. Which store offers the better deal? Why?

S: The second store, because the price (per brush) is lower there.

T: When would you ever buy paint brushes from the first store?

S: . . . if I only wanted to buy three brushes.

S: . . . if I had more than 40¢ but less than 48¢.

S: If I wanted to buy five brushes, I would buy three from one store and two from the other.

Erase the board before going on to Exercise 2.

Exercise 2

Tell the following or a similar story to your class. Choose one of your students to star in this story. Substitute the student's name for "Matthew" in the following dialogue.

T: Matthew wants to order scissors for the craft center. He looks at two catalogs and finds their prices for scissors. Both catalogs offer exactly the same scissors, but one catalog charges $1.20 for a package of five scissors and the other catalog charges $2.10 for a package of eight scissors.

Record this information on the board.

Catalog #1 5 scissors for $1.20
Catalog #2 8 scissors for $2.10

T: Which catalog offers the better deal? From which catalog should Matthew order the scissors?

Let the students think about this problem and discuss it collectively. Your students probably will not be able to figure out the price of one pair of scissors.

T: The first catalog sells the scissors in packages of five and the second catalog sells them in packages of eight. Could we buy the same number of scissors at each store? (Yes) What is the smallest such number? (Forty)

N 23

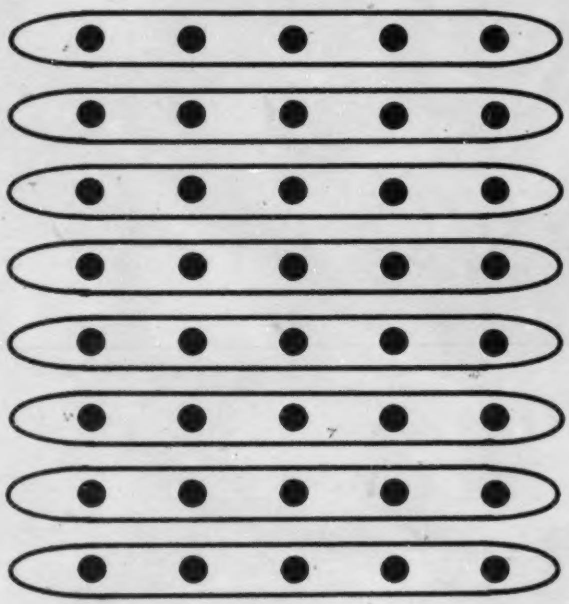
If necessary, encourage the students to think of numbers which are multiples of 5 and multiples of 8. You might even need to ask which quantities you could buy from the second catalog, for example ; 8, 16, 24, 32, . . . ; and then ask if you could buy any of those quantities from the first catalog.

T: To get forty scissors, how many packages would we order from the first catalog?

S: Eight packages of five scissors.

Draw this picture on the board.

Catalog #1 5 scissors for #1.20
Catalog #2 8 scissors for #2.10

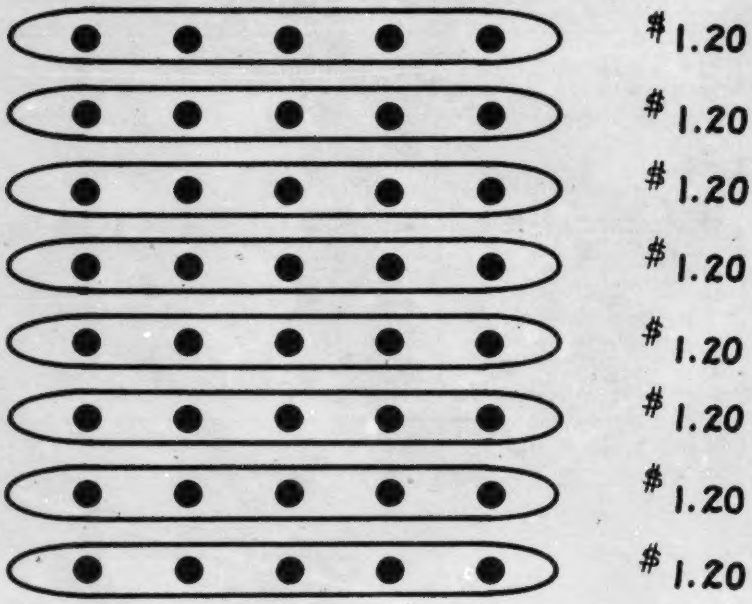


T: How much would forty scissors cost if we order them from the first catalog? How do you know?

S: $9.60. Five scissors cost $1.20, so I calculated "8 × $1.20". 8 × $1 = $8 and 8 × 20¢ = $1.60; $8 + $1.60 = $9.60.

Record this information on the board.

Catalog #1 5 scissors for #1.20 or 40 scissors for #9.60
Catalog #2 8 scissors for #2.10



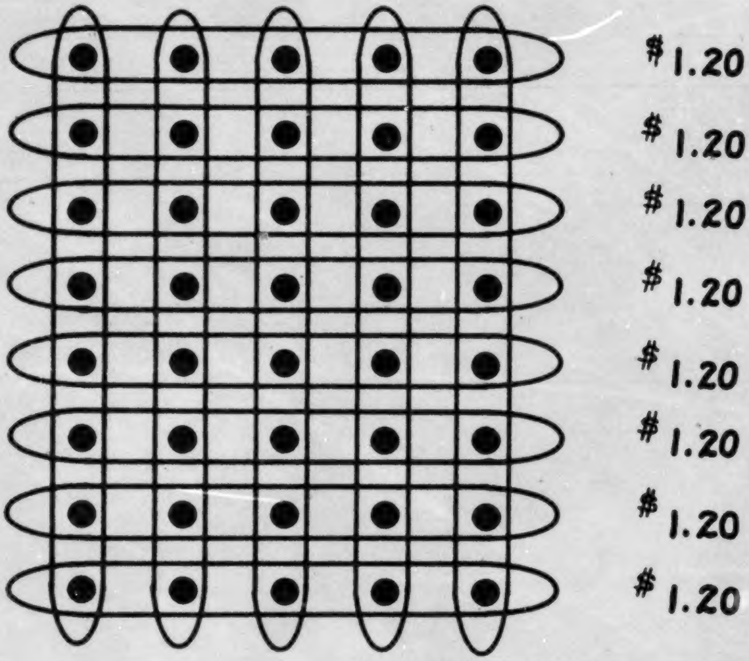
T: To get forty scissors, how many packages would we order from the second catalog?

S: Five packages of eight scissors.

N 23

Draw red strings around groups of eight dots.

Catalog #1 5 scissors for $1.20 or 40 scissors for $9.60
Catalog #2 8 scissors for $2.10

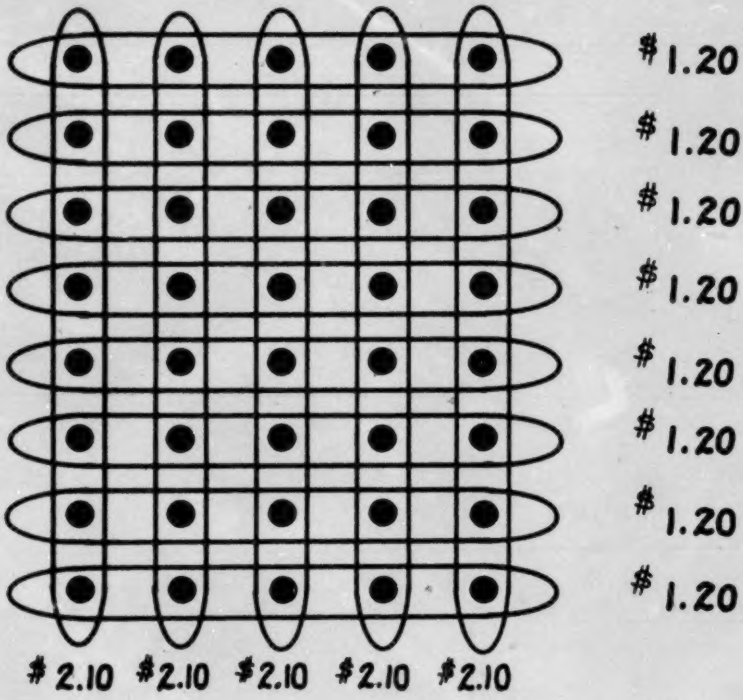


T: How much would forty scissors cost if we order them from the second catalog? How do you know?

S: $10.50. We would need to buy five packages of eight scissors, so I calculated "5 × $2.10". 5 × $2 = $10 and 5 × 10¢ = 50¢.

Record this information on the board.

Catalog #1 5 scissors for $1.20 or 40 scissors for $9.60
Catalog #2 8 scissors for $2.10 or 40 scissors for $10.50



T: Which catalog offers the better deal if we are buying forty pairs of scissors? (Catalog #1)

How much would we save if we order forty scissors from the first catalog instead of from the second catalog? (90¢)

From which catalog should Matthew order the scissors for the craft center? Why?

S: The first catalog, because its price (per pair of scissors) is lower.

T: When would you ever buy scissors from the second catalog?

S: . . . if I needed only eight scissors.

S: If I needed thirteen scissors, I would buy five from the first catalog and eight from the second.

Erase the board before going on to Exercise 3.

Exercise 3

Tell the following or a similar story to your class. Choose one of your students to star in this story. Substitute the student's name for "Rusty" in the following dialogue.

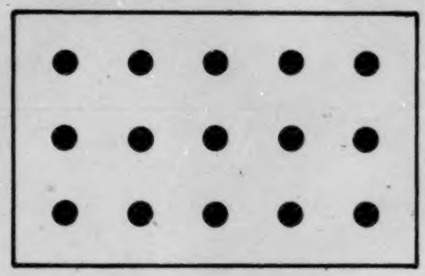
T: Rusty and his brother often go to see movies at the city library. The library sells cards good for three admissions and cards good for five admissions. Each time Rusty or his brother go to the library to see a movie, the librarian punches their card. Since they go so often, Rusty would like to know which card is a better deal. A three-punch card costs 80¢ and a five-punch card costs $1.30.

Record this information on the board.

3 admissions for $0.80
5 admissions for $1.30

T: Can you help Rusty? Try to solve this problem on your own.

Look at the students' work. After a few minutes, show the class the paper of a student who has drawn a 3 by 5 array of dots.



T: How many dots has Carine drawn on her paper? (Fifteen) How could these dots help us solve Rusty's problem?

S: We need to find out how much it costs for fifteen admissions with each type of card.

3 admissions for $0.80 15 admissions for ____
5 admissions for $1.30 15 admissions for ____

Allow the students to continue working on this problem for a few more minutes and then ask students to explain how they calculated the price of fifteen admissions with the two types of cards. Illustrate the students' calculations on the board with a dot picture. A possible dialogue is given here.

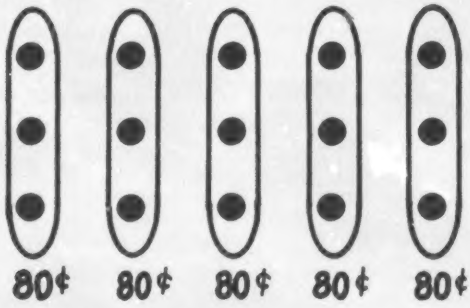
S: It costs $4.00 to buy fifteen admissions with the three-punch cards.

T: How do you know?

S: Five three-punch cards are good for fifteen admissions and 5 × $0.80 = $4.00.

Draw this dot picture on the board and complete the information for the three-punch cards.

3 admissions for $0.80 15 admissions for $4.00
5 admissions for $1.30 15 admissions for ____



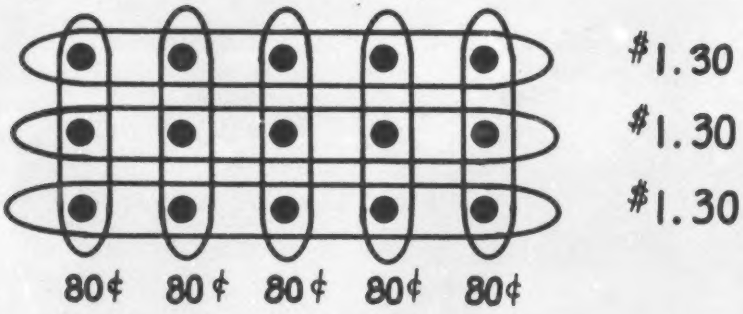
T: How much does it cost to buy fifteen admissions with the five-punch cards? How do you know?

S: $3.90, because three five-punch cards are good for fifteen admissions and 3 × $1.30 = $3.90.

Complete the dot picture and record the cost of fifteen admissions using five-punch cards.

3 admissions for $0.80
5 admissions for $1.30

15 admissions for $4.00
15 admissions for $3.90



T: Which card is the better deal?

S: The five-punch card.

T: When would you ever buy a three-punch card instead of a five-punch card?

S: . . . if I only wanted to go to the movies a couple of times.

S: . . . if I only had enough money for a three-punch card.

S: If I wanted to go to the movies six times, I would buy two three-punch cards.

Erase the board before going on to Exercise 4.

Exercise 4

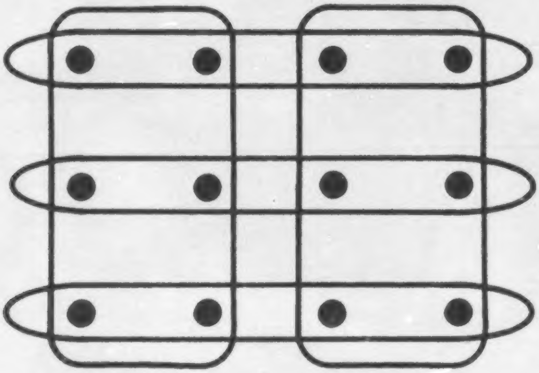
T: Carolyn frequently buys pencils for school. At her favorite store, they sell four pencils for 13¢ or six pencils for 20¢. Carolyn would like to know which is the better deal.

Write this information on the board.

4 pencils for 13¢
6 pencils for 20¢

T: Can you help Carolyn? Try to solve this problem on your own.

After a few minutes of independent work, ask the students what is the smallest number of pencils that Carolyn could buy at either price. Some students may suggest twenty-four, but others should observe that it is possible to buy twelve pencils at either price. Suggest to students who need help that they draw this dot picture on their papers.



NOTE: The students can also determine which is the better deal by calculating the cost of twenty-four (or any multiple of 12) pencils at each price.

Continue as in Exercise 3. Conclude that four pencils for 13¢ is the better deal.

N 24

N24 Composition of Functions #3

CAPSULE LESSON SUMMARY

With the support of the Minicomputer, decide that the composition $10\times$ followed by $\frac{1}{2}\times$ is $5\times$. Draw the corresponding arrow picture. Do several $5\times$ calculations using the method suggested by the arrow picture. In a more elaborate arrow picture with $10\times$ and $\frac{1}{2}\times$ arrows, label the dots and draw $5\times$ arrows.

MATERIALS

Teacher: Minicomputer set; colored chalk
Student : Worksheets N24*, **, ***, and ****; colored pencils

DESCRIPTION OF LESSON

Exercise 1

Begin the lesson with a short mental arithmetic activity involving the function $10\times$. Write these problems on the board, asking for and recording the answer to a problem before posing the next. The answers are in boxes.

$$10 \times 10 = \boxed{100} \qquad 10 \times 59 = \boxed{590}$$
$$10 \times 12 = \boxed{120} \qquad 10 \times 73 = \boxed{730}$$
$$10 \times 17 = \boxed{170} \qquad 10 \times 90 = \boxed{900}$$
$$10 \times 25 = \boxed{250} \qquad 10 \times 89 = \boxed{890}$$
$$10 \times 48 = \boxed{480} \qquad 10 \times 146 = \boxed{1{,}460}$$

Continue this activity with the following calculations. Allow several students to whisper their answers to you before completing the number story on the board. The answers are in boxes.

$$10 \times 203 = \boxed{2{,}030}$$
$$10 \times 300 = \boxed{3{,}000}$$
$$10 \times 465 = \boxed{4{,}650}$$
$$10 \times 1{,}697 = \boxed{16{,}970}$$

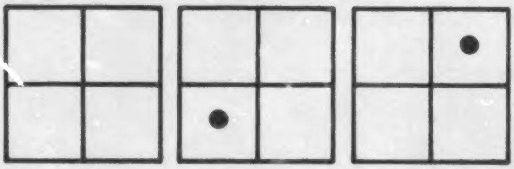
T: How did you calculate these answers so quickly? Do you know a short-cut?

S: When we multiply a number by 10, all we need to do is add a "0" to the right of the number.[*]

Erase the board before going on to Exercise 2.

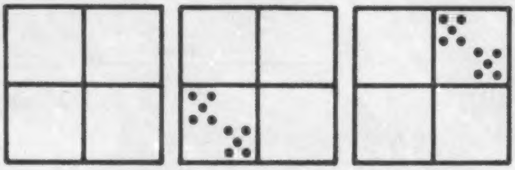
Exercise 2

Ask a student to put 24 on the Minicomputer.



T: How could we put 10 × 24 on the Minicomputer?

Accept correct suggestions and then put this configuration on the Minicomputer.



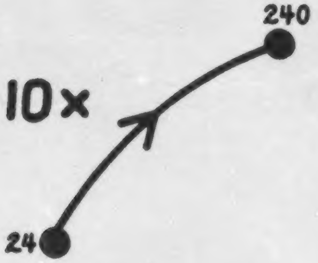
T: Is this number 10 × 24? (Yes)
What number is 10 × 24? (240)

---

[*] This is true only in the case of integers because $10 \times 0.37 \neq 0.370$ and $2.5 \times 10 \neq 2.50$.

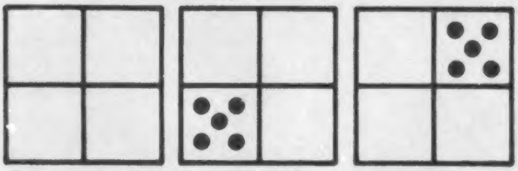
N 24

T: I'm going to show that we have multipled 24 by 10 in an arrow picture.

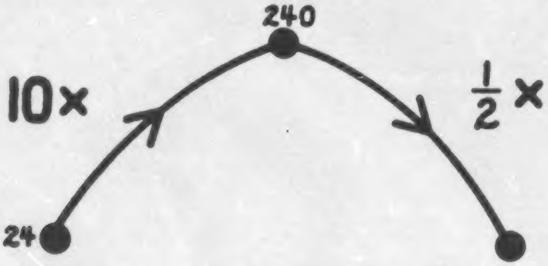


T: By removing some checkers from the Minicomputer, we could have one-half of the number. Which checkers should we remove?

S: Take off five of the checkers in each group of ten checkers.



Add a $\frac{1}{2}$x arrow to the picture.

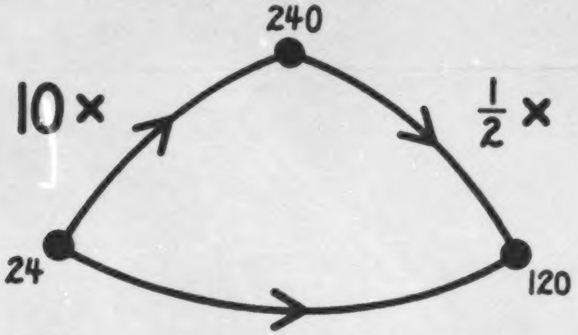


T: What number is on the Minicomputer? How do you know?

S: 120, because 5 x 20 = 100 and 5 x 4 = 20.

S: I know it's 120 because $\frac{1}{2}$ x 240 = 120.

Label the dot for 120 and draw a green arrow from 24 to 120.



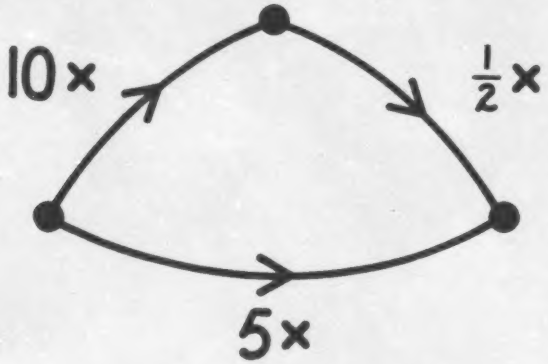
T: What could the green arrow be for? How do you know?

S: The green arrow could be for 5× because there are five 24's on the Minicomputer.

Label the green arrow "5×". Trace the appropriate arrows as you say:

T: A 10× arrow followed by a $\frac{1}{2}$× arrow is the same as a 5× arrow.

Erase the labels for the dots.



T (pointing to the starting dot of the green arrow): What number is 5 × 14? How do you know?

S: We could calculate "10 × 14" and then $\frac{1}{2}$ × that number.

N 24

T: What number is 10 × 14? (140)

So what number is 5 × 14? (70)

$10 \times 14 = 140$ and $\frac{1}{2} \times 140 = 70$, so $5 \times 14 = 70$.

T: What number is 5 × 26?

Take whispers from several students before asking a student to answer aloud.

T: How did you know that 5 × 26 = 130?

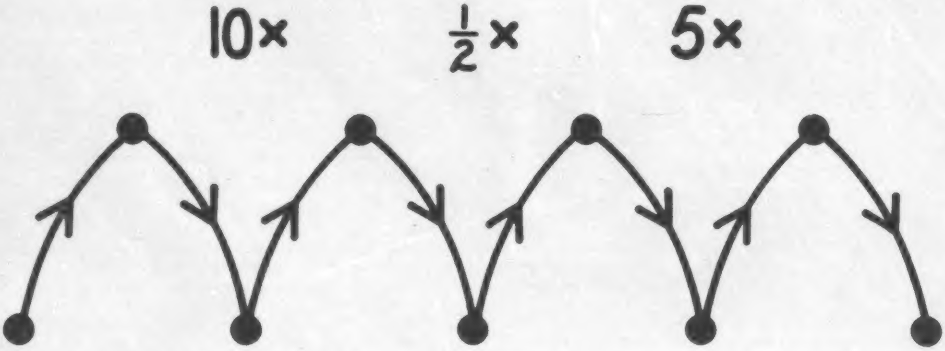
S: $10 \times 26 = 260$; $\frac{1}{2} \times 200 = 100$ and $\frac{1}{2} \times 60 = 30$; so $\frac{1}{2} \times 260 = 130$.

Continue this activity with these calculations.

5 × 12 (60)   5 × 42 (210)

5 × 45 (225)   5 × 81 (405)

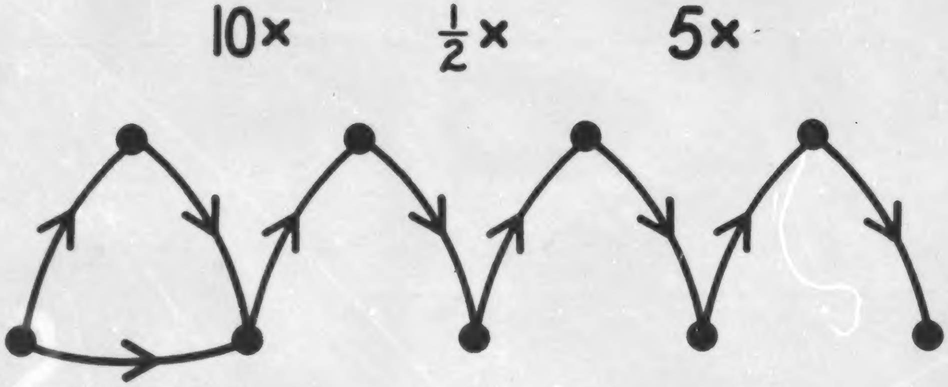
Erase the board and distribute copies of Worksheet N24*. Copy the arrow picture from the worksheet onto the board.



T: Where could we draw a green 5× arrow in this arrow picture?

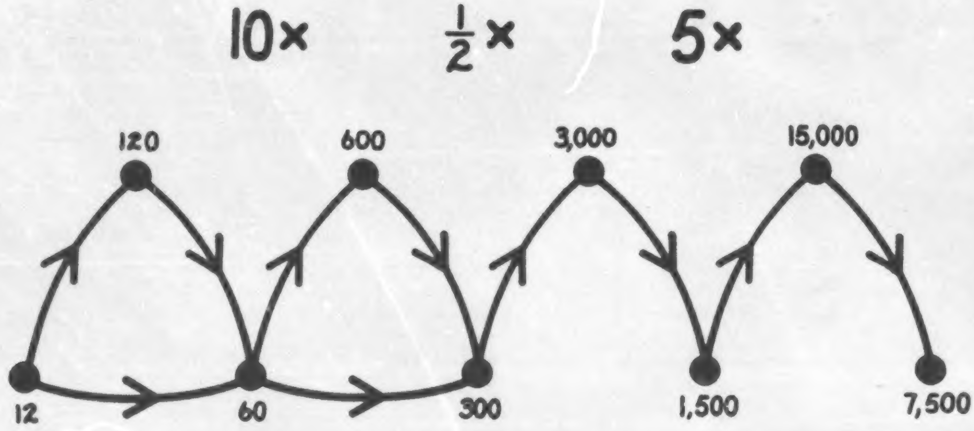
Invite a student to trace an arrow; if it is correct, let the student draw it in green. Encourage the students to draw the same 5× arrow on their worksheets.

Perhaps a student will draw this green arrow.



Call for another volunteer to draw a 5 × arrow in this arrow picture and ask the students to copy it on their worksheets. Label the starting dot 12; ask the students to label all the dots on their worksheets. Students who finish quickly should draw all the possible 5 × arrows.

After several minutes of individual work, invite students to label the dots in the arrow picture on the board and to explain how they calculated the answers. Continue until all the dots are labeled. Perhaps your arrow picture will look like this one.



N 24

T: Where could we draw some more 5 × arrows in this arrow picture?

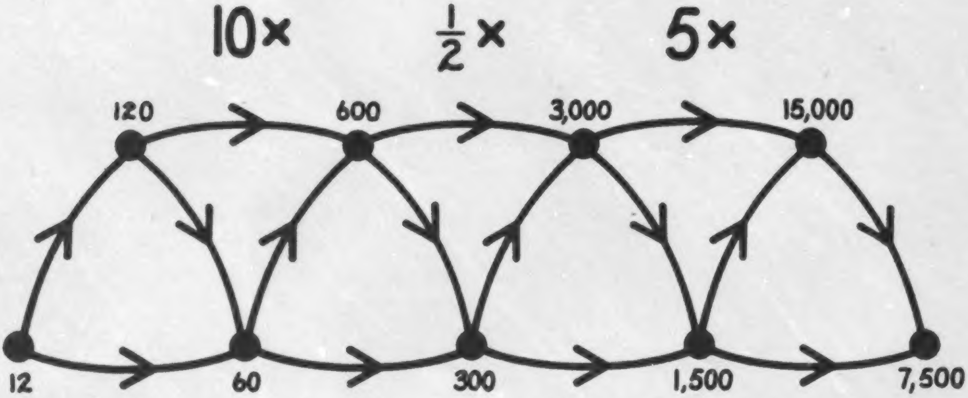
When a green arrow is drawn in the lower part of the picture, trace the appropriate arrows as you say:

T: A 10 × arrow followed by a $\frac{1}{2}$ × arrow is the same as a 5 × arrow.

When a green arrow is drawn in the upper part of the picture, trace the appropriate arrows as you say:

T: A $\frac{1}{2}$ × followed by a 10 × arrow is the same as a 5 × arrow.

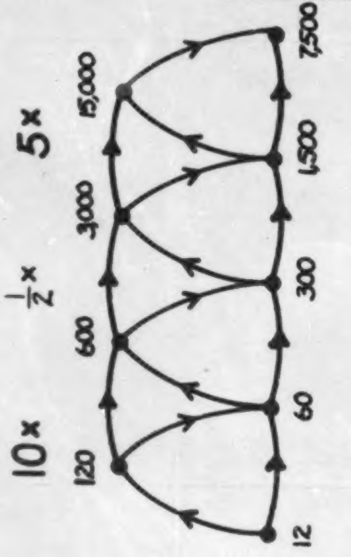
Continue until all the possible 5 × arrows have been drawn.



Worksheets N24**, ***, and **** are available for individual work.

N 24 *

Name

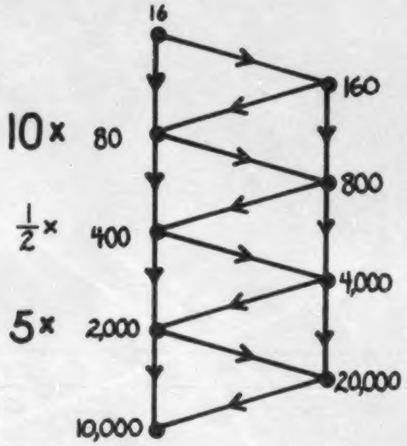

10x
$\frac{1}{2}$x
5x
120
600
3000
15,000
12
60
300
1,500
7,500

Name

N 24 **

Label the dots.

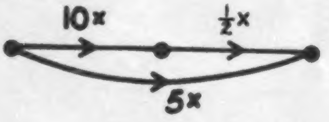
Draw all the possible 5 x arrows.


10x
$\frac{1}{2}$x
5x
16
160
80
800
400
4,000
2,000
20,000
10,000

Name

N 24 ***

Complete this table.

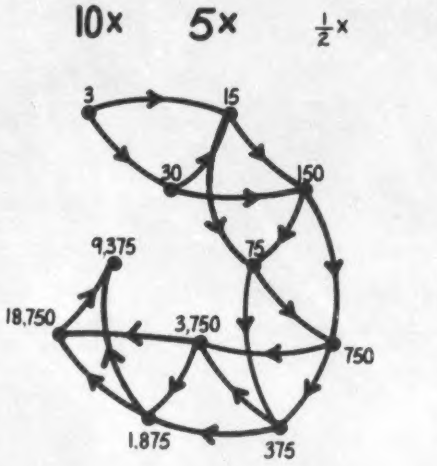

10x
$\frac{1}{2}$x
5x

Starting Number	10x	5x
25	250	125
82	820	410
41	410	205
63	630	315
85	850	425
94	940	470

Name

N 24 ****

Draw all the possible 10 x, 5 x, and $\frac{1}{2}$ x arrows.


10x
5x
$\frac{1}{2}$x
3
15
30
150
9,375
75
18,750
3,750
750
1,875
375

N 25

N25 Subtraction Algorithm #3

CAPSULE LESSON SUMMARY

Do some multiplication calculations on the Minicomputer using positive and negative checkers; for example,

$$3 \times 198 = 3 \times (200 - 2) = 3 \times (200 + \hat{2})$$

Label the dots in an arrow road using a subtraction algorithm to do the necessary calculations.

MATERIALS

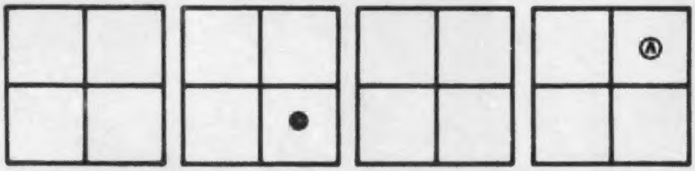
Teacher: Minicomputer set; colored chalk

Student: Paper; colored pencils; SUBTRACTION PROBLEMS booklet

DESCRIPTION OF LESSON

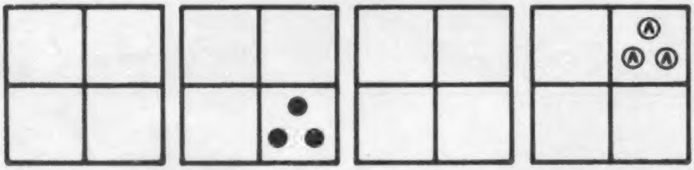
Exercise 1

Put this configuration on the Minicomputer.



T: What number is this? How do you know?

S: 96, because 100 - 4 = 96.



T: What number is 3 x 96? Write the answer on a piece of paper.

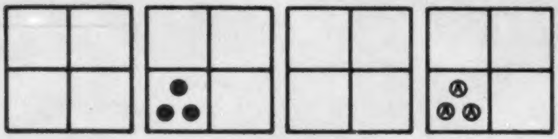
Look at many of the students' papers before asking a student to give and explain the answer.

S: 288. 96 is 4 less than 100, so I subtracted 12 from 300.

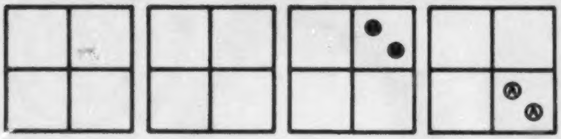
Write this subtraction problem on the board and ask a student to solve it using Nick's method.

$$\begin{array}{r} {}^{10\;\;10} \\ 300 \\ {}^{1\;\;1\;\;} \\ -\;\;12 \\ \hline 288 \end{array}$$

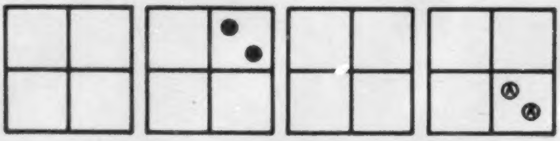
Continue this activity with the following configurations. Frequently ask several students to explain how they did a calculation.



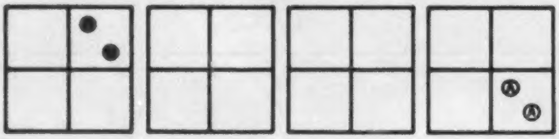
$= 3 \times 198 = 594$



$= 2 \times 39 = 78$

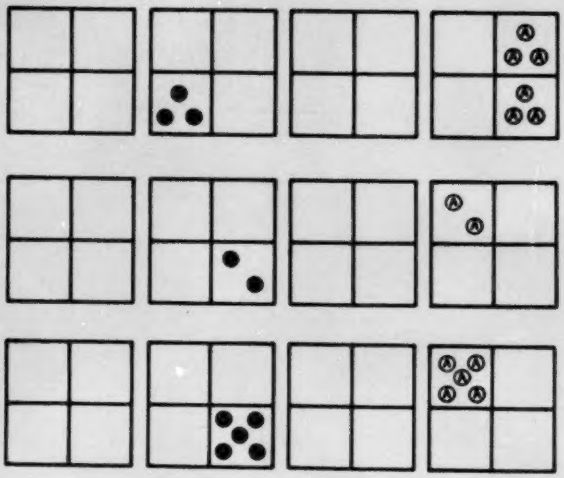


$= 2 \times 399 = 798$



$= 2 \times 3{,}999 = 7{,}998$

N 25



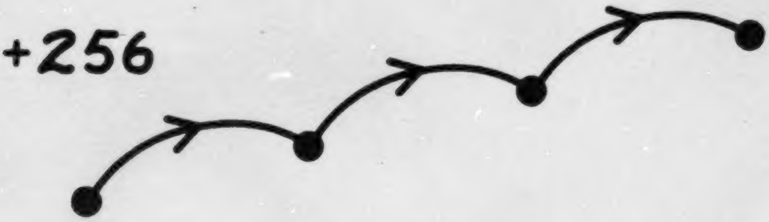
= 3 × 195 = 585

= 2 × 92 = 184

= 5 × 92 = 460

Exercise 2

Draw this arrow picture on the board and ask the students to copy it on their papers.



T: Where is the largest number in this arrow picture?

The students should point to the ending dot of the road.

T: Label that dot "2,662" on your papers.

Ask a student who labeled the dot correctly to label the dot on the board.

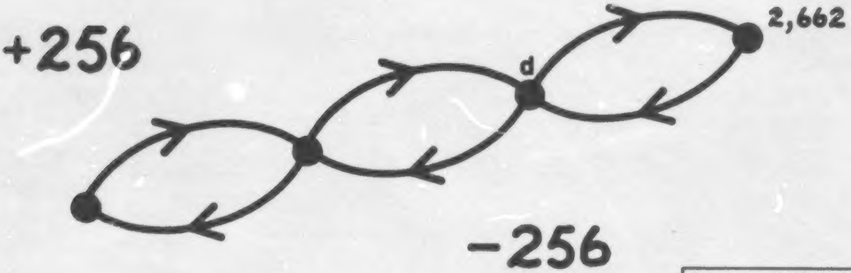
T: I would like to label the other dots in this arrow picture, but the arrows are going in the wrong direction. What could I do?

S: Draw the return arrows.

Ask the students to draw the return arrows on their papers and invite one student to draw the return arrows on the board.

T: What are the return arrows for?

S: - 256.



Do not write the letter on the board. It is here just to make the description of the lesson easier to follow.

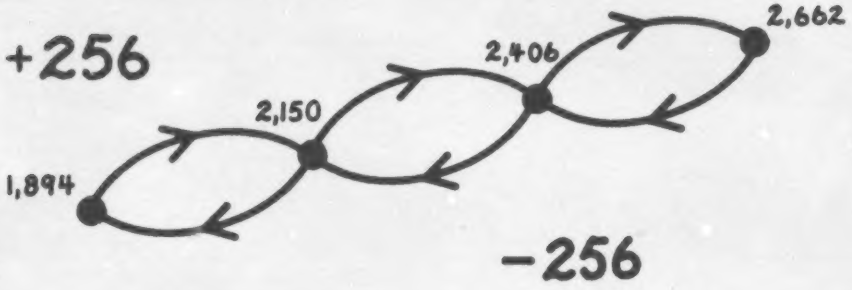
T (pointing to d): What calculation do we need to do before we can label this dot?

S: "2,662 - 256".

Ask the students to do the calculation on their papers and help students who have difficulty. Then ask a student to write the problem on the board, to solve it explaining each step, and finally to label the dot.

N 25

Continue this activity until all of the dots are labeled.



Your students will have done the following calculations in order to label the dots.

$$\begin{array}{r} 2{,}66\overset{10}{2} \\ -\ \ 2\overset{1}{5}6 \\ \hline 2{,}406 \end{array} \qquad \begin{array}{r} 2{,}40\overset{10}{6} \\ -\ \ 2\overset{1}{5}6 \\ \hline 2{,}150 \end{array} \qquad \begin{array}{r} 2{,}\overset{10}{1}\overset{10}{5}\overset{10}{0} \\ -\ {}^{1}\ \overset{1}{2}\overset{1}{5}6 \\ \hline 1{,}894 \end{array}$$

When calculating "2,150 - 256", emphasize that ten hundreds is the same as one thousand.

Distribute the SUBTRACTION PROBLEMS booklets and allow 10 to 15 minutes for individual work. Encourage the students to correct any errors that they made in previous weeks. After the lesson, collect the booklets, check them, and have them ready for Lesson N32, the last lesson in UPG-IV on the subtraction algorithm.

N26 Roads #3

CAPSULE LESSON SUMMARY

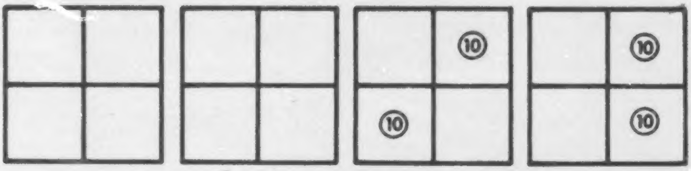
Calculate 10 x 65 on the Minicomputer using ⑩-checkers as a review before individually doing some 10 x calculations. Build a road between 5 and 29 using +10, +1, and -1 arrows and using fewer than ten arrows.

MATERIALS

Teacher:	Minicomputer set; ⑩-checkers; colored chalk
Student:	Paper; pencil; colored pencils; Worksheets N26*, **, ***, and ****

DESCRIPTION OF LESSON

Put this configuration on the Minicomputer.



T: What calculation do these checkers suggest?

S: "10 x 65".

T: What number is 10 x 65?

S: 650.

T: How do you know?

S: Whenever you multiply a number by 10, the answer ends in zero.[*]

---

[*] This comment is true for the integers; however, it is not true in general. For example, 10 x 3.8 = 38.

S: All you need to do is put a "0" on the end of "65".

T: Who could write the number story on the board?

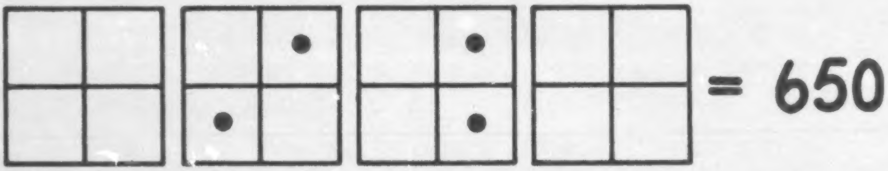
$$10 \times 65 = 650$$

T: This is another way to write this number story.

$$\begin{array}{r} 65 \\ \times\, 10 \\ \hline 650 \end{array}$$

NOTE: Both of these number stories should be read, "Ten times sixty-five equals six hundred fifty."

Invite a student to make plays on the Minicomputer to put the number in its standard configuration.



T: There aren't any checkers on the ones' board, so the last digit in this number is 0.

Continue this activity with these calculations.

10 x 93 (930)

10 x 71 (710)

10 x 208 (2,080)

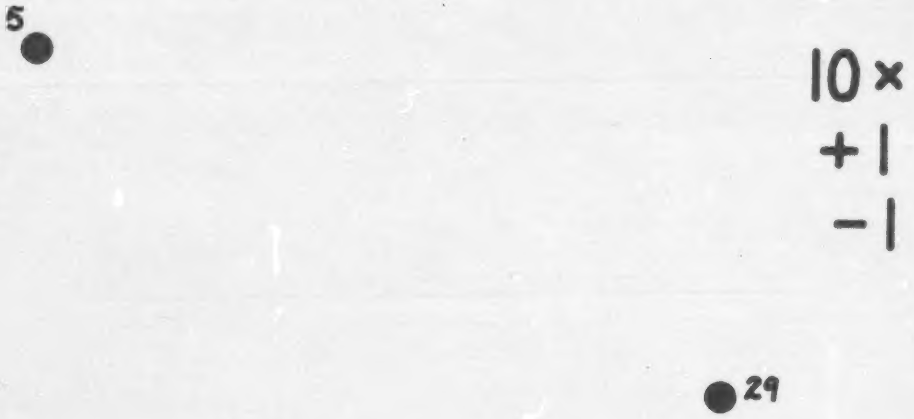
10 x 280 (2,800)

Write the following calculations on the board and ask the students to solve them.

10 × 35	10 × 24
10 × 305	10 × 204
10 × 350	10 × 240
10 × 300	10 × 200
10 × 3,005	10 × 2,004
10 × 3,000	10 × 2,400

When a few students have solved all the problems, begin letting students write the answers on the board and read the completed number stories. Emphasize that the commas make the numbers easier to read and that we put a comma between the thousands' digit and the hundreds' digit when we write a number.

Erase the board and ask the students to copy on their papers what you draw on the board.



T: Build a red-blue-green road between 5 and 29, but try to use less than ten arrows.

N 26

Walk around the room and look at the students' work. Students who finish quickly may begin working on Worksheets N26* and **. Students who build a road with ten or more arrows should try again. After five to ten minutes, ask the students to stop working. Point to the picture on the board.

T: Which multiple of 10 is closest to 29?

S: 30.

Draw a dot for 30 near the dot for 29.

T: Will it be easy for us to build a road between 30 and 29?

S: Yes, all we need is one green arrow because 30 - 1 = 29.

T: How can we build a road between 5 and 30?

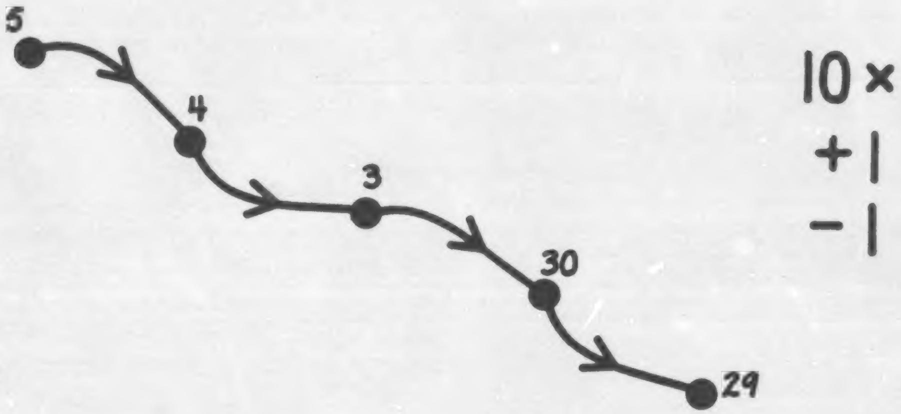
S: We need to go from 5 to 3 and then from 3 to 30.

T: Which arrows should we use?

S: Start with two green arrows and then use one red arrow. 5 - 1 = 4, 4 - 1 = 3, and 10 x 3 = 30.

Do not draw any arrows on the board at this time, but help any students who are having trouble getting started by telling them to draw a dot for 30 on their papers and to follow the suggestions that were given by the other students. When most of the students have solved this problem, ask one of them to put the solution on the board.

Here is one possible solution.



Worksheets N26*, **, ***, and **** are available for individual work. Emphasize to students that they are to use less than ten arrows to build each road.

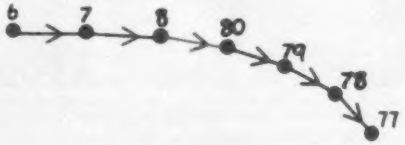
If a student has difficulty with the * or ** problems, you may wish to give a hint. Ask which multiple of 10 is closest to the larger of the two numbers in the problem and then draw a dot for that number on the student's worksheet. Encourage the student to try to build a road between the two numbers through that number.

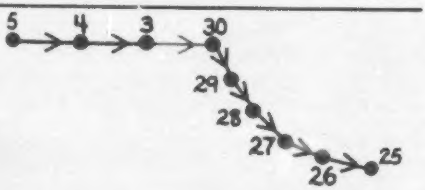
If many of the students are having difficulty with the same problem, discuss that problem collectively.

Name ______________ **N 26 ***

10× +1 −1

Build a red-blue-green road between each pair of numbers. Use less than ten arrows to build each road.


6
7
8
80
79
78
77

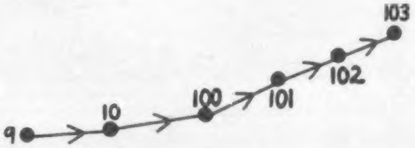

5
4
3
30
29
28
27
26
25

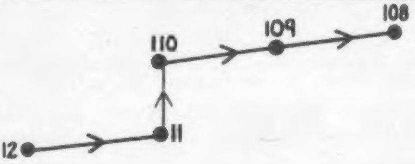
Other solutions are possible; however, these are shortest roads.

Name ______________ **N 26 ****

10× +1 −1

Build a red-blue-green road between each pair of numbers. Use less than ten arrows to build each road.


103
102
101
100
10
9

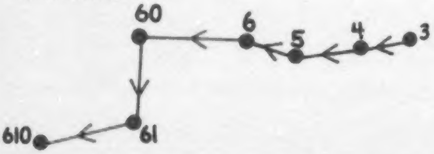

108
109
110
11
12

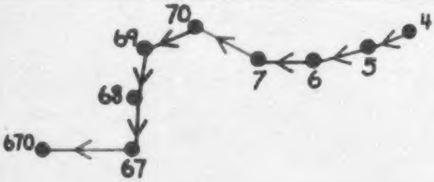
Other solutions are possible; however, these are shortest roads.

Name ______________ **N26 *****

10× +1 −1

Build a red-blue-green road between each pair of numbers. Use less than ten arrows to build each road.


60
6
5
4
3
610
61

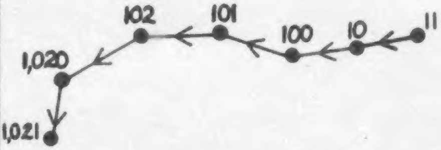

70
69
68
670
67
7
6
5
4

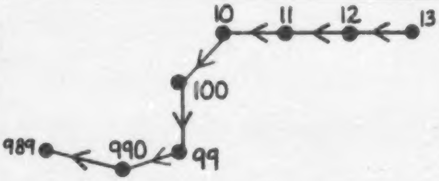
Other solutions are possible; however, these are shortest roads.

Name ______________ **N26 ******

10× +1 −1

Build a red-blue-green road between each pair of numbers. Use less than ten arrows to build each road.


102
101
100
10
11
1,020
1,021


10
11
12
13
100
989
990
99

Other solutions are possible; however, these are shortest roads.

N27 Mystery Arrows

CAPSULE LESSON SUMMARY

Do some multiplication calculations on the Minicomputer using positive and negative checkers; for example,

$$2 \times 38 = 2 \times (40 - 2) = 2 \times (40 + \hat{2}) = 80 + \hat{4} = 76$$

Find possible labels for the two arrows in an arrow road given its starting and ending numbers.

MATERIALS

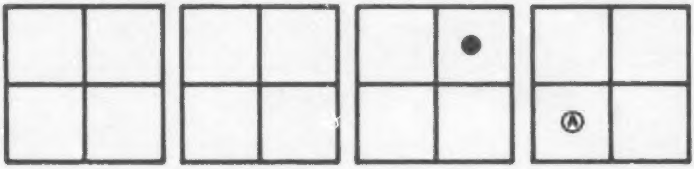
Teacher: Minicomputer set; colored chalk

Student: Paper; colored pencils

DESCRIPTION OF LESSON

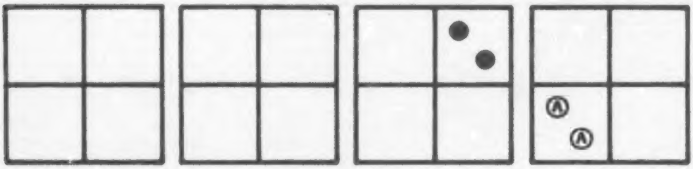
Exercise 1

Put this configuration on the Minicomputer.



T: What is this number? How do you know?

S: 38, because 40 - 2 = 38.



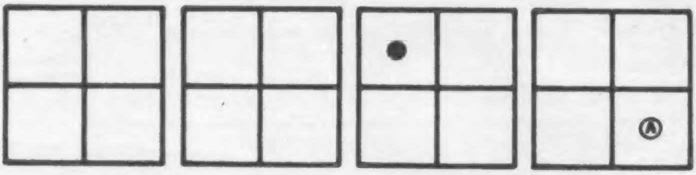
T: What number is 2 x 38?

S: 76.

T: How did you know the number so quickly?

S: I calculated "80 - 4 = 76."

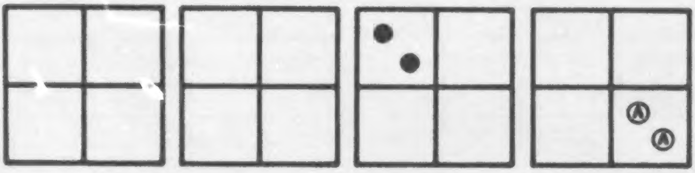
Put this configuration on the Minicomputer.



T: What number is this? How do you know?

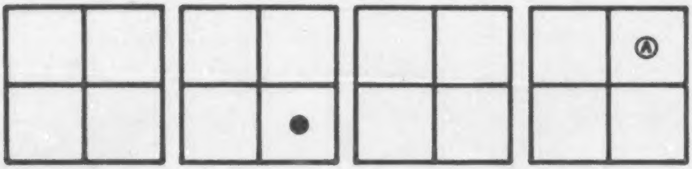
S: 79, 80 - 1 = 79.

Double the number on the Minicomputer.



T: What number is 2 x 80 ? (160)
160 - 2 = 158, so 2 x 79 = 158.

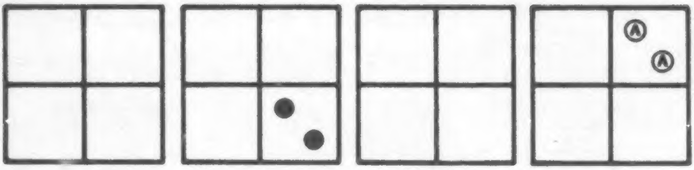
Put this configuration on the Minicomputer.



T: What number is this? How do you know?

S: 96, 100 - 4 = 96.

Double the number on the Minicomputer.



T: What number is 2 x 96?

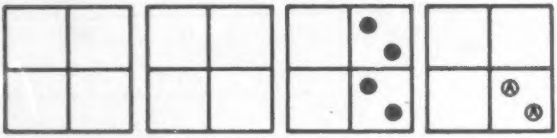
Allow several students to whisper their answers to you before asking someone to answer aloud.

S: 192.

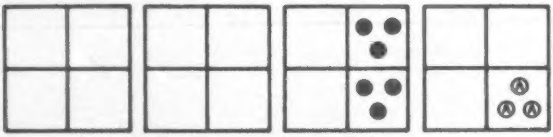
T: How did you do the calculation?

S: I calculated "200 - 4 = 196" and "196 - 4 = 192."

Continue this activity with these configurations.

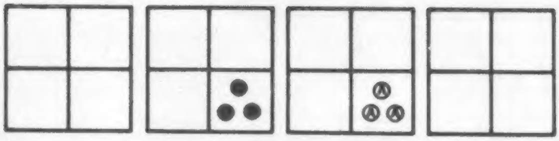


= 2 × 49 = 98

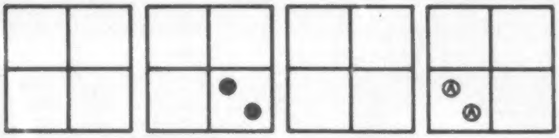


= 3 × 49 = 147

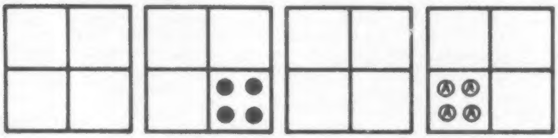
N 27



= $3 \times 90 = 270$



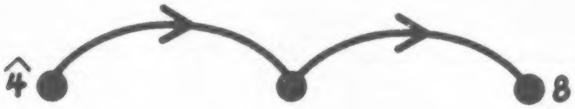
= $2 \times 98 = 196$

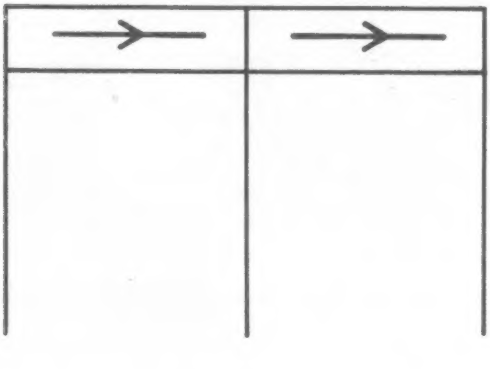


= $4 \times 98 = 392$

Exercise 2

T: Copy the picture that I draw on the board.





Ask the students to find possible labels for the arrows. Walk around the room and look at students' solutions. After about five minutes, call the class back together to share their findings. Record appropriate addition, subtraction, multiplication, and division relations in the table. If students correctly label the arrows with relations such as "is less than", "is greater than", "is a multiple of", and "is a divisor of", accept these as correct suggestions but do not record them in the table. A sample dialogue follows.

T: In this arrow picture, what could the blue arrow and the red arrow be for?

S: The blue arrow could be for +4.

T: If the blue arrow were for +4, what could the red arrow be for?

S: +8.

T: $\hat{4} + 4 = 0$ and $0 + 8 = 8$.

→	→
+4	+8

T: Is this the only solution? (No)

S: The blue arrow could be for +8.

T: If the blue arrow were for +8, what could the red arrow be for?

S: +4.

N 27

T: $\hat{4} + 8 = 4$ and $4 + 4 = 8$.

If the blue arrow were for + 8, what else could the red arrow be for?

S: 2 x.

T: $\hat{4} + 8 = 4$ and $2 \times 4 = 8$.

→	→
+4	+8
+8	+4 or 2×

T (looking at Melinda's paper): Melinda says the blue arrow could be for +6.

In this case, what could the red arrow be for? (+6)

$\hat{4} + 6 =$ . . . ? (2)

And $2 + 6 =$ . . . ? (8)

Is there anything else the red arrow could be for? (4 x)

→	→
+4	+8
+8	+4 or 2×
+6	+6 or 4×

T (looking at Jack's paper): Jack says the blue arrow could be for + 20. What number is $\hat{4} + 20$? (16)

In this case, what could the red arrow be for? (- 8)

$16 - 8 = 8$.

S: The red arrow could also be for $\frac{1}{2} \times$ because $\frac{1}{2} \times 16 = 8$.

→	→
+4	+8
+8	+4 or 2×
+6	+6 or 4×
+20	−8 or $\frac{1}{2}$×

T (looking at Sheila's paper): Sheila thinks the red arrow could be for +2. Is this possible? What would the blue arrow be for?

S: The middle dot would be for 6, so the blue arrow could be for + 10.

→	→
+4	+8
+8	+4 or 2×
+6	+6 or 4×
+20	−8 or $\frac{1}{2}$×
+10	+2

T: Anthony is "stuck." Let's see if we can help him. Anthony's blue arrow is for +30, but he can't figure out what the red arrow is for. Do you know? Write the solution on your paper.

Look at several of the students papers before continuing.

T: What number is $\hat{4} + 30$? (26)
Now we need to go from 26 to 8.

S: We could subtract 18.

$$\begin{array}{r} 26 \\ -18 \\ \hline \end{array}$$

T: Who can use Nick s method to solve this problem?

$$\begin{array}{r} \overset{10}{} \\ 2\,6 \\ -\overset{1}{1}\,8 \\ \hline 8 \end{array}$$

→	⇢
+4	+8
+8	+4 or 2×
+6	+6 or 4×
+20	−8 or $\frac{1}{2}$×
+10	+2
+30	−18

NOTE: Students might suggest subtraction or multiplication relations for the blue arrow, although it is unlikely since subtracting or multiplying 4 by whole numbers gives numbers smaller than 4 and they are trying to get to 8. Students are unlikely to suggest division relations simply because they have no experience with division involving a negative number.

N28 Division Problems #3

CAPSULE LESSON SUMMARY

Solve division problems suggested by story situations. Use string pictures to find and record the solutions.

MATERIALS

Teacher: Colored chalk
Student: Paper; pencil

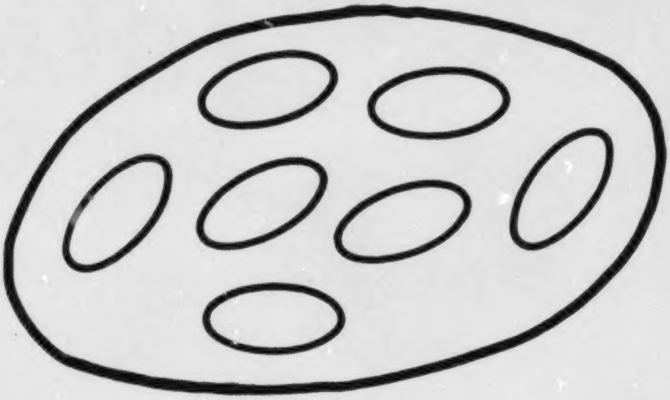
DESCRIPTION OF LESSON

Exercise 1

Draw a large string on the board.

T: This string is for one hundred tulips which are to be shared equally among seven people.

Draw seven smaller strings inside the large string.



T: Each of these blue strings is for one person's share of the tulips. How many tulips should each person get?

Let the students think for awhile. A possible dialogue is given here.

N 28

S: Each person should get at least ten tulips.

Write "10" inside each of the blue strings.

T: How many tulips are left?

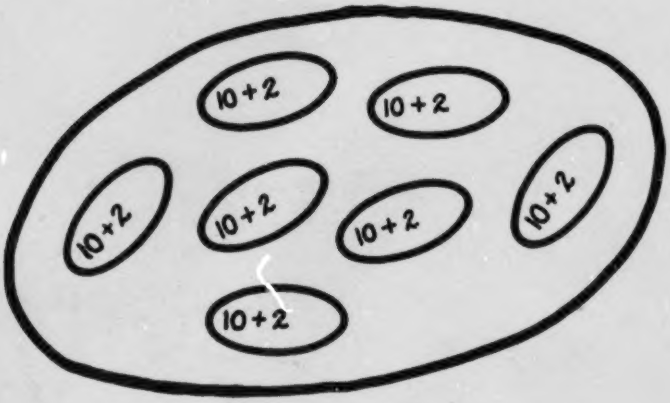
S: Thirty.

S: Give five more tulips to each person.

S: 7 x 5 = 35, so five is too much.

S: Give two more tulips to each person.

Write "+2" inside each of the blue strings as illustrated below.



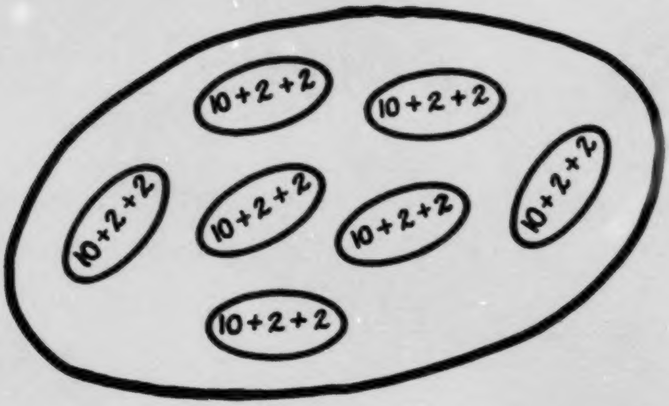
T: Now each person has twelve tulips. How many tulips are left?

S: Sixteen.

You may need to remind the students that there were thirty tulips left before and that fourteen tulips were just distributed.

S: Give each person two more tulips. Then there will be two tulips left over.

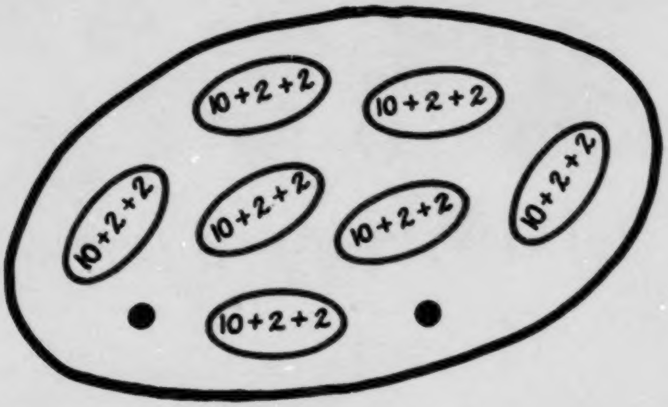
Write another "+ 2" inside each of the blue strings as illustrated below.


10 + 2 + 2
10 + 2 + 2
10 + 2 + 2
10 + 2 + 2
10 + 2 + 2
10 + 2 + 2
10 + 2 + 2

S: We could cut up the two extra tulips.

S: That's no good. No one wants part of a flower.

T: There are two extra tulips. I'll draw dots for them in the string picture.


10 + 2 + 2
10 + 2 + 2
10 + 2 + 2
10 + 2 + 2
10 + 2 + 2
10 + 2 + 2
10 + 2 + 2

T: How many tulips does each person get?

S: Fourteen.

T: Who can write a number story about this problem?

N 28

Perhaps a student will write this number story on the board.

$$7 \times 14 = 98$$

T: This number story doesn't tell everything. There are one hundred tulips.

You may need to suggest this number story.

$$(7 \times 14) + 2 = 100$$

Continue letting students write number stories on the board.

$$\begin{array}{r} 70 \\ 14 \\ 14 \\ +\ 2 \\ \hline 100 \end{array}$$

$$7 \times (10 + 2 + 2) + 2 = 100$$

$$100 - (7 \times 14) = 2$$

$$(7 \times 10) + (7 \times 4) + 2 = 100$$

$$\begin{array}{r} 100 \\ -\ 70 \\ \hline 30 \\ -\ 28 \\ \hline 2 \end{array}$$

T: We could also record the problem and its answer like this.

$$\begin{array}{r} 14 \\ 7\overline{)100} \end{array} \ \text{Remainder} = 2$$

T: 100 divided by 7 is 14 with a remainder of 2. "Remainder" is a word we use to show what is left over. We usually abbreviate the word "Remainder" by just writing "R".

$$\begin{array}{l} \quad\;\; 14 \;\text{R}=2 \\ 7\overline{)100} \end{array}$$

Erase the board before going on to Exercise 2.

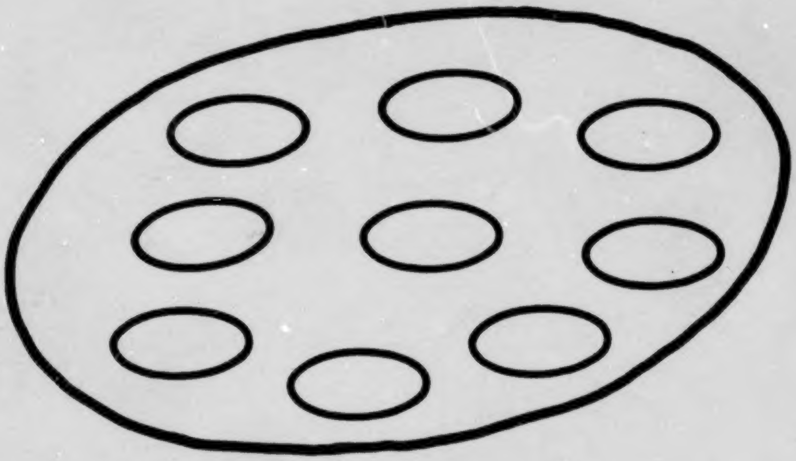
Exercise 2

Draw a large string on the board.

T: This string is for two hundred baseball cards which are to be shared equally among nine children. How many small strings should we draw?

S: Nine.

Ask the students to draw this string picture on their papers and to try to determine how many baseball cards each of the children should receive.



N 28

Look at the students' work. When all the students have at least copied the string picture, interrupt them for a brief discussion of the problem.

T: We are trying to find out how many baseball cards each of the nine children should receive. What are your suggestions?

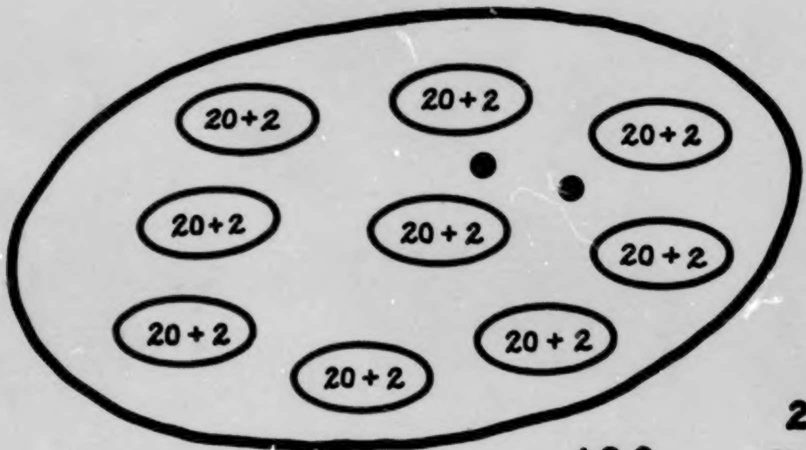
S: 9 x 20 = 180, so let's begin by giving each child twenty baseball cards.

Indicate that each child has twenty baseball cards in the string picture by writing "20" inside each of the blue strings.

S: Now there are twenty baseball cards left.

T: Try to finish this problem on your own. When you have solved the problem, write a number story about it.

The students who finish quickly may be given the problem of sharing two hundred cookies among seven children while the other students continue to work on the original problem. After a few minutes, ask students to complete the string picture on the board and to write their number stories on the board.



$$(9 \times 22) + 2 = 200$$

$$200 - (9 \times 22) = 2$$

$$\begin{array}{r} 180 \\ 18 \\ +\ 2 \\ \hline 200 \end{array} \qquad \begin{array}{r} 22 \\ \times\ 9 \\ \hline 198 \\ +\ 2 \\ \hline 200 \end{array}$$

Under the number stories write this problem as you ask:

$$9\overline{)200}$$

T: 200 divided by 9 is . . . ?

S: 22 with a remainder of 2.

Ask a student to record the answer.

$$\begin{array}{l} \;\;\;\;22 \;\; R=2 \\ 9\overline{)200} \end{array}$$

Conclude that each of the nine children should receive twenty-two baseball cards and that there will be two cards left over.

Erase the board and then write these three problems. Ask the students to solve as many of them as they can.

Share 100 marbles among 9 children.
Share 200 cookies among 7 children.
Share 300 walnuts among 8 children.

N29 Multiplication Problems

CAPSULE LESSON SUMMARY

Put three checkers on the 4-square, then move the checkers to the 40-square, to the 400-square, and to the 4,000-square, each time asking which number is on the Minicomputer and recording the appropriate number story. Repeat the activity with other configurations. Multiply 3 x 148 by putting on 148 three times and figuring out the number on each board and also by computing 3 x (150 - 2). Multiply 2 x 247 by doubling the ones, the tens, and the hundreds in 247. Record the solution like this:

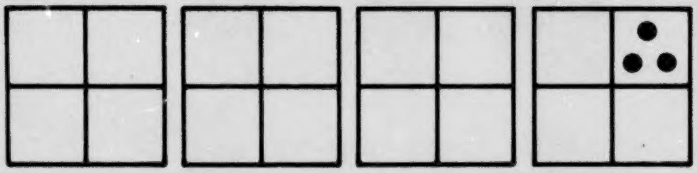
$$\begin{array}{r} 247 \\ \underline{\times\ 2} \\ 14 \\ 80 \\ \underline{400} \\ 494 \end{array}$$

MATERIALS

Teacher: Minicomputer set
Student: Paper; pencil

DESCRIPTION OF LESSON

Put this configuration on the Minicomputer.



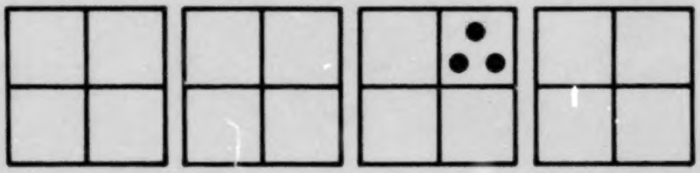
T: What calculation is suggested by these checkers?

S: "3 x 4".

T: What number is 3 x 4?

S: 12.

Write "3 x 4 = 12" on the board. Move the checkers to the 40-square.



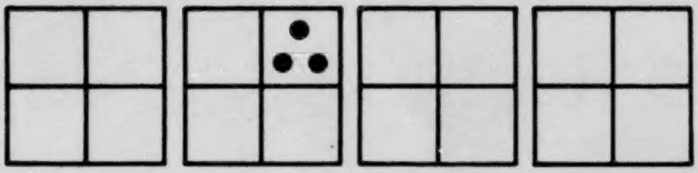
T: What calculation is suggested by the checkers on the Minicomputer now?

S: "3 x 40".

T: What number is 3 x 40?

S: 120.

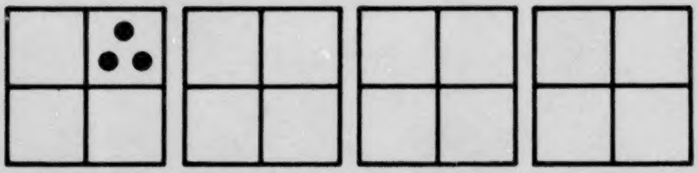
Write "3 x 40 = 120" on the board. Move the checkers to the 400-square.



T: And now . . . ?

S: "3 x 400"; 3 x 400 = 1,200.

Write "3 x 400 = 1,200" on the board. Move the checkers to the 4,000-square.



T: And now . . . ?

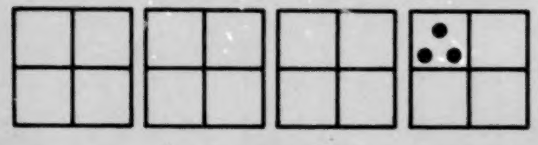
S: "3 x 4,000"; 3 x 4,000 = 12,000.

$$\begin{array}{rcrcr} 3 & \times & 4 & = & 12 \\ 3 & \times & 40 & = & 120 \\ 3 & \times & 400 & = & 1,200 \\ 3 & \times & 4,000 & = & 12,000 \end{array}$$

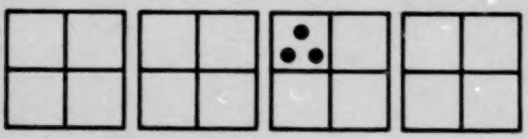
T: What patterns do you see in this sequence of number stories?

Let students comment freely. Someone very likely will comment on the number of zeros on each side of the equal sign.

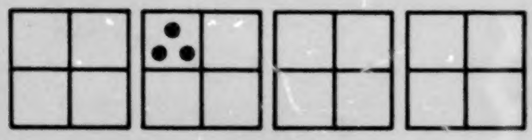
Erase the board and repeat the activity with these configurations.



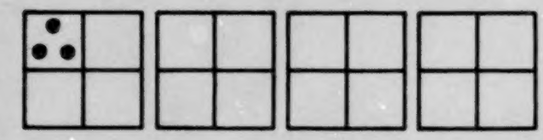
$= 3 \times 8 = 24$



$= 3 \times 80 = 240$



$= 3 \times 800 = 2,400$



$= 3 \times 8,000 = 24,000$

If your students are unsure about any of these numbers, suggest that they do the corresponding addition problem on their papers. For example, if your students are not convinced that 3 x 800 = 2,400, ask them to do this addition calculation.

$$\begin{array}{r} 800 \\ 800 \\ \underline{+800} \end{array}$$

Write this problem on the board.

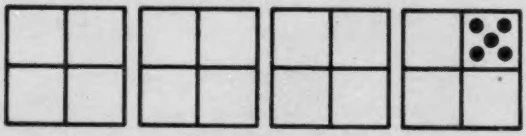
$$3 \times 8{,}000{,}000$$

T: What number is 3 x 8,000,000?

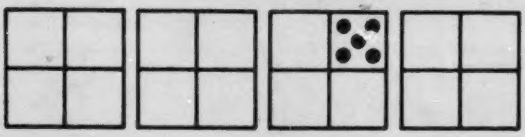
Invite a student to record the answer and to read it aloud.

$$3 \times 8{,}000{,}000 = 24{,}000{,}000$$

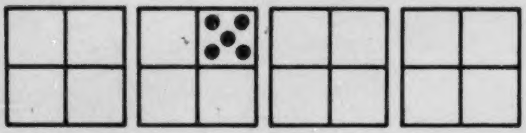
Erase the board and repeat the activity with these configurations.



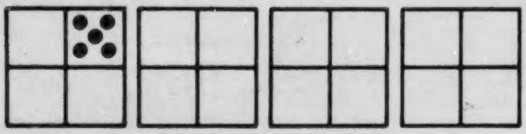
$= 5 \times 4 = 20$



$= 5 \times 40 = 200$



$= 5 \times 400 = 2{,}000$



$= 5 \times 4{,}000 = 20{,}000$

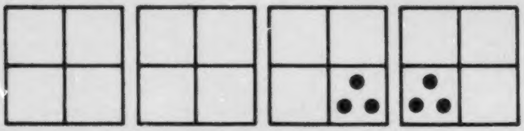
N 29

T: What number is 5 x 4,000,000?

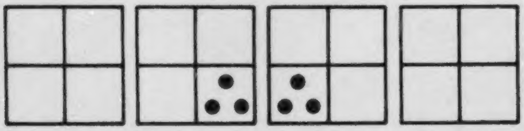
Write "5 x 4,000,000" on the board and ask a student to record the answer and to read it aloud.

$$5 \times 4{,}000{,}000 = 20{,}000{,}000$$

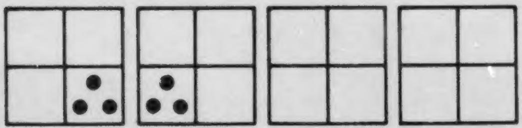
Erase the board and repeat the activity with these configurations.



$= 3 \times 12 = 36$



$= 3 \times 120 = 360$



$= 3 \times 1{,}200 = 3{,}600$

You should have this sequence of number stories on the board.

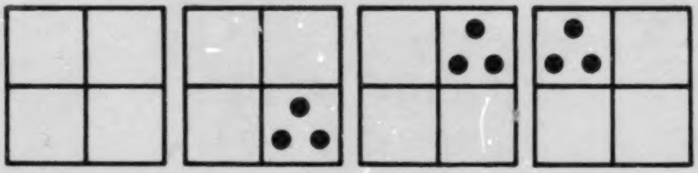
$$\begin{aligned} 3 \times 12 &= 36 \\ 3 \times 120 &= 360 \\ 3 \times 1{,}200 &= 3{,}600 \end{aligned}$$

Continue this sequence with these calculations. Answers are in the boxes.

$$3 \times 12{,}000 = \boxed{36{,}000}$$

$$3 \times 12{,}000{,}000 = \boxed{36{,}000{,}000}$$

Erase the board and put this configuration on the Minicomputer.



T: What calculation is suggested by these checkers? ("3 x 148")
What number is this? Write it on your paper.

Walk around the room and look at the students' papers. Then discuss the problem.

T: Which number is on the hundreds' board? How do you know?

S: 300, 3 x 100 = 300.

Record "3 x 100 = 300" on the board.

T: What number is on the tens' board. How do you know?

S: 120, because 40 + 40 = 80 and 80 + 40 = 120.

Record "3 x 40 = 120" on the board.

T: What number is on the ones' board? How do you know?

S: 24, because 3 x 8 = 24.

Record "3 x 8 = 24" on the board.

$$\begin{aligned} 3 \times 100 &= 300 \\ 3 \times 40 &= 120 \\ 3 \times 8 &= 24 \end{aligned}$$

N 29

T: What number is 3 x 148? How do you know?

S: It's 444, because 300 + 120 + 24 = 444.

$$\begin{array}{r} 3\times 100 = 300 \\ 3\times 40 = 120 \\ 3\times 8 = 24 \\ \hline 3\times 148 = 444 \end{array}$$

T: Does anyone see another way to calculate "3 x 148"?

Encourage the students to explain their calculations. If no one suggests calculating "3 x (150 - 2)", suggest this method yourself.

T: 148 is almost 150.

Write this number story on the board.

$$148 = 150 - ?$$

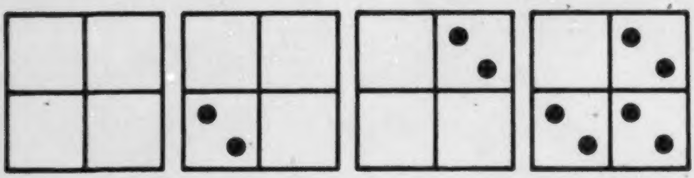
T: Who can complete this number story for us?

$$148 = 150 - 2$$

T: How does this help us to calculate "3 x 148"?

S: 3 x 150 = 450 and 450 - 6 = 444.

Erase the board and put this configuration on the Minicomputer.



T: What number is this? Write it on your paper.

Look at the students' papers before asking several students to answer aloud and to explain how they did the calculation.

S: I added "247 + 247" and got 494.

Write this calculation on the board.

$$\begin{array}{r} 247 \\ +247 \\ \hline 494 \end{array}$$

S: I calculated "2 x 7 = 14", "2 x 40 = 80", and "2 x 200 = 400." Then I calculated "14 + 80 + 400 = 494."

Record these calculations on the board.

$$\begin{array}{rcr} 2 \times 7 &=& 14 \\ 2 \times 40 &=& 80 \\ 2 \times 200 &=& 400 \\ \hline 2 \times 247 &=& 494 \end{array}$$

T: This is a shorter way to write all this down.

$$\begin{array}{r} 247 \\ \times\ 2 \\ \hline \end{array}$$

T: What number is 2 x 7?

$$\begin{array}{r} 247 \\ \times\ 2 \\ \hline 14 \end{array}$$

S: 14.

N 29

T: What number is 2 x 40?

S: 80.

$$\begin{array}{r} 247 \\ \times\ 2 \\ \hline 14 \\ 80 \end{array}$$

T: What number is 2 x 200?

S: 400.

$$\begin{array}{r} 247 \\ \times\ 2 \\ \hline 14 \\ 80 \\ 400 \\ \hline 494 \end{array}$$

T: 14 + 80 + 400 = 494, so 2 x 247 = 494.

S: I calculated "2 x 250 = 500" and "500 - 6 = 494."

T: You used the fact that 247 is 3 less than 250.
2 x 250 = 500 and 2 x 3 = 6, so 2 x 247 = 500 - 6.

Erase the board and then write this problem.

$$\begin{array}{r} 356 \\ \times\ 3 \\ \hline \end{array}$$

T: What number is 3 x 356? Write the answer on your paper.

Look at many of the students' papers and observe the methods they are using to find the answer. Invite several students to explain how they did the problem. When appropriate, record their calculations on the board. For example,

$$\begin{array}{r} 356 \\ 356 \\ +356 \\ \hline 1{,}068 \end{array} \qquad \begin{array}{r} 356 \\ \times\ 3 \\ \hline 18 \\ 150 \\ 900 \\ \hline 1{,}068 \end{array}$$

N30 Division Problems #4

CAPSULE LESSON SUMMARY

Solve division problems suggested by story situations. Use string pictures and arrow pictures to find and record the solutions.

MATERIALS

Teacher: Colored chalk
Student: Paper; colored pencils

DESCRIPTION OF LESSON

T: Suppose our class is going on a picnic and we have 147 sandwiches. I would like to share these 147 sandwiches equally among the students in our class. How many students are in our class?

For the purpose of this lesson description, we will assume that there are twenty-six students in the class. You should use the actual number of students in your class instead of twenty-six.

Share 147 sandwiches among 26 students.

T: How many sandwiches will each of you get and how many sandwiches will be left over? Draw an arrow picture or a string picture on your paper to show your solution to this problem.

Allow a few minutes for individual work. Look at the students' work and when several students have solved the problem, begin sharing their solutions with the class. A possible dialogue is given here.

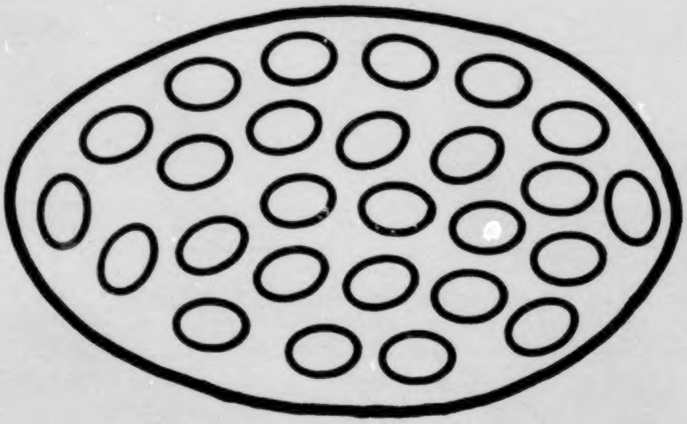
T: Randy has drawn a string picture.

Draw a large string on the board.

T: Randy, is this the string for the 147 sandwiches? (Yes)
How many strings did you draw inside of the large string?

S: Twenty-six, because there are twenty-six students in our class.

Draw twenty-six strings inside the large string.



T: What are the blue strings for?

S: The sandwiches that each of us will get.

T: How many sandwiches will each of you get?

S: Five.

T: How can we be sure this is correct?

S: We could calculate 5 x 26.

S: We could count by fives.

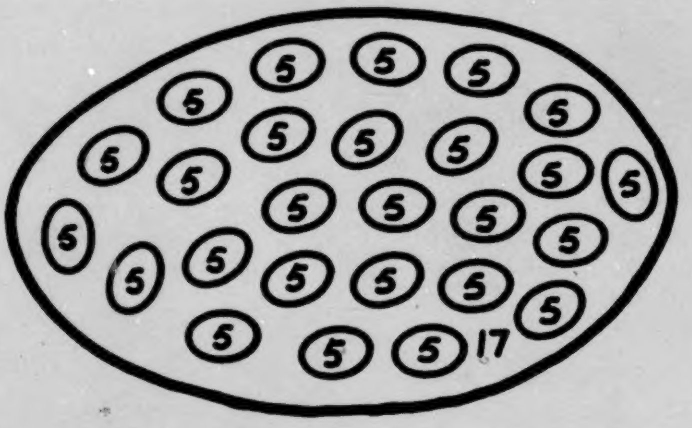
T: What number is 5 x 26? (130)

T: Can we give each of you five sandwiches? (Yes)
How many sandwiches will be left over? (Seventeen)
Could we give each of you one more sandwich?

S: No, because there are twenty-six of us and only seventeen sandwiches left over.

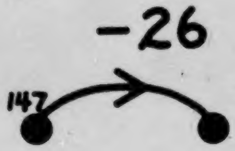
T: Each of you will receive five sandwiches and there will be seventeen sandwiches left over.

Complete the string picture.



T: Who drew an arrow picture about this problem? (Mike)
Mike drew a dot for 147 and then he drew a - 26 arrow.

On the board draw a - 26 arrow starting at 147.



T: Why did you draw a - 26 arrow?

N 30

S: I gave one sandwich to each of us.

T: How many sandwiches were left?

S: We need to calculate "147 - 26".

Invite a student to write this problem on the board and to solve it.

$$\begin{array}{r}147\\-26\\\hline 121\end{array}$$

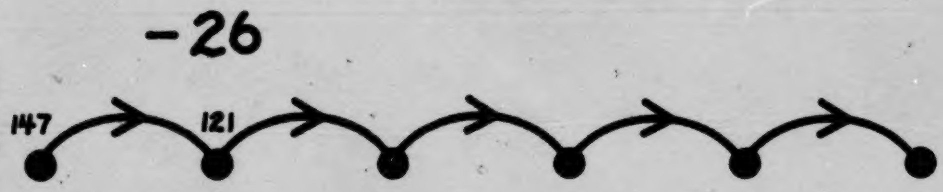
Label the dot for 121.

T: Mike, what did you do next?

S: I drew another - 26 arrow.

T: How many - 26 arrows are in your arrow picture? (Five)

Draw the corresponding arrow road on the board.



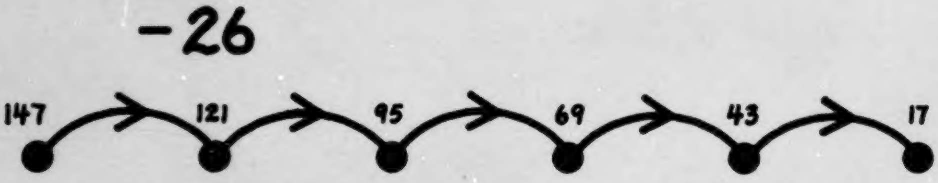
T: What calculation do we need to do to label the next dot?

S: "121 - 26".

Invite a student to do the subtraction calculation at the board.

$$\begin{array}{rrr} 1 & \overset{10}{2} & \overset{10}{1} \\ -\overset{1}{} & \overset{1}{2} & 6 \\ \hline & 9 & 5 \end{array}$$

Continue this activity until all the dots are labeled.



$$\begin{array}{rrr} 1 & 4 & 7 \\ - & 2 & 6 \\ \hline 1 & 2 & 1 \end{array} \qquad \begin{array}{rrr} 1 & \overset{10}{2} & \overset{10}{1} \\ -\overset{1}{} & \overset{1}{2} & 6 \\ \hline & 9 & 5 \end{array} \qquad \begin{array}{rr} 9 & \overset{10}{5} \\ -\overset{1}{2} & 6 \\ \hline 6 & 9 \end{array} \qquad \begin{array}{rr} 6 & 9 \\ -2 & 6 \\ \hline 4 & 3 \end{array} \qquad \begin{array}{rr} 4 & \overset{10}{3} \\ -\overset{1}{2} & 6 \\ \hline 1 & 7 \end{array}$$

T: Look carefully at this arrow picture. How many sandwiches will each of you receive and how many sandwiches will be left over? How do you know?

S: We'll each get five sandwiches because there are five blue arrows. I know that there will be seventeen sandwiches left over because the ending number is 17.

T: Did anyone draw a different arrow picture or string picture to show how they solved this problem?

If students have solved this problem in different ways, draw their arrow pictures or string pictures on the board.

T: What number stories can we write about this problem?

Perhaps the following number stories will be suggested.

$$(26 \times 5) + 17 = 147$$

$$147 - (26 \times 5) = 17$$

$$147 - 130 = 17$$

$$\begin{array}{r} 26 \\ 26 \\ 26 \\ 26 \\ 26 \\ +17 \\ \hline 147 \end{array}$$

Under the number stories write this problem as you ask:

$$26\overline{)147}$$

T: 147 divided by 26 is . . . ?

S: 5 with a remainder of 17.

Ask a student to record the answer.

$$\begin{array}{r} 5 \quad \text{R} = 17 \\ 26\overline{)147} \end{array}$$

Erase the board and then present the following problem to your class.

T: Suppose that our class helps clean up the park and collects 105 soda bottles. We put all the empty bottles into cartons that hold eight bottles each. How many cartons do we fill?

Record the information on the board.

105 soda bottles
8 bottles per carton

T: Try to solve this problem on your own and then draw a string picture or an arrow picture to show your solution.

Look at the students' work. Tell students who are having trouble to draw a dot for 105 and then to draw a - 8 arrow road, adding arrows until they reach a number smaller than 8. The number of - 8 arrows tells how many cartons are filled. While the students are working, briefly describe some of the students' work to the class. For example,

T: Mindy started at 105 and is building a - 16 arrow road. Why would Mindy draw - 16 arrows?

S: A - 16 arrow is the same as two - 8 arrows.

T: Frankie has drawn a + 40 arrow. Why did you draw a + 40 arrow, Frankie?

S: 5 x 8 = 40, so 40 bottles will fill five cartons and 40 is easier to add than 8.

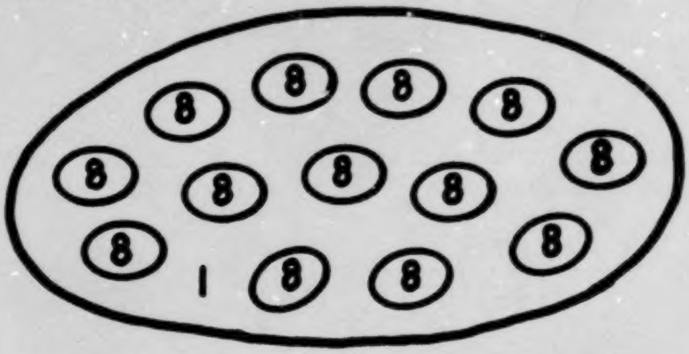
T: Nicole has drawn a - 80 arrow. Why do you think she chose a - 80 arrow?

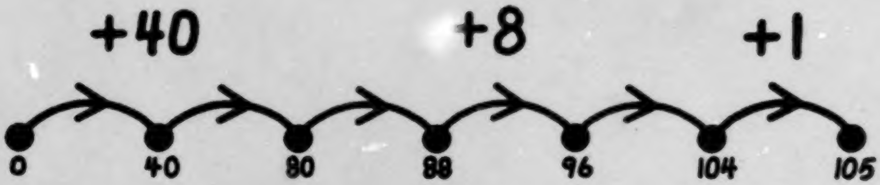
S: 10 x 8 = 80, so ten cartons will hold 80 bottles.

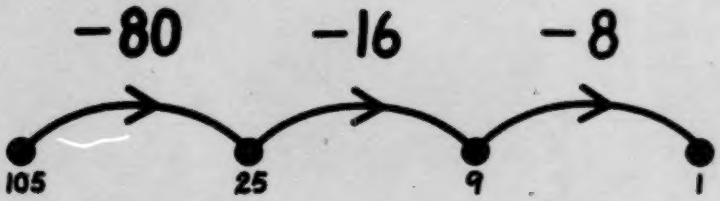
Encourage the students to solve the problem in their own ways, to show their solutions in arrow pictures or in string pictures, and to write number stories about the problem. Students who finish quickly may be asked to determine how many cartons are needed for 307 bottles.

N 30

When most of the students have solved the problem of putting 105 soda bottles into cartons that hold eight bottles each, ask two or three students to show their solutions on the board. A few of the possible arrow pictures and string pictures are shown below.







Conclude that 105 bottles will fill 13 cartons and there will be one bottle left over. Record on the board several of the students' number stories and the division problem with its answer.

$$(13 \times 8) + 1 = 105$$

$$105 = 80 + 24 + 1$$

$$\begin{array}{r} 13 \text{ R}=1 \\ 8\overline{)105} \end{array}$$

N31 Hand-Calculator Games #2

CAPSULE LESSON SUMMARY

With the display of the hand-calculator hidden, anticipate the effect of a sequence of operations on a starting number. Go from one number to another on the hand-calculator using the operations +, -, x, and ÷ and using one-digit positive integers.

MATERIALS

Teacher: Colored chalk
Student: Hand-calculator; paper; pencil; Worksheets N31*, **, ***, and ****

NOTE: Only two students use hand-calculators in the first exercise. Each student uses a hand-calculator in the second exercise.

DESCRIPTION OF LESSON

Exercise 1

Give hand-calculators to only two students. Tell them to follow your directions exactly.

T: Turn on the calculators. What number is on the display? (0)
Cover the display and press $\boxed{4}\boxed{\times}\boxed{3}\boxed{+}\boxed{8}\boxed{+}\boxed{2}\boxed{=}$.

When you are telling the students which keys to press, you should pause after each numeral key so the students will have an opportunity to do mental calculations.

T: Don't look! What number should be on the display? (22)

Allow several students to answer before asking the students with the calculators to look at the displays. If either of the calculators has a number other than 22 on the display, ask the student with that calculator to press $\boxed{C}$ and then to put 22 on the display.

N 31

T: Cover the display again. Press $\boxed{\times}\boxed{3}\boxed{+}\boxed{1}\boxed{0}\boxed{+}\boxed{1}\boxed{0}\boxed{=}$. Don't look. Which number should be on the display? (86)

Let several students answer before asking the students with the calculators to look at the displays. Ask the students with calculators to be sure that 86 is on the display before continuing.

Continue this activity with the following sequences.

$\boxed{\div}\boxed{2}\boxed{=}$ (43)

$\boxed{-}\boxed{1}\boxed{0}\boxed{\div}\boxed{3}\boxed{=}$ (11)

$\boxed{\times}\boxed{5}\boxed{-}\boxed{6}\boxed{=}$ (49)

$\boxed{\times}\boxed{2}\boxed{+}\boxed{2}\boxed{=}$ (100)

$\boxed{-}\boxed{1}\boxed{\times}\boxed{3}\boxed{=}$ (297)

Exercise 2

NOTE: It is best if every student has a hand-calculator for this exercise. If this is not possible, the exercise could be adapted. Students could build arrow roads on paper versus finding an equivalent to them on the calculator.

T: Today we are going to try something new on the calculator. We will start with one number and use only the one-digit numbers 1 through 9 to get to another number.

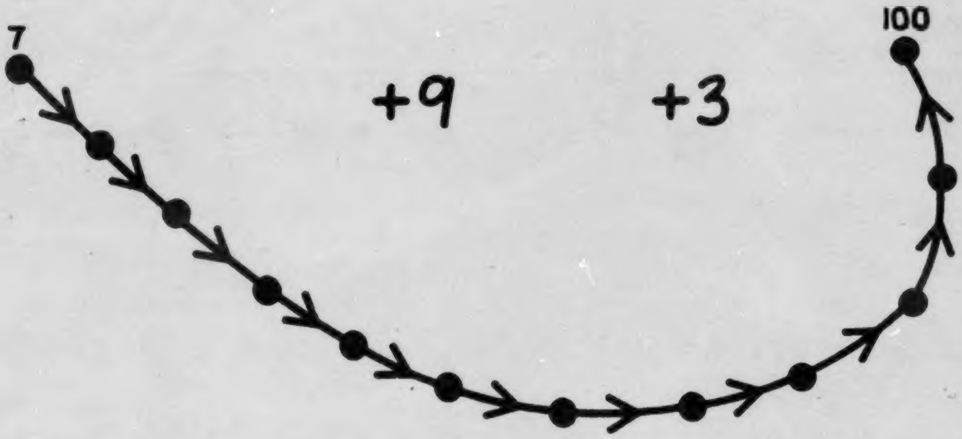
Draw two dots on the board. Label one of them "7" and the other "100".

T: Put 7 on your display. Try to get to 100. You can add or subtract any of the one-digit numbers 1 through 9. You can multiply or divide by any of the one-digit numbers 1 through 9.

You might suggest that students press $\boxed{=}$ after pressing a numeral key so that they can see the result before deciding which operation to use next. Allow the students a few minutes to work on this problem before letting a student share one solution with the class. As the student describes the solution, draw the corresponding arrow road on the board. A possible dialogue is given here.

S: I pressed $\boxed{+}\boxed{9}\boxed{=}\boxed{=}\boxed{=}$ [*] . . . until I got 97, then I pressed $\boxed{+}\boxed{3}\boxed{=}$.

T: How many times did you add 9? (Ten times)



T: How many arrows are in this arrow road? (Eleven)
Did anyone get to 100 in another way that would give us a shorter road?

S: I pressed $\boxed{+}\boxed{3}$ and then $\boxed{+}\boxed{9}\boxed{0}$.

T: But 90 is not a one-digit number. Try again.

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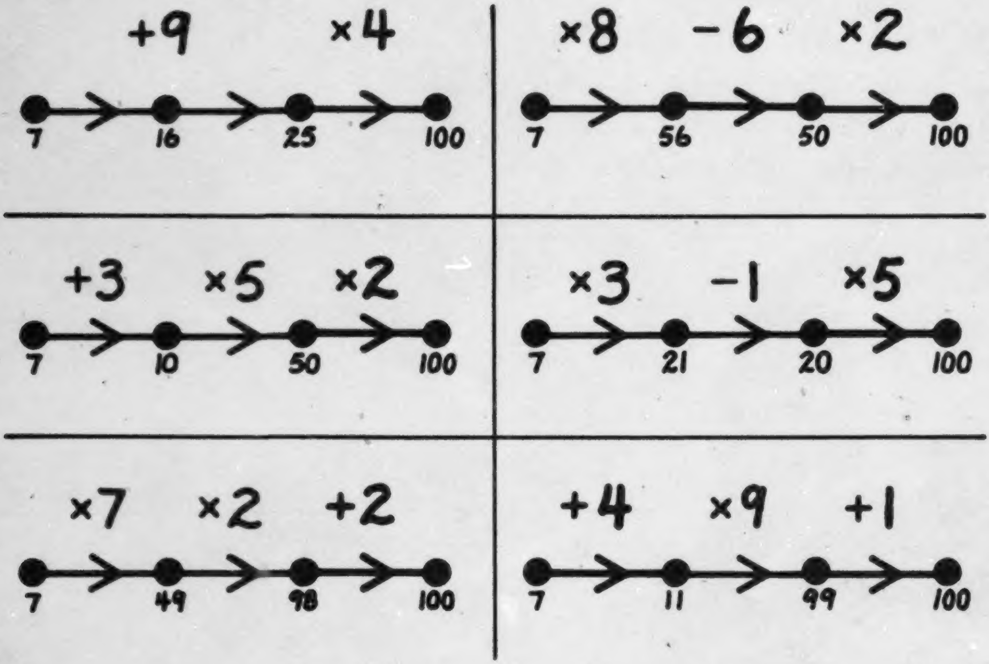
[*] See the Note on the Role and Use of Hand-Calculators in the General Introduction.

N 31

Continue this activity until at least three arrow roads are drawn on the board. Include at least one three-arrow road. Six of the many possible three-arrow roads are shown here for your information.

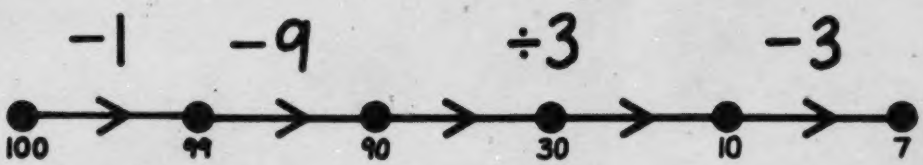


Encourage the students to try and find other solutions which would give even shorter roads than those which are on the board. Perhaps your class will see that it is not possible to go from 7 to 100 using only one or two such arrows.

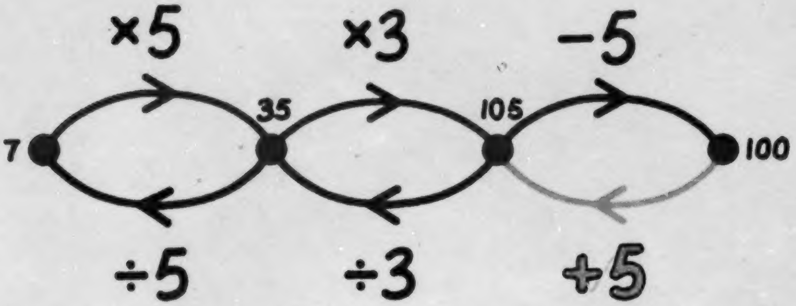
NOTE: If your students have hand-calculators, do not require the students to record their solutions to these problems on paper. Some students will want to keep track of each step by noting them on paper while other students would find working simultaneously with pencil and calculator very awkward and inhibiting.

Do not erase the arrow roads. Ask the students to go from 100 to 7 with the same restrictions. Allow the students to work on this problem for a few minutes before sharing one of the students' solutions with the class. As the solution is described, draw the corresponding arrow road on the board.

One arrow road is shown here.



Encourage the students to find solutions which give even shorter roads than the road on the board. One of the students should notice that to build a road from 100 to 7, you only need to go backward on a road from 7 to 100. For example, if this arrow road was suggested from 7 to 100, then the return arrows form a road from 100 to 7.



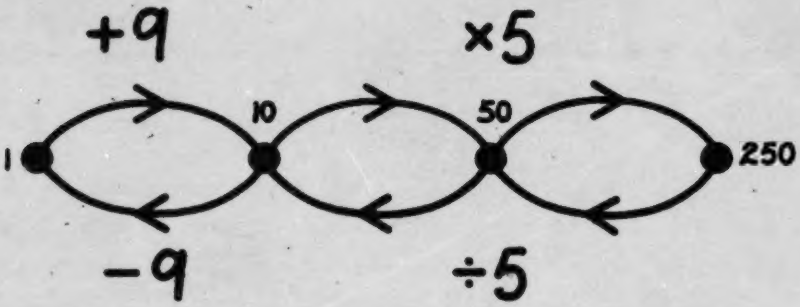
If necessary, suggest using return arrows for the arrow roads from 7 to 100 that are on your board.

Erase the arrow pictures and repeat this activity going from 1 to 250 and from 250 to 1 on the hand-calculator.

320

N 31

For your information, shortest possible roads from 1 to 250 and from 250 to 1 are shown below.

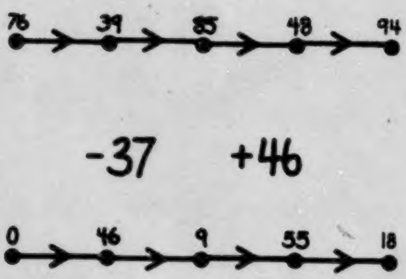


You may wish to ask the students who finish quickly to go from 17 to 400 with the hand-calculator.

Worksheets N31*, **, ***, and **** are available for individual work. Allow the students to use hand-calculators while they are doing the worksheets.

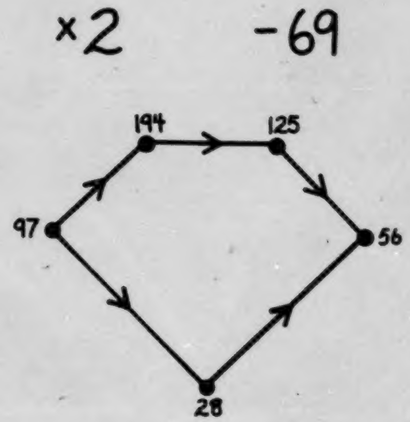
Name ____________________ N 31 *

Label the dots.


76
39
85
48
94
-37
+46
0
46
9
55
18

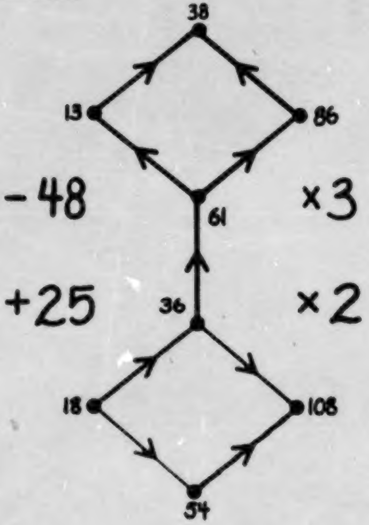
Name ____________________ N 31 **

Label the dots.


×2
-69
194
125
97
56
28

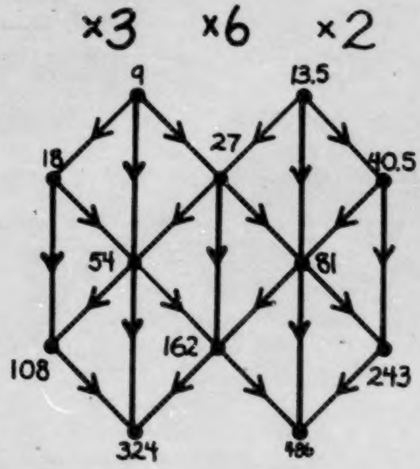
Name ____________________ N 31 ***

Label the dots.


38
13
86
-48
61
×3
+25
36
×2
18
108
54

Name ____________________ N 31 ****

Label the dots.
Draw the seven missing green arrows.


×3
×6
×2
9
13.5
18
27
40.5
54
81
108
162
243
324
486

N 32

N32 Division Problems #5

CAPSULE LESSON SUMMARY

Begin the lesson with some subtraction calculations. Label the dots in an arrow road that starts at 200 and that has - 48 and - 8 arrows. Decide how many cartons two hundred eggs will fill by using the arrow picture. Ask how many adults should accompany the class to a swimming pool if there is to be at least one adult for every five students.

MATERIALS

Teacher: Colored chalk

Student: Paper; colored pencils; SUBTRACTION PROBLEMS booklet

DESCRIPTION OF LESSON

Exercise 1

Write these problems on the board and ask the students to solve as many of them as they can.

$$\begin{array}{r} 782 \\ -\,526 \\ \hline \end{array} \qquad \begin{array}{r} 8{,}429 \\ -\,156 \\ \hline \end{array} \qquad \begin{array}{r} 9{,}587 \\ -\,2{,}954 \\ \hline \end{array} \qquad \begin{array}{r} 4{,}603 \\ -\,1{,}345 \\ \hline \end{array}$$

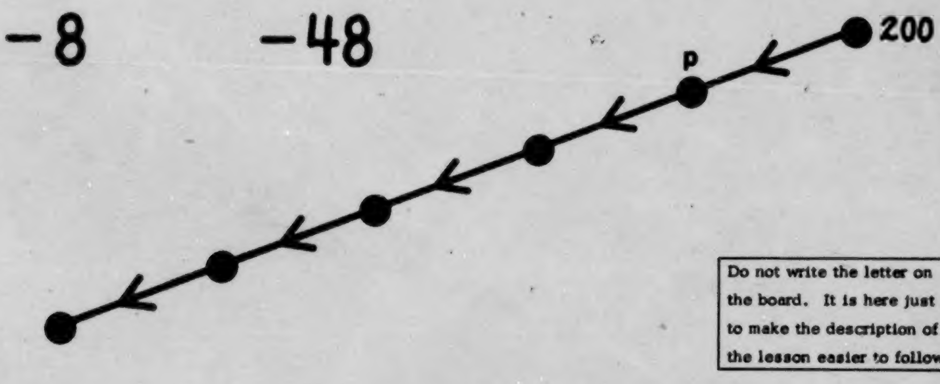
Allow several minutes for individual work. Help students who have difficulty with subtraction. When a few students have solved all the problems, ask students to solve the problems at the board and to explain each step.

$$\begin{array}{r} 78\overset{10}{2} \\ -\,5\overset{1}{2}6 \\ \hline 256 \end{array} \qquad \begin{array}{r} 8{,}4\overset{10}{2}9 \\ -\,\overset{1}{1}56 \\ \hline 8{,}273 \end{array} \qquad \begin{array}{r} 9{,}\overset{10}{5}87 \\ -\,\overset{1}{2}{,}954 \\ \hline 6{,}633 \end{array} \qquad \begin{array}{r} 4{,}6\overset{10}{0}\overset{10}{3} \\ -\,1{,}3\overset{1}{4}\overset{1}{5} \\ \hline 3{,}258 \end{array}$$

Erase the board before going on to Exercise 2.

Exercise 2

Draw this arrow picture on the board. Ask the students to copy it on their papers and to label all the dots.



Do not write the letter on the board. It is here just to make the description of the lesson easier to follow.

Allow several minutes for this activity. Observe the students' work and help individual students with the subtraction problems. When a few students have labeled all the dots, continue with the collective lesson.

T (pointing to p): What calculation do we need to do before we can label this dot?

S: "200 - 48".

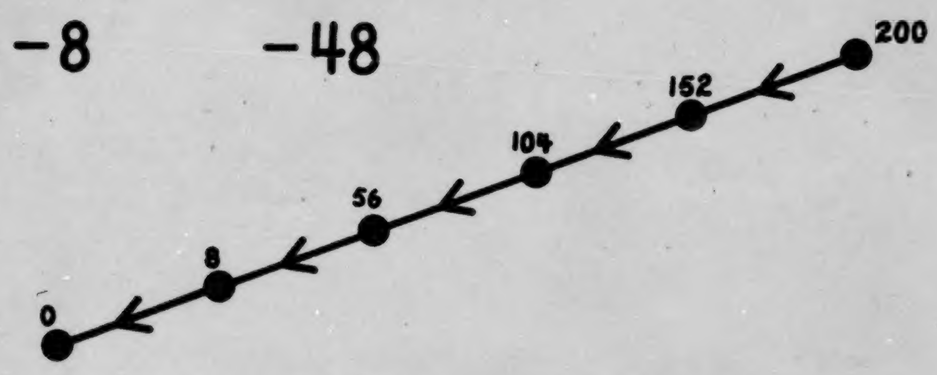
Ask a student to solve the problem at the board and to label p.

$$\begin{array}{r} {}^{10\ 10} \\ 200 \\ {}^{1\ 1\ \ } \\ -\ 48 \\ \hline 152 \end{array}$$

N 32

Continue this activity until all the dots are labeled. Your students will have done the following calculations in order to label the dots in this arrow picture.

$$\begin{array}{r} \underset{1}{2}\,\underset{1}{\overset{10}{0}}\,\overset{10}{0} \\ -\;4\;8 \\ \hline 1\;5\;2 \end{array} \qquad \begin{array}{r} 1\,\underset{1}{5}\,\overset{10}{2} \\ -\;4\;8 \\ \hline 1\;0\;4 \end{array} \qquad \begin{array}{r} \underset{1}{1}\,\underset{1}{\overset{10}{0}}\,\overset{10}{4} \\ -\;4\;8 \\ \hline 5\;6 \end{array} \qquad \begin{array}{r} \underset{1}{5}\,\overset{10}{6} \\ -\;4\;8 \\ \hline 8 \end{array} \qquad \begin{array}{r} 8 \\ -\;8 \\ \hline 0 \end{array}$$



Keep the arrow picture on the board for the rest of the exercise.

T: Pretend that we have a case of two hundred eggs and we want to put them in egg cartons. How many eggs can we put in each carton?

S: Twelve.

T: Each carton holds one dozen or twelve eggs. How many cartons can we fill? Will there be any eggs left over?

Allow the students to think about this problem for a minute or two.

T: Look at the arrow picture. Perhaps it can help you solve this problem.

S: We can fill four cartons with forty-eight eggs. There will be eight eggs left over.

T: Each of these blue arrows tells us that we can fill four cartons.

S: We can fill sixteen cartons, because 4 x 4 = 16.

T: Why did you calculate "4 x 4"?

S: Because there are four cartons for each blue arrow and there are four blue arrows.

T: What number stories can we write about this problem?

Let several students record number stories on the board. If no one suggests a number story involving multiplication, suggest one yourself. These are some of the possible number stories involving multiplication.

$$(16 \times 12) + 8 = 200 \qquad (4 \times 48) + 8 = 200$$

$$200 - (16 \times 12) = 8$$

Under the number stories write this problem as you ask:

$$12\overline{)200}$$

T: 200 divided by 12 is . . . ?

S: 16 with a remainder of 8.

Ask a student to record the answer.

$$\begin{array}{r} 16 \\ 12\overline{)200} \end{array} \text{R} = 8$$

Erase the board before going on to Exercise 3.

N 32

Exercise 3

Tell the following or a similar story to your class.

T: Pretend that this weekend our class is going swimming. The beach where we are going requires that we have at least one adult for each group of five students. Everyone in our class is going, so how many adults do we need to go with us?

Encourage your students to discuss this problem. The calculations involved are fairly simple, but your students may have some difficulty agreeing on how many adults are needed. A sample dialogue is given here for a class of twenty-six students.

S: We will need five adults.

S: There is an extra student.

T: How many students may go to the beach if we have five adults? (Twenty-five)
Remember that there are twenty-six of you and all of you are going.

S: We will need six adults so that everyone can go swimming.

S: If we had six adults, we could invite four more children to come with us.

A class of twenty-six students should conclude that they will need at least six adults to accompany them to the beach.

Distribute the SUBTRACTION PROBLEMS booklets and encourage the students to correct any errors made in previous weeks before attempting any new problems. This is the last lesson using this booklet.

N33 Comparing Prices #2

CAPSULE LESSON SUMMARY

Compare prices of items packaged in two different quantities by finding the cost of a certain number of them that can be purchased with either packaging. Do a similar activity that involves comparing four prices.

MATERIALS

Teacher: Colored chalk
Student: Paper; colored pencils

DESCRIPTION OF LESSON

Exercise 1

Tell the following or a similar story to your class.

T: I have a friend named Jeremy who lives near a stable. Do you know what a stable is?

If necessary, tell the students that a stable is a place where horses live and are cared for. Many people who own horses pay a stable fee to care for the horses because they do not have enough land for a horse or because they do not have enough time to care for a horse properly.

T: Jeremy loves to go horseback riding, and the stable has several horses which they will rent by the hour to a person like Jeremy who doesn't own a horse. The stable sells three-punch cards and five-punch cards to the people who come to the stable often. Each time Jeremy rents a horse for an hour, his card is punched. Since he goes so often, Jeremy would like to know which card is a better deal. The three-punch card costs $7 and the five-punch card costs $11.

Record this information on the board.

3 one-hour rides $7
5 one-hour rides $11

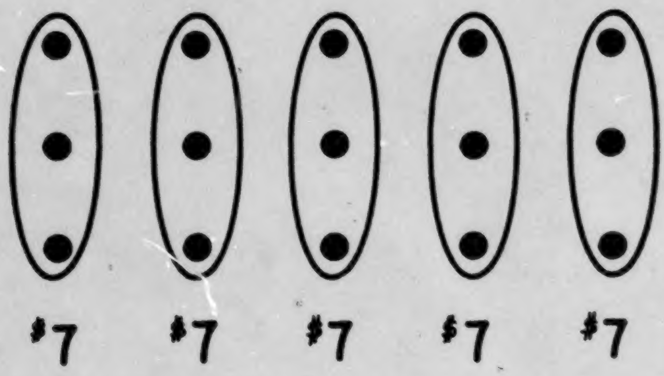
T: Which card is the better deal? How could we solve this problem?

Let the students think about this problem and then discuss it collectively. If no one suggests finding the cost of the same number of rides at each price, suggest this approach yourself. Ask the students for a good number to use. Any common multiple of 3 and 5 is an appropriate choice. In the dialogue that follows, the cost of fifteen rides at each price is computed.

T: How many three-punch cards would Jeremy need to buy in order to get fifteen rides? How do you know?

S: He would need to buy five three-punch cards because $3 + 3 + 3 + 3 + 3 = 15$.

Draw this picture on the board.



T: Each three-punch card costs $7. How much does it cost to buy enough three-punch cards to get fifteen rides? How do you know?

S: 5 x 7 = 35, so fifteen rides cost $35.

Record this information on the board.

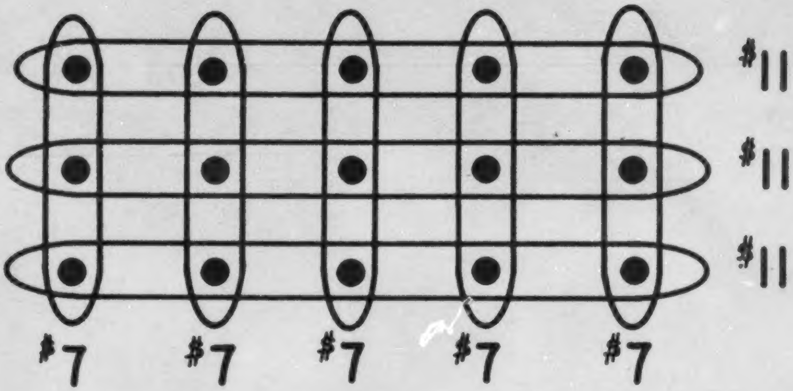
3 rides for $7 or 15 rides for $35

5 rides for $11

T: How many five-punch cards would Jeremy need to buy in order to get fifteen rides?

S: Three.

Draw red strings around groups of five dots.



T: Each five-punch card costs $11. How much does it cost to buy enough five-punch cards to get fifteen rides? How do you know?

S: 3 x 11 = 33, so fifteen rides cost $33.

N 33

Record this information on the board.

3 rides for \$7 or 15 rides for \$35
5 rides for \$11 or 15 rides for \$33

T: Which card is the better deal?

S: The five-punch card.

T: When would you ever buy a three-punch card instead of a five-punch card?

S: . . . if I only wanted to ride three times.

S: . . . if I had enough money to buy a three-punch card but not enough to buy a five-punch card.

S: If I wanted to ride eight times, I would buy one five-punch card and one three-punch card.

Exercise 2

Tell the following or a similar story to your class.

T: This summer the municipal swimming pool is going to sell three-punch tickets and ten-punch tickets. There are many children who would like to know which ticket is a better deal. The three-punch ticket sells for \$2 and the ten-punch ticket sells for \$5.50.

Record this information on the board.

3 admissions for \#2
10 admissions for \#5.50

Encourage the students to try to solve this problem on their own. After a few minutes, discuss approaches. Perhaps someone will recognize that if ten admissions cost $5.50, then one costs 55¢. So three admissions would cost $1.15 since 55 + 55 + 55 = 115. Otherwise, compute the cost of the same number of rides at each price, choosing a number which is a common multiple of 3 and 10. In the following dialogue, the cost of thirty admissions at each price is computed.

3 admissions for \#2 or 30 admissions for ____
10 admissions for \#5.50 or 30 admissions for ____

Allow the students to continue working on this problem for a few more minutes while you draw a 3 by 10 array of dots on the board. Ask students to explain how they calculated the price of thirty admissions with the two types of tickets. Illustrate the students' calculations with the dot picture on the board. A possible dialogue is given here.

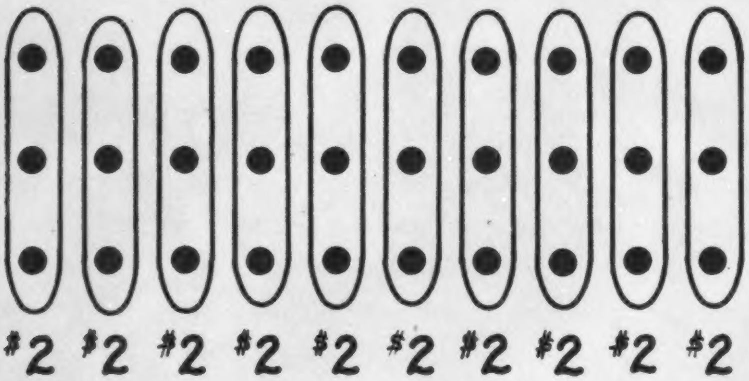
T: How much does it cost to buy thirty admissions with three-punch tickets? How do you know?

S: $20, because ten three-punch tickets are good for thirty admissions and 10 x $2 = $20.

N 33

3 admissions for #2 or 30 admissions for #20

10 admissions for #5.50 or 30 admissions for ____



T: How much does it cost to buy thirty admissions with ten-punch tickets? How do you know?

S: $16.50, because three ten-punch tickets are good for thirty admissions and 3 × $5.50 = $16.50.

T: How did you calculate "3 × $5.50"?

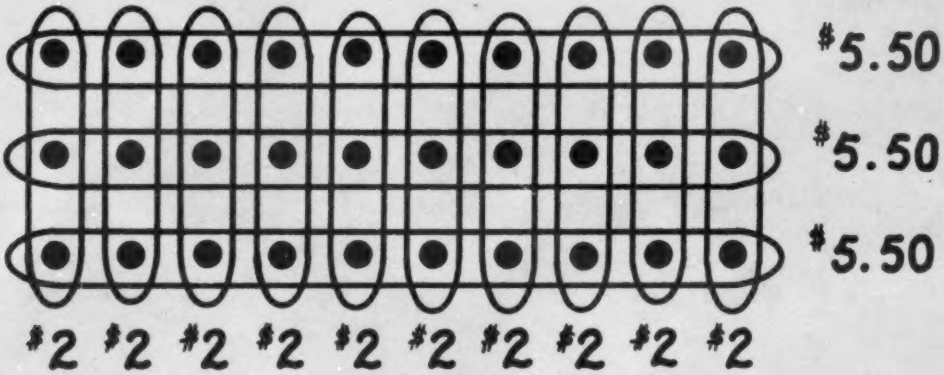
S: I wrote the addition problem on my paper.

Write this problem on the board and ask a student to solve it.

$$\begin{array}{r} \overset{1}{5}.50 \\ 5.50 \\ +5.50 \\ \hline 16.50 \end{array}$$

Complete the dot picture and the information on the board.

3 admissions for #2 or 30 admissions for #20
10 admissions for #5.50 or 30 admissions for #16.50



T: Which ticket is a better deal?

S: The ten-punch ticket.

T: How much do you save on thirty admissions if you buy the ten-punch ticket instead of the three-punch ticket?

S: $3.50.

T: Does anyone see an easier way to tell which of these tickets gives a lower admission price?

Perhaps someone will mention that nine admissions with the three-punch tickets would be $6 and that you can get ten admissions with the ten-punch tickets for only $5.50. If necessary, point this out to your class.

Erase the board before going on to Exercise 3.

N 33

Exercise 3

Tell the following or a similar story to your class.

T: Robin's grandfather loves flowers, so she decides to buy some petunias for his window box. At the flower shop petunias are selling at two for 13¢ or seven for 50¢.

Record this information on the board.

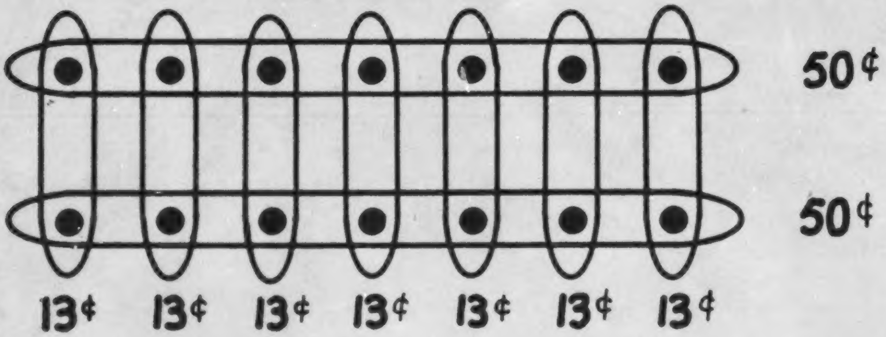
2 petunias for 13¢

7 petunias for 50¢

Continue as in Exercise 2. Conclude that two petunias for 13¢ is the better price. You should have the following dot picture and information on the board.

2 petunias for 13¢ or 14 petunias for 91¢

7 petunias for 50¢ or 14 petunias for $1.00



Erase the board before going on to Exercise 4.

Exercise 4

Tell the following or a similar story to your class.

T: When I went to the butcher shop recently, they had the following prices for large quantities of ground beef.

Record this information on the board.

1 kg for $3.00
2 kg for $5.50
3 kg for $8.00
4 kg for $10.75

Explain to the class that a kilogram is a unit of weight just as a pound is another unit of weight, and that "kg" is an abbreviation for kilogram.

T: Sometimes the more you buy the less you pay for each kilogram of meat. Is that true if these are the prices?

Let the students think about this problem and discuss it collectively. Very likely a student will observe that the price for one kilogram of ground beef is the highest because at this price, two kilograms cost $6, three kilograms cost $9, and four kilograms cost $12. If none of the students mention this, you should ask if it is possible that the price for one kilogram of ground beef is the best one. When your class had concluded that the price for a single kilogram is the most expensive, erase this price from your list.

T: Is it possible that the price for two kilograms of ground beef is the least expensive of these prices?

S: If you bought four kilograms at that price, it would cost you $11 because 2 x $5.50 = $11, so that price is more expensive than the price for four kilograms of ground beef.

Erase the price for two kilograms of ground beef.

3 kg for $8.00
4 kg for $10.75

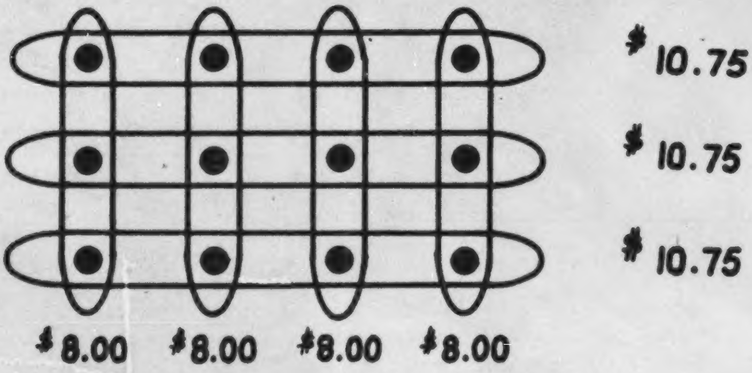
T: Which of these prices is the better deal?

Let the students express their opinions. Very likely someone will suggest calculating the price of 12 kilograms of beef at each of these prices so a comparison can be made. If necessary, suggest this yourself.

3 kg for $8.00 or 12 kg for ______
4 kg for $10.75 or 12 kg for ______

Continue as in Exercise 2. Conclude the lowest price (per kilogram) is three kilograms for $8.00. Point out that buying the largest package is not the best deal in this case. You should have the following information on the board.

3 kg for $8.00 or 12 kg for $32.00
4 kg for $10.75 or 12 kg for $32.25



N34 Roads with Cords

CAPSULE LESSON SUMMARY

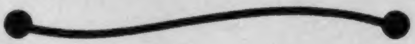
Consider two relations: one pairs two numbers if and only if one number is 1 more than the other; the other pairs two numbers if and only if one number is the double of the other. Label the dots in some pictures involving both of these relations.

MATERIALS

Teacher: Colored chalk
Student: Paper; colored pencils; Worksheets N34*, ** ***, and ****

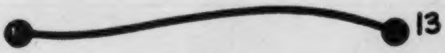
DESCRIPTION OF LESSON

Draw a blue cord on the board.



T: These two dots are for numbers. Two dots can be connected by a blue cord if and only if one of the numbers is 1 more than the other

Label one of the dots "13".



T: If this dot were for 13, what could the other dot be for?

S: 14.

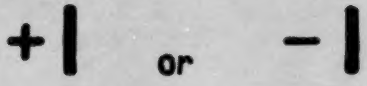
T: 13 + 1 = 14. Is there another possibility?

N 34

S: 12.

T: 13 - 1 = 12. The blue cord is for + 1 or - 1.

On the board draw this key for the cord.



Erase the "13" and label one of the dots "100".

T: If this dot were for 100, what could the other dot be for?

S: 101, because 100 + 1 = 101.

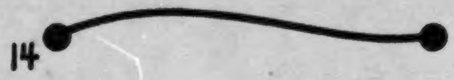
S: 99, because 100 - 1 = 99.

Repeat this activity with 80; 360; 1,000; 1,010; and $\hat{S}$.

Draw a red cord on the board.

T: These two dots are for numbers. Two dots can be connected by a red cord if and only if one of the numbers is the double of the other number.

Label one of the dots "14".



T: If this dot were for 14, what could the other dot be for?

S: 28, because 2 x 14 = 28.

T: Are there any other possible answers?

S: 7, because 2 x 7 = 14.

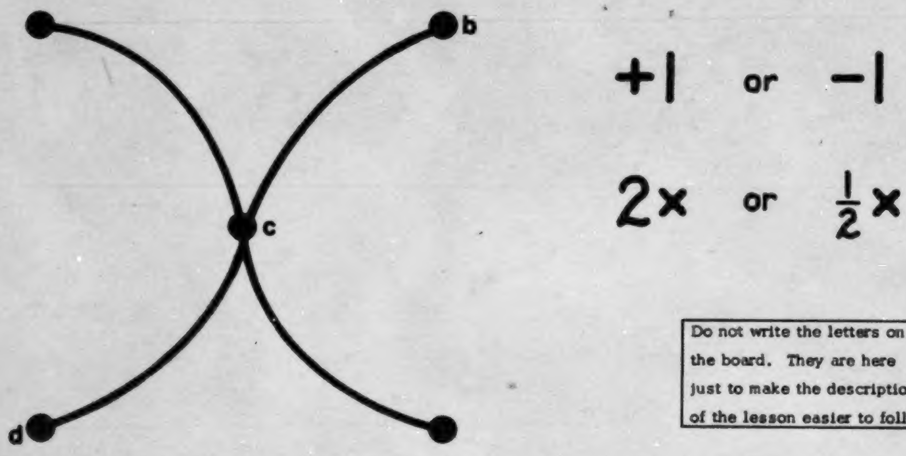
T: The red cord is for 2 x or $\frac{1}{2}$ x.

On the board draw this key for the red cord.

$$2\times \quad \text{or} \quad \frac{1}{2}\times$$

Repeat this activity with 22, 30, 50, 5, and any other whole numbers that are appropriate for the abilities of your students.

Draw this picture on the board and ask the students to copy it.

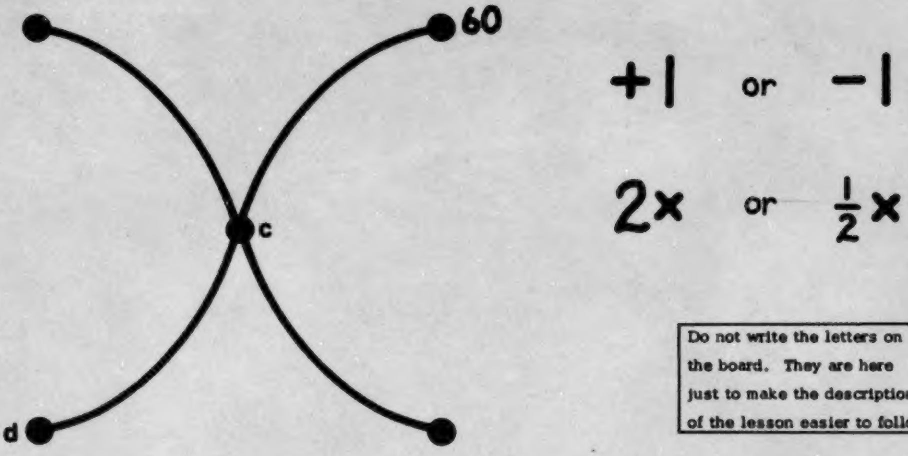


Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: Who would like to label one of these dots? You may choose any whole number.

N 34

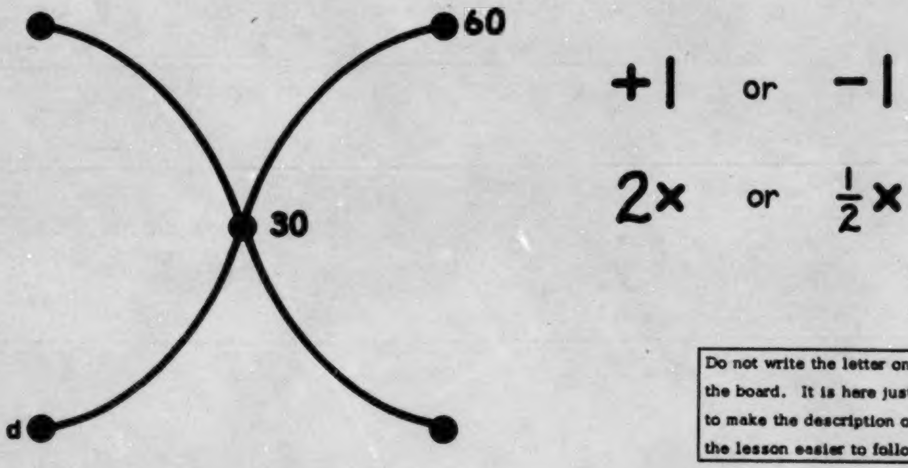
Suppose a student labels b "60".



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: If this dot (b) were for 60, what could the other dots be for?

Invite students to label the other dots. Discuss the students' choices. A sample dialogue follows.

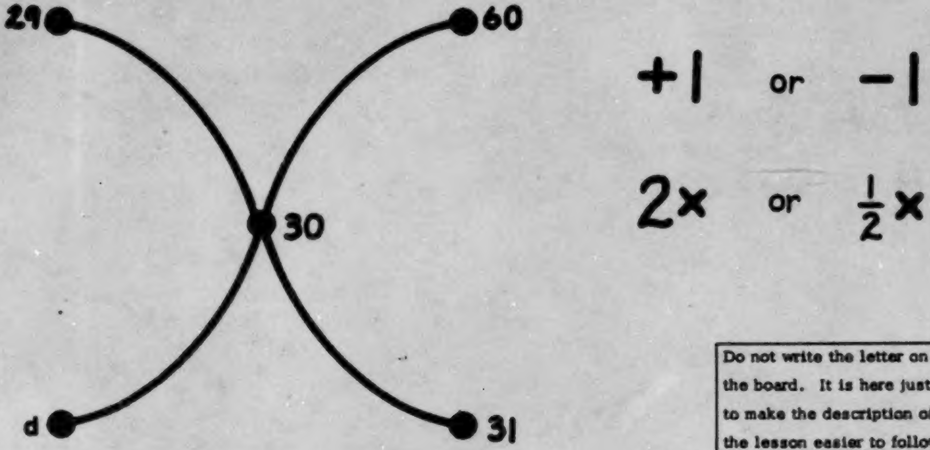


Do not write the letter on the board. It is here just to make the description of the lesson easier to follow.

T: $\frac{1}{2} \times 60 = 30$.
Are there any other numbers this dot (c) could be for?

S: 120.

T: Right, this dot could also be for 120 because 2 x 60 = 120. So this number could be 30 or 120 and we've chosen it to be 30.

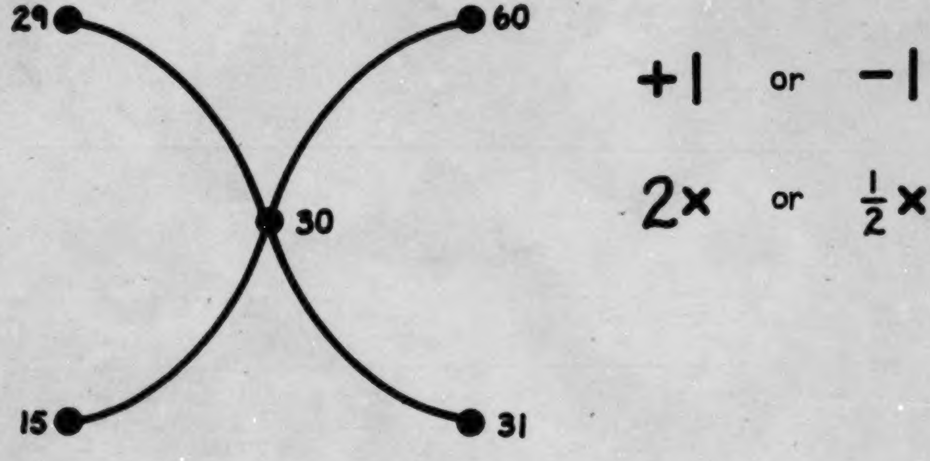


Do not write the letter on the board. It is here just to make the description of the lesson easier to follow.

T: 30 - 1 = 29 and 30 + 1 = 31.
There is one dot left to label. What could this dot (d) be for?

S: 15.

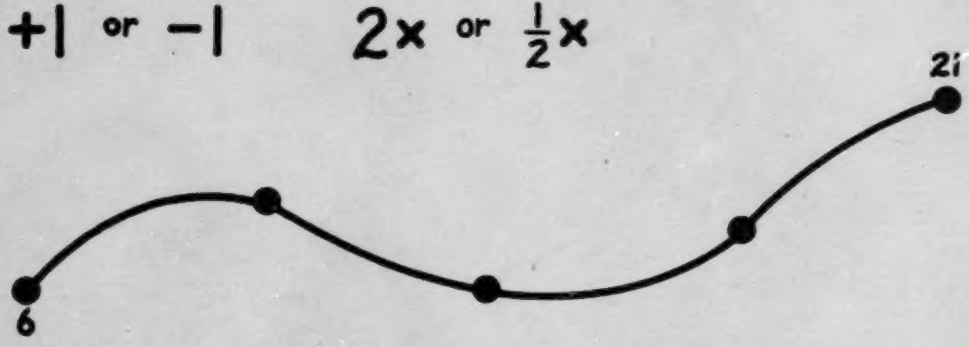
If a student suggests 60, remind the class that 60 is already in this cord picture and ask if there are any other possibilities.



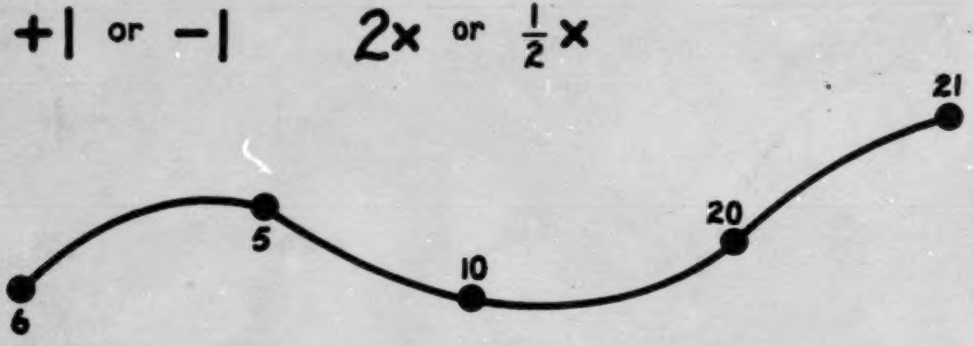
N 34

Erase the numerals in the cord picture on the board and ask the students to label the dots on their papers starting with any whole number they wish. Some students may be able to complete the cord picture in several ways.

Erase the board. Draw this picture on the board and ask the students to copy it and to label all the dots.



Allow the students to work independently. When students have labeled the dots correctly they should begin working on the worksheets immediately. When all the students have labeled the dots in this cord picture, allow a student to label the dots on the board.

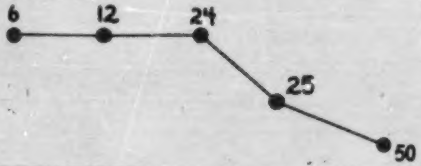


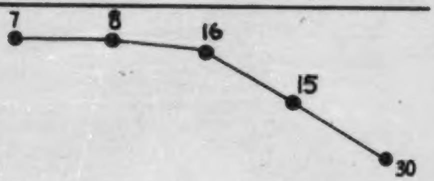
Worksheets N34*, **, ***, and **** are available for individual work. Emphasize to students that they are to use less than ten cords to build each road on the * and ** worksheets.

Name ____________ **N 34 ***

$2x$ or $\frac{1}{2}x$ $+1$ or -1

Build a red-blue road between each pair of numbers. Use less than ten cords to build each road.



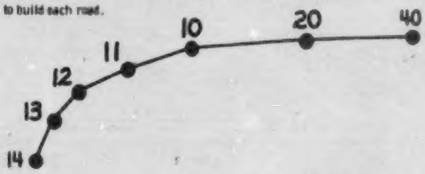


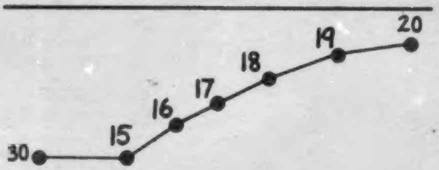
Other solutions are possible; however, these are shortest roads.

Name ____________ **N 34 ****

$2x$ or $\frac{1}{2}x$ $+1$ or -1

Build a red-blue road between each pair of numbers. Use less than ten cords to build each road.


14
13
12
11
10
20
40

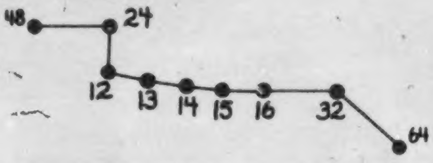

30
15
16
17
18
19
20

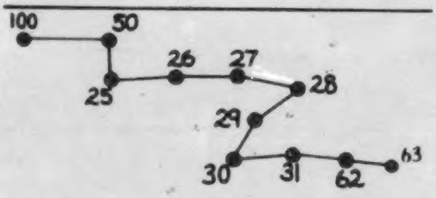
Other solutions are possible; however, these are shortest roads.

Name ____________ **N 34 *****

$2x$ or $\frac{1}{2}x$ $+1$ or -1

Build a red-blue road between each pair of numbers. Use as few cords as possible to build each road.


48
24
12
13
14
15
16
32
64

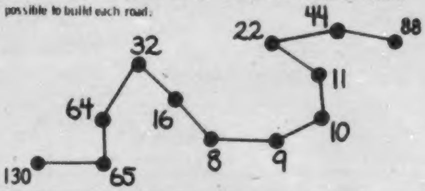

100
50
25
26
27
28
29
30
31
62
63

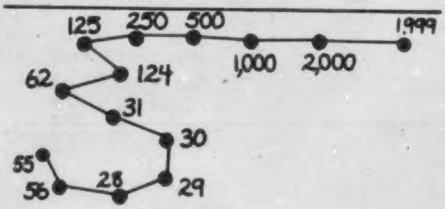
Other solutions are possible.

Name ____________ **N 34 ******

$2x$ or $\frac{1}{2}x$ $+1$ or -1

Build a red-blue road between each pair of numbers. Use as few cords as possible to build each road.


130
65
64
32
16
8
9
10
11
22
44
88


125
250
500
1,000
2,000
1,999
124
62
31
30
29
28
56
55

Other solutions are possible,

N 35

N35 Minicomputer Golf #2

CAPSULE LESSON SUMMARY

Explore the effect of moving various checkers in a configuration of both regular and negative checkers on the Minicomputer — after a move, is there a larger, a smaller, or the same number on the Minicomputer? Play Minicomputer Golf twice. Include a negative checker in the starting configuration of the second game.

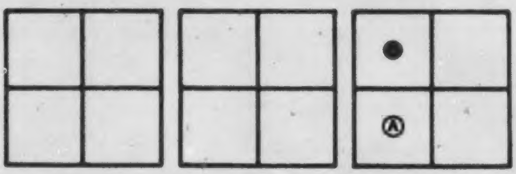
MATERIALS

Teacher: Minicomputer set; colored chalk
Student: None

DESCRIPTION OF LESSON

Exercise 1

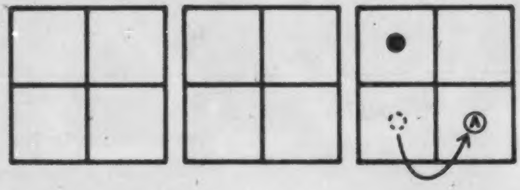
Put this configuration on the Minicomputer.



T: Which number is on the Minicomputer?

S: 6.

Move the negative checker to the 1-square.



T: What number is this?

S: 7.

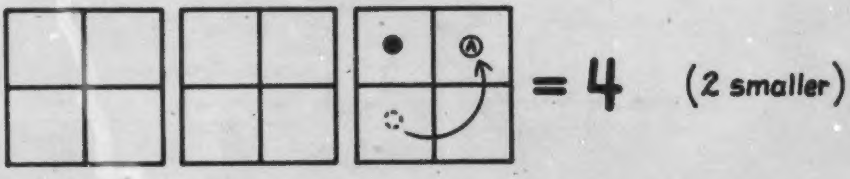
T: By moving the negative checker from the 2-square to the 1-square, did I get a larger or smaller number?

S: Larger.

T: . . . larger by how much?

S: 1.

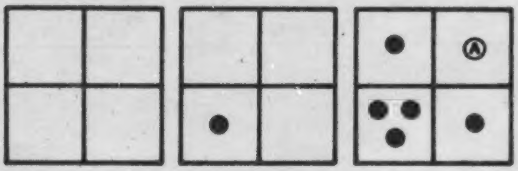
Move the negative checker back to the 2-square. Repeat the activity, only this time move the negative checker to the 4-square.



Clear the Minicomputer.

T: I'll put a number on the Minicomputer. As I'm putting on checkers, see if you can figure out which number it is.

Put checkers on the Minicomputer gradually, allowing the students to calculate mentally, until you have this configuration displayed.



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N 35

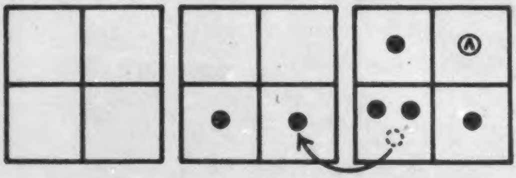
T: Which number did I put on the Minicomputer?

S: 31.

Check that the number is 31 by pointing to the checkers, one at a time, as the class computes the total. Write "31" to the right of the Minicomputer boards.

T: Now I'll move a checker. Tell me whether the new number is smaller or larger and by how much. Also tell me the new number.

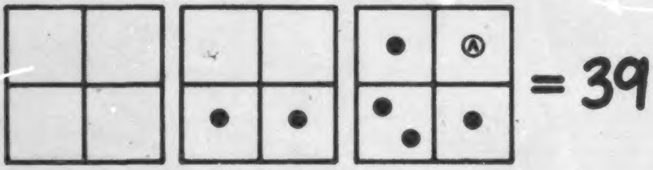
Move a checker from the 2-square to the 10-square.



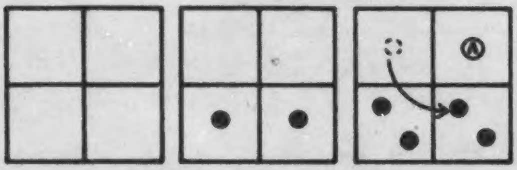
S: Larger by 8.

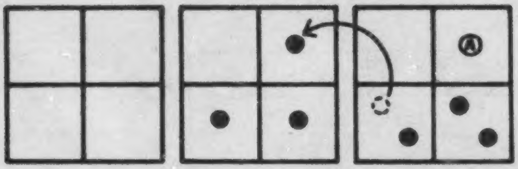
S: 39 is on the Minicomputer.

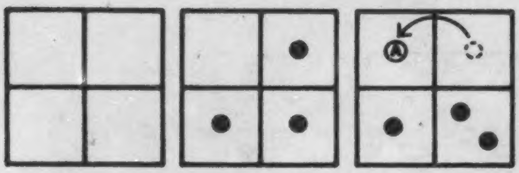
Erase "31" and write "39" beside the Minicomputer.

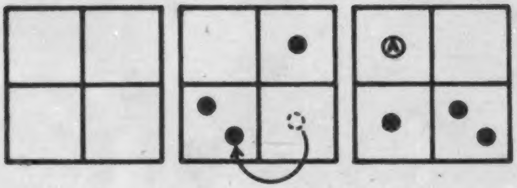


Continue this activity with the moves illustrated below.


Ⓐ
= 32
(7 smaller)

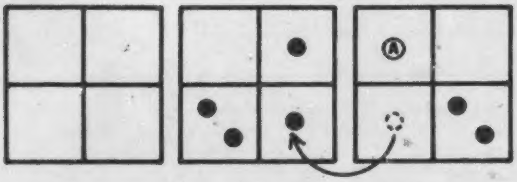

Ⓐ
= 70
(38 larger)


Ⓐ
= 66
(4 smaller)


Ⓐ
= 76
(10 larger)

T: Who can move one checker and get a new number that is 8 larger than this number?

A volunteer should move a checker from the 2-square to the 10-square.



T: Who can move one checker and get a new number that is 6 larger than this number?

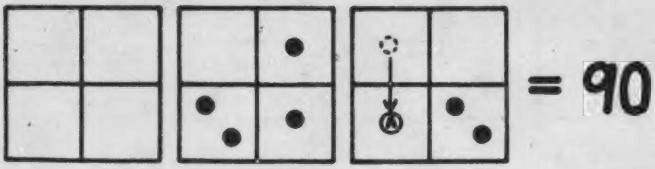
Your students might first notice that moving any of the regular checkers will not accomplish the desired change. This should focus their attention to the negative checker. A sample dialogue follows.

S: We can't do it because we don't have a checker on the 2-square.

T: Where would we move it if we did?

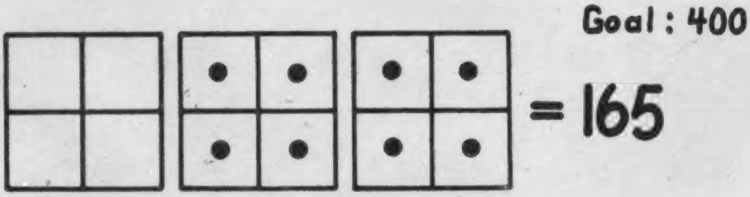
S: To the 8-square.

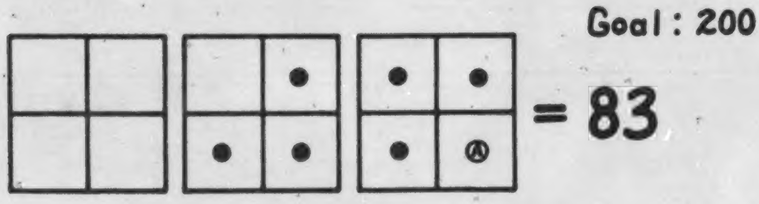
S: Move the negative checker from the 8-square to the 2-square.



Exercise 2: Minicomputer Golf

Play Minicomputer Golf as described in Lesson N15. Starting configurations and goals for two games are suggested below.





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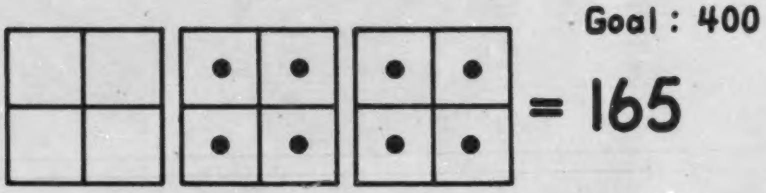
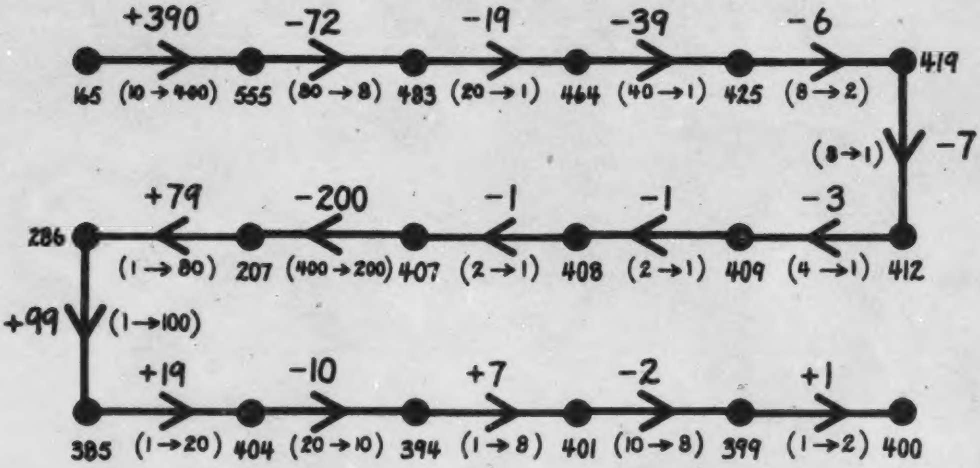
You may wish to play Minicomputer Golf as a competition between two teams. Teams would take turns making moves. A sample game with a Red Team and a Blue Team is recorded in the arrow picture below.


Goal : 400
= 165

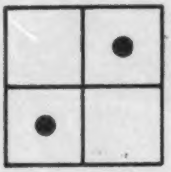

+390
165
(10 → 400)
-72
555
(80 → 8)
-19
483
(20 → 1)
-39
464
(40 → 1)
-6
425
(8 → 2)
419
-7
(8 → 1)
412
-3
(4 → 1)
409
-1
(2 → 1)
408
-1
(2 → 1)
407
-200
(400 → 200)
207
+79
(1 → 80)
286
+99
(1 → 100)
385
+19
(1 → 20)
404
-10
(20 → 10)
394
+7
(1 → 8)
401
-2
(10 → 8)
399
+1
(1 → 2)
400

The Red Team wins!

APPENDIX A: STANDARD CONFIGURATION AND STANDARD MINICOMPUTER MOVES

A number can be represented by putting checkers[*] on the squares of the Minicomputer. Each square has a numerical (positional) value, so that the number represented by a checker depends on which square the checker is placed. If several checkers are put on the Minicomputer, the number shown is the sum of the numbers represented by the checkers. A number can be put on the Minicomputer in a variety of ways, but the representation which uses at most one checker on each square and uses checkers to represent a digit 9 or less on each board is usually the easiest to read. In this case, we say that the number is in standard configuration. Standard configurations for the numbers 1-9 soon become as familiar to the students as the usual numerals so that they no longer need to do any mental calculation for such configurations.

For example,



. . . is seen as a name for 6 without doing the calculation "4 + 2 = 6".

Doing calculations on the Minicomputer involves putting one or more checkers on the various squares and making "plays" to get a specific configuration. In this context, a play is an exchange of checkers in a configuration which does not change the number represented. Some plays are natural to the design of the Minicomputer; we call these standard or usual plays (these are described in detail in this section). There are advantages to establishing such a set of plays:

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[*] In this appendix we refer only to whole numbers and, therefore, use only regular (positive) checkers.

- They become routine, and a large group of students with varying abilities can follow a calculation without needing to verify every play which is made.
- They lead to standard configurations of numbers.

These are the standard Minicomputer moves.

1 + 1 = 2
10 + 10 = 20
100 + 100 = 200
1,000 + 1,000 = 2,000

two checkers on the white square
of a board
=
one checker on the red square
of the same board

2 = 1 + 1
20 = 10 + 10
200 = 100 + 100
2,000 = 1,000 + 1,000

one checker on the red square
of a board
=
two checkers on the white square
of the same board

2 + 2 = 4
20 + 20 = 40
200 + 200 = 400
2,000 + 2,000 = 4,000

two checkers on the red square
of a board
=
one checker on the purple square
of the same board

4 = 2 + 2
40 = 20 + 20
400 = 200 + 200
4,000 = 2,000 + 2,000

one checker on the purple square
of a board
=
two checkers on the red square
of the same board

$4 + 4 = 8$
$40 + 40 = 80$
$400 + 400 = 800$
$4,000 + 4,000 = 8,000$

two checkers on the purple square
of a board
=
one checker on the brown square
of the same board

$8 = 4 + 4$
$80 = 40 + 40$
$800 = 400 + 400$
$8,000 = 4,000 + 4,000$

one checker on the brown square
of a board
=
two checkers on the purple square
of the same board

$8 + 2 = 10$
$80 + 20 = 100$
$800 + 200 = 1,000$
$8,000 + 2,000 = 10,000$

one checker on each of the brown
and red squares of a board
=
one checker on the white square
of the next board to the left

$10 = 8 + 2$
$100 = 80 + 20$
$1,000 = 800 + 200$
$10,000 = 8,000 + 2,000$

one checker on the white square
of a board
=
one checker on each of the brown
and red squares of the next board
to the right[*]

---

[*] This provides the link that motivates the introduction of the decimal numbers. In particular, we have that one checker on the white square of the ones' board equals one checker on each of the brown and red squares of a special board (the tenths' board — or as it is often referred to in the lessons, the dimes' board) located to the right of the ones' board and separated from it by a special barrier.

After many experiences with the Minicomputer, some students might suggest "non-standard" plays, e.g. "8 + 4 = 10 + 2". When students suggest such plays, ask them to convince the class that the plays are indeed correct. There will be a natural acceptance or rejection of non-standard plays by the students, sometimes depending on the situation.

Some "non-standard" plays will be introduced for specific purposes; for example, ten checkers on a square can be exchanged for one checker on the same color square of the next board to the left. Such plays serve to reinforce the support that the Minicomputer gives to decimal place value or to provide insight into a certain kind of problem. They are inviting in particular situations, but do not play the same "routine" role that the usual plays do.

APPENDIX B: MENTAL ARITHMETIC

Mental arithmetic activities are short, fast-moving question and answer sessions where students are asked to calculate mentally, preferably without the aid of paper and pencil. Often a mental arithmetic session consists of several sequences in which the answers to one or more questions lead to the answer of another, more difficult question; for example,

2 × 6 = ? (12)
2 × 10 = ? (20)
2 × 16 = ? (32)

There are specific lessons which call for mental arithmetic; however, we suggest you try to include such activities for five to ten minutes at least twice a week. Use mental arithmetic as a warm-up activity to a lesson, as a transition between two exercises in a lesson, as a conclusion to a lesson, or as a quick change of pace at any time during the day.

Why is Mental Arithmetic Important?

1) It develops mental computational skills.
2) It provides an opportunity to involve and help students at all ability levels.
3) It creates an awareness of patterns and mathematical relationships.
4) It helps students to be able eventually to recall arithmetic facts easily.
5) It provides an opportunity to keep alive and to reinforce concepts previously introduced.
6) It helps the students to see that there is often more than one way to solve a problem.

Hints for Mental Arithmetic Activities

1) Vary the level of difficulty of questions throughout a session.

2) Involve many students.

3) Follow a pattern for awhile and then start a new pattern.

4) Keep a brisk pace, although you should allow students time for thought.

5) Give extra attention to a student who shows signs of improvement.

6) Illustrate, whenever possible, the value of estimation in making calculations.

7) Occasionally ask a student to explain an answer.

8) Occasionally use the chalkboard to emphasize a pattern.

9) Occasionally ask students to whisper an answer to you so that many students will have an opportunity to answer.

SUGGESTED MENTAL ARITHMETIC ACTIVITIES

This section contains specific examples that illustrate patterns in arithmetic. You should elaborate and expand upon ideas depending upon the abilities and interests of your students. As you work with your class on the regularly scheduled mathematics lessons, you might make a note of computational skills you would like to improve; then plan a short mental arithmetic activity which develops those skills. Do not expect everyone of your students to be good at all of these suggested activities, even at the end of the year.

Counting Activities

1) Count by twos, threes, fours, fives, or tens collectively; everyone counting together.

2) Count by twos, threes, fours, fives, or tens as a group activity with each student saying one number. Use a natural seating order in your

class (such as going up and down rows) and ask each student in turn to say the number which comes next. If you are counting by threes, the first student will say "3"; the second will say "6"; and so on. Repeat the counting but start with a different student.

3) Count by twos, threes, fours, fives, or tens, starting at 1 or any number other than 0.

4) Count backward by twos, fives, or tens. Do this immediately following a similar forward count.

5) Play "BUZZ". The students count by ones starting at 0 but skip every fourth number by saying "buzz". This should be done rapidly. This can be done collectively or as a group activity with each student in turn saying the number that comes next or "buzz".

More challenging variations of this activity involve:

- skipping every sixth number rather than every fourth number;
- starting the counting at 2 and skipping every fourth number;
- counting by twos or fives and skipping every fourth number.

Addition and Subtraction

The first fourteen examples are lists of calculations to be done mentally. Each list suggests a pattern in arithmetic which students should begin to use in doing such calculations.

1)	6 + 4	8 + 2	30 + 20
	6 + 5	8 + 3	30 + 21
	6 + 6	8 + 4	30 + 22
	6 + 7	8 + 5	30 + 23

7 + 4
7 + 14
7 + 24
7 + 34

9 + 5
9 + 15
9 + 25
9 + 35

21 + 35
21 + 45
21 + 55
21 + 65

702 + 100
702 + 200
702 + 300

536 + 100
536 + 200
536 + 300

600 + 213
600 + 313
600 + 413

2) 10 - 4
10 - 5
10 - 6
10 - 7

40 - 1
40 - 2
40 - 3
40 - 4

100 - 10
100 - 11
100 - 12
100 - 13

44 - 10
44 - 20
44 - 30
44 - 40

52 - 3
52 - 13
52 - 23
52 - 33

186 - 10
186 - 20
186 - 30
186 - 40

307 - 100
307 - 200
307 - 300

520 - 105
520 - 205
520 - 305

1,000 - 100
1,000 - 200
1,000 - 300

3) 2 + 5
20 + 50
200 + 500
2,000 + 5,000
2,000,000 + 5,000,000

8 + 8
80 + 80
800 + 800
8,000 + 8,000
8,000,000 + 8,000,000

23 - 10
230 - 100
2,300 - 1,000

51 - 10
510 - 100
5,100 - 1,000

98 - 20
980 - 200
9,800 - 2,000

4) 4 + 10
14 + 10
24 + 10
44 + 10

7 + 10
17 + 10
117 + 10
1,017 + 10

6 + 10
18 + 10
33 + 10
71 + 10

5 + 20
15 + 20
25 + 20
35 + 20

3 + 20
13 + 20
113 + 20
1,013 + 20

19 + 20
100 + 20
600 + 20
619 + 20

16 - 10
26 - 10
36 - 10
46 - 10

100 - 10
200 - 10
600 - 10
1,000 - 10

40 - 20
50 - 20
60 - 20
90 - 20

5) 10 + 6
20 + 6
30 + 6
40 + 6

13 + 14
23 + 14
33 + 14
43 + 14

19 + 3
29 + 3
39 + 3
49 + 3

7 - 5
17 - 5
27 - 5
37 - 5

10 - 3
20 - 3
30 - 3
40 - 3

100 - 8
200 - 8
300 - 8
400 - 8

6) 5 + 3
8 - 3
7 + 6
13 - 6

10 + 20
30 - 20
90 + 40
130 - 40

106 + 105
211 - 105
150 + 150
300 - 150

7) 16 + 5
5 + 16

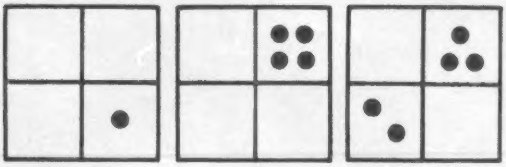
54 + 12
12 + 54

3 + 2 + 5
3 + 5 + 2
5 + 3 + 2

8)	6 + 10	8 + 10	27 + 100
	6 + 9	8 + 9	27 + 99
	20 - 10	43 - 10	126 - 100
	20 - 9	43 - 9	126 - 99
9)	50 + 40	30 + 20	40 + 60
	8 + 2	5 + 7	9 + 4
	58 + 42	35 + 27	49 + 64
	50 - 20	90 - 30	60 - 10
	8 - 6	7 - 2	9 - 4
	58 - 26	97 - 32	69 - 14
10)	5 + 2	16 + 2	64 + 2
	5 + 1	16 + 1	64 + 1
	5 + 0	16 + 0	64 + 0
	$5 + \hat{1}$	$16 + \hat{1}$	$64 + \hat{1}$
	$5 + \hat{2}$	$16 + \hat{2}$	$64 + \hat{2}$
11)	12 - 5	21 - 4	145 - 30
	$12 + \hat{5}$	$21 + \hat{4}$	$145 + \widehat{30}$
12)	$2 + \hat{3}$	$\hat{5} + \hat{1}$	$8 + \hat{5}$
	$20 + \widehat{30}$	$\widehat{50} + \widehat{10}$	$80 + \widehat{50}$
	$200 + \widehat{300}$	$\widehat{500} + \widehat{100}$	$800 + \widehat{500}$
13)	10 - 6	11 - 4	27 - 13
	6 - 10	4 - 11	13 - 27
14)	$7 + \hat{9}$	$3 + \hat{8}$	$12 + \widehat{15}$
	$\hat{7} + 9$	$\hat{3} + 8$	$\widehat{12} + 15$

15) Choose some numbers and ask the students to add 1 to each number. For example, if you say "108", a student responds "109". You can vary this activity by asking the students to add 2, 3, 5, or 10 to each of your numbers or to subtract 1, 2, 3, or 10 from each of your numbers.

16) Conduct an activity similar to Exercise 1 in Lesson N15 Minicomputer Golf #1. Put six to ten checkers on the ones' and the tens' boards. Move one checker from some square (it makes no difference which square you choose) to another square. Ask the students how much larger or smaller the new number is. For example, suppose this configuration is on the Minicomputer.



Move a checker from the 4-square to the 10-square. When the students tell you that the new number is larger, ask them how much larger this number is than the previous number. (6 larger)

17) Ask the students to give various names for a particular number. Encourage them to suggest names involving subtraction and also names involving addition of three numbers. For example, names for 20 include: 16 + 4; 30 - 10; 4 + 4 + 12; 21 + $\hat{1}$; 4 × 5; $\frac{1}{2}$ × 40.

18) Ask for several subtraction problems with the same answer.

Multiplication and Division

The first seventeen examples are lists of calculations to be done mentally. Each list suggests a pattern in arithmetic which students should begin to use in doing such calculations.

1)

3 + 3
2 × 3
5 + 5
2 × 5

7 + 7
2 × 7
24 + 24
2 × 24

100 + 100
2 × 100
200 + 200
2 × 200

4 + 4 + 4
3 × 4
5 + 5 + 5
3 × 5

8 + 8 + 8
3 × 8
12 + 12 + 12
3 × 12

2 + 2 + 2 + 2
4 × 2
11 + 11 + 11 + 11
4 × 11

2)

1 × 5
2 × 5
3 × 5
4 × 5

1 × 20
2 × 20
3 × 20
4 × 20

5 × 3
6 × 3
7 × 3
8 × 3

3)

2 × 3
2 × 30
2 × 300
2 × 3,000
2 × 3,000,000

2 × 7
2 × 70
2 × 700
2 × 7,000
2 × 7,000,000

3 × 6
3 × 60
3 × 600
3 × 6,000
3 × 6,000,000

4)

2 × 4
2 × 44
2 × 444
2 × 4,444

3 × 3
3 × 33
3 × 333
3 × 3,333

2 × 5
2 × 55
2 × 555
2 × 5,555

5) 2 × 3
3 × 2
2 × 30
30 × 2

8 × 5
5 × 8
8 × 50
50 × 8

2 × 16
16 × 2
3 × 12
12 × 3

6) 2 × 10
2 × 9
2 × 20
2 × 19

2 × 50
2 × 49
2 × 100
2 × 99

3 × 10
3 × 9
3 × 20
3 × 19

7) 2 × 10
2 × 3
2 × 13

2 × 40
2 × 2
2 × 42

2 × 20
2 × 6
2 × 26

2 × 60
2 × 4
2 × 64

2 × 100
2 × 7
2 × 107

2 × 200
2 × 30
2 × 1
2 × 231

8) 10 × 5
10 × 50
10 × 500

10 × 8
10 × 80
10 × 800

10 × 12
10 × 120
10 × 1,200

20 × 3
20 × 30
20 × 300

20 × 5
20 × 50
20 × 500

20 × 8
20 × 80
20 × 800

9) 6 + 6
2 × 6
$\frac{1}{2} \times 12$

10 + 10
2 × 10
$\frac{1}{2} \times 20$

50 + 50
2 × 50
$\frac{1}{2} \times 100$

$11 + 11 + 11$
3×11
$\frac{1}{3} \times 33$

$20 + 20 + 20$
3×20
$\frac{1}{3} \times 60$

$13 + 13 + 13 + 13$
4×13
$\frac{1}{4} \times 52$

10) $\frac{1}{2} \times 10$
$\frac{1}{2} \times 100$
$\frac{1}{2} \times 1{,}000$
$\frac{1}{2} \times 10{,}000$
$\frac{1}{2} \times 10{,}000{,}000$

$\frac{1}{2} \times 24$
$\frac{1}{2} \times 240$
$\frac{1}{2} \times 2{,}400$
$\frac{1}{2} \times 24{,}000$
$\frac{1}{2} \times 24{,}000{,}000$

$\frac{1}{3} \times 6$
$\frac{1}{3} \times 60$
$\frac{1}{3} \times 600$
$\frac{1}{3} \times 6{,}000$
$\frac{1}{3} \times 6{,}000{,}000$

11) $\frac{1}{2} \times 20$
$\frac{1}{2} \times 30$
$\frac{1}{2} \times 40$
$\frac{1}{2} \times 50$

$\frac{1}{2} \times 6$
$\frac{1}{2} \times 16$
$\frac{1}{2} \times 26$
$\frac{1}{2} \times 36$

$\frac{1}{2} \times 10$
$\frac{1}{2} \times 100$
$\frac{1}{2} \times 108$
$\frac{1}{2} \times 148$

12) 2×13
$\frac{1}{2} \times 26$

3×10
$\frac{1}{3} \times 30$

2×35
$\frac{1}{2} \times 70$

3×8
$\frac{1}{3} \times 24$

2×104
$\frac{1}{2} \times 208$

3×105
$\frac{1}{3} \times 315$

13) 2×43
$86 \div 2$

3×20
$60 \div 3$

2×17
$34 \div 2$

3×6
$18 \div 3$

2×108
$216 \div 2$

3×201
$603 \div 3$

14)	$\frac{1}{2} \times 40$	$\frac{1}{2} \times 100$	$\frac{1}{3} \times 90$
	$\frac{1}{2} \times 8$	$\frac{1}{2} \times 12$	$\frac{1}{3} \times 6$
	$\frac{1}{2} \times 48$	$\frac{1}{2} \times 112$	$\frac{1}{3} \times 96$
15)	$20 \div 2$	$100 \div 2$	$30 \div 3$
	$6 \div 2$	$10 \div 2$	$9 \div 3$
	$26 \div 2$	$110 \div 2$	$39 \div 3$
16)	$\frac{1}{2} \times 14$	$\frac{1}{2} \times 7$	$\frac{1}{2} \times 100$
	$\frac{1}{2} \times 15$	$\frac{1}{2} \times 9$	$\frac{1}{2} \times 101$
	$\frac{1}{2} \times 20$	$\frac{1}{2} \times 11$	$\frac{1}{2} \times 102$
	$\frac{1}{2} \times 21$	$\frac{1}{2} \times 13$	$\frac{1}{2} \times 103$
17)	2×3	2×30	3×3
	2×2	2×20	3×2
	2×1	2×10	3×1
	2×0	2×0	3×0
	$2 \times \hat{1}$	$2 \times \widehat{10}$	$3 \times \hat{1}$
	$2 \times \hat{2}$	$2 \times \widehat{20}$	$3 \times \hat{2}$
	$2 \times \hat{3}$	$2 \times \widehat{30}$	$3 \times \hat{3}$

18) Ask for a multiple of 3 (or 4) between two other numbers; for example, ask for a multiple of 3 between 20 and 25.

Estimation

Ask the students to estimate a calculation and to explain their answers. For example,

T: Is 218 + 157 more than 300? How do you know?

S: Yes, because 200 + 100 = 300.

T: Is it more than 500? How do you know?

S: No, because 300 + 200 = 500. 218 is less than 300 and 157 is less than 200, so 218 + 157 must be less than 500.

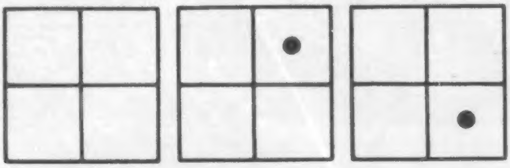
T: Is it more than 400?

S: No, because 18 + 57 is less than 100.

During the lessons, students are often asked to estimate a number displayed on the Minicomputer.

Other Mental Arithmetic Activities

1) Use the Minicomputer to give visual clues for addition, subtraction, and multiplication problems. For example, suppose this configuration is on the Minicomputer.



T: What number is on the Minicomputer? (41)

366

Hold a checker over the 8-square.

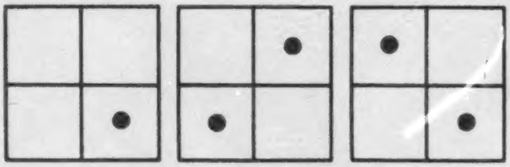
T: What number is 41 + 8? (49)

Hold a checker over the 10-square.

T: What number is 41 + 10? (51)

After a few such clues, hold a checker over the square only if the class has difficulty.

The next example involves subtraction.



T: What number is on the Minicomputer? (169)

Lift the checker temporarily from the 8-square.

T: What number is 169 - 8? (161)

Lift the checker temporarily from the 20-square.

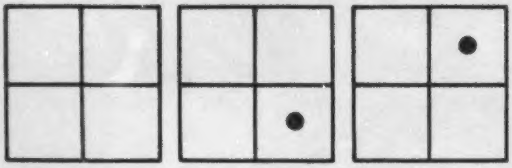
T: What number is 169 - 20? (149)

Lift the checkers temporarily from the 20-square and the 1-square.

T: What number is 169 - 21? (148)

After a few such clues, lift the checkers only if the class has difficulty.

The next example involves multiplication.



T : What number is on the Minicomputer? (14)

Lift the checkers.

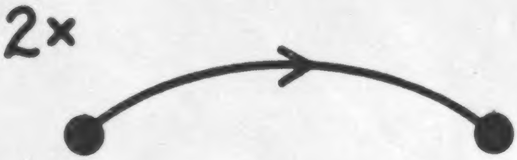
T: What number is 10 × 14?

S: 140.

Move the checker on the 10-square to the 100-square; move the checker on the 4-square to the 40-square.

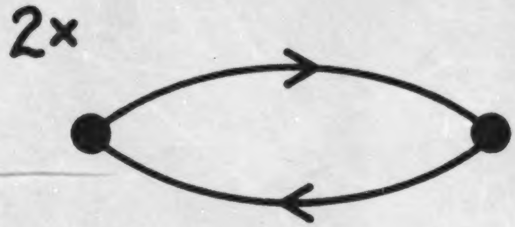
Repeat this activity starting with other starting configurations.

2) Use an arrow picture to pose mental arithmetic questions. For example, draw this arrow picture on the board.



Ask what number is at the right if the number at the left is, for example, 7.

Repeat this activity several times and then draw the return arrow.



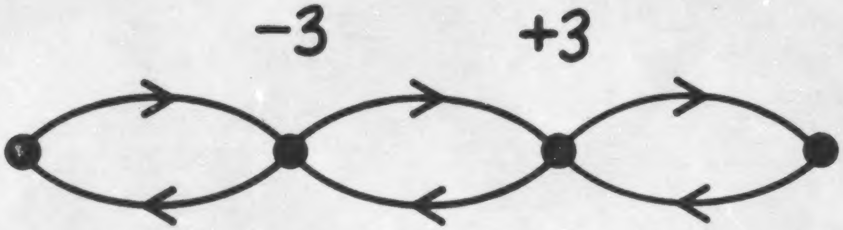
T: What could this blue arrow be for?

S: $\frac{1}{2}$x.

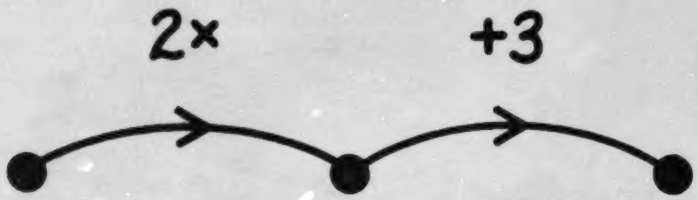
Ask the students what number is at the left if the number at the right is, for example, 10. Repeat this activity several times.

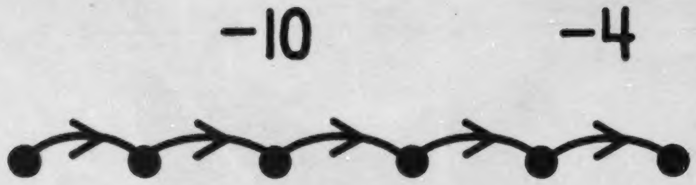
There are many variations of this activity; for example,

- use larger starting and ending numbers;
- use functions other than 2x and $\frac{1}{2}$x;
- use two or more arrows of the same function;



- use a composition of two or more functions (two examples are shown below).





3) Use the suggestion of an arrow picture to pose place value questions.

- Imagine an arrow road starting at 0 and having four +10 arrows and seven +1 arrows. What is the ending number?
- Imagine an arrow road starting at 0 and having two +100 arrows, five +10 arrows, and two +1 arrows. What is the ending number?

4) Play "Guess My Rule". Choose a secret rule such as 3×. A student suggests a number to you and you answer with 3× that number. This continues until the students discover your rule; for example,

S: 5.

T: 15.

S: 1.

T: 3.

S: 100.

T: 300.

S: 50.

T: 150. Does anyone know my secret rule?

S: 3 ×.

T: How do you know?

S: Each time you answered a number which was 3 × the number we gave you.

Choose other secret rules such as +3, +5, -2, -4, $\frac{1}{2}$×, 4 ×, and 10 ×.

Hand-Calculator Activities

Use the hand-calculator (i) to develop and reinforce counting skills; (ii) to emphasize counting patterns; and (iii) to improve estimation skills.

1) Use one or more hand-calculators in the class to support counting activities in which patterns are generated. Start with any whole number on the display of your calculator and press $\boxed{+}$ $\boxed{5}$ $\boxed{=}$ $\boxed{=}$ $\boxed{=}$ . . . . Record on the board the number that appears on the display of the hand-calculator each time you press $\boxed{=}$. As soon as most of your students are able to predict the sequence quickly, you can abandon the calculator. Ask students to explain the pattern.

There are many variations of this activity, such as:

- press $\boxed{+}$ $\boxed{2}$ (or $\boxed{3}$ or $\boxed{4}$ or $\boxed{1}$ $\boxed{0}$) $\boxed{=}$ $\boxed{=}$ $\boxed{=}$ . . . ;
- start with a large number and press $\boxed{-}$ $\boxed{5}$ (or $\boxed{2}$ or $\boxed{1}$ $\boxed{0}$ or $\boxed{1}$ $\boxed{0}$ $\boxed{0}$) $\boxed{=}$ $\boxed{=}$ $\boxed{=}$ . . . .

2) Play "Let's Concentrate" using one or more hand-calculators in the class. Ask the students who have the calculators to put a number of your choice on the display and then to hide the display with one hand. Give a sequence of two operations for the students to do by pressing the appropriate keys and without looking at the display. Then ask what number is on the display and let the students check to see if they are correct. Continue with the new number and another sequence of two operations. For example,

Start with 4 on the display.

Press $\boxed{+}$ $\boxed{3}$ $\boxed{+}$ $\boxed{2}$ $\boxed{=}$. The resulting number is 9.
(Students check that 9 is on the display.)

Press $\boxed{+}$ $\boxed{1}$ $\boxed{\times}$ $\boxed{2}$ $\boxed{=}$. The resulting number is 20.
(Students check that 20 is on the display.)

Press $\boxed{-}$ $\boxed{3}$ $\boxed{+}$ $\boxed{1}$ $\boxed{=}$. The resulting number is 18.
(Students check that 18 is on the display.)

3) Play "Hand-Calculator Golf" with a small group of students, each one having a calculator. Ask the students to put a small whole number of your choice on the display of the calculators. Then let the students take turns suggesting some numbers to add or to subtract until a particular goal, such as 200, is reached.

A possible game in which 200 is the goal and four students are playing is described below.

Start with 12 on the display.		12
First player:	+ 9	21
Second player:	+72	93
Third player:	+17	110
Fourth player:	+ 7	117
First player:	+93	210
Second player:	-10	200

Some variations of this game include:

- changing the starting number;
- making the goal a larger multiple of 100;
- allowing the use of $\boxed{\times}$ $\boxed{\text{some number between 0 and 10}}$.

mprehensive School Mathematics Program
CEMREL, Inc.
Final Experimental Version
3-44201

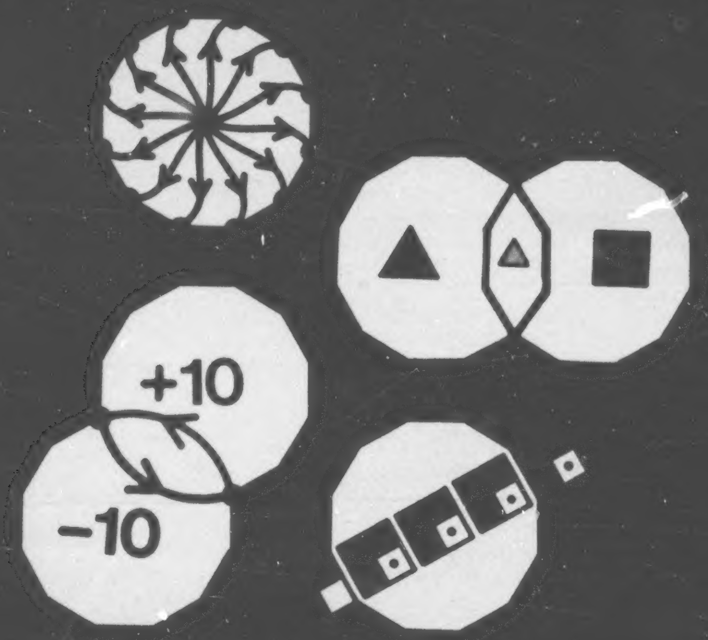
CSMP Mathematics for the Upper Primary Grades

SF

Part IV

Teacher's Guide

The Languages of Strings and Arrows
Geometry and Measurement
Workbooks



CEMREL, Inc.
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THE LANGUAGES OF STRINGS AND ARROWS

Comprehensive School Mathematics Program

CSMP Mathematics
for the
Upper Primary Grades
Part IV

The Languages of Strings and Arrows

CEMREL, Inc.

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THE LANGUAGES OF STRINGS AND ARROWS

INTRODUCTION

The language of strings is the nonverbal language of sets, that is, collections of objects put together in some way. The language of arrows is the nonverbal language of relations among objects. These languages permeate the CSMP content strands and are used, separately or together, to present an unending variety of numerical and nonnumerical situations.

Classification: The Language of Strings

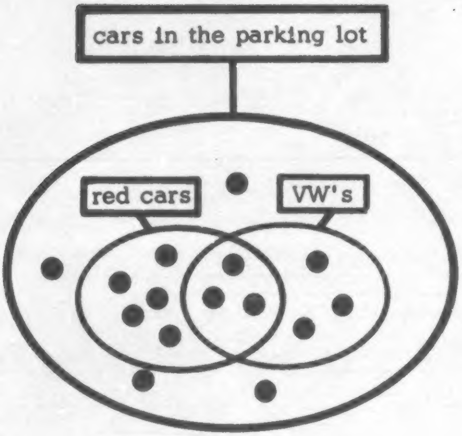
As the word implies, classifying means putting things into classes, or as the mathematician says: sets. If the mathematics of sets has any place in an elementary school mathematics curriculum, it is precisely because it can help students to understand and use the ideas of classification.

The basic idea is simple: given a set S and any object x, either x belongs to S (x is in S) or x does not belong to S (x is not in S). We represent this simple act of sorting--"in" or "out"--neatly by using string pictures to illustrate "in" and "out" in a dramatic way. Objects to be sorted are represented by dots, and the sets into which they are sorted are represented by drawing "strings" around dots. A dot inside (the region delineated by) a set's string is for an object in the set and a dot outside a set's string is for an object not in the set.

This language of strings and dots provides a precise (and nonverbal) way of recording and communicating thoughts about classification. The abilities to classify, to reason about classification, and to extract information from a classification are important skills for everyday life, for intellectual activity, in general, and for the pursuit and understanding of mathematics, in particular. Ordinarily, we are not expected to deal with such problems until our command of our natural language is fully developed and until we face classification problems in adult life. It is not that an ability to classify requires special intelligence, but that it requires a special mode of thinking. The unique quality of the language of strings is that it provides a nonverbal language that is particularly suited to the mode of thinking involved in classification. It frees young minds to think logically and creatively about situations which involve classifying and to report their thinking long before they have extensive verbal skills.

In this strand we present situations and ask carefully phrased questions to continue to advance skills in classification, always remembering that the skills grow out of such experiences. To be able to draw strings and dots is not an objective in itself; to develop the mode of thinking involved is the objective. Thus it is important for us to construct the situations carefully. The sets into which we ask students to classify objects must be determined by well-defined attributes; otherwise, the students have the added problem of deciding whether a certain object does or does not have a certain attribute. For this reason we make extensive use of A-Blocks (Attribute Blocks) and particular sets of numbers in designing classification situations. Students can immediately say whether or not a block is red, whether or not it is a square, and so on. They know whether or not a given number is less than 15, whether or not it is odd, and so on.

One reason for classifying objects is to count the objects that have a certain attribute. Suppose for example, in a parking lot there are eight red cars and six Volkswagons. If four cars in the lot are neither red nor Volkswagons, must there be eighteen cars in the lot? Suppose we count the cars in the parking lot and find there are exactly fifteen. Is this possible? A string picture immediately settles this apparent paradox.



There are three red Volkswagons.

Relations: The Language of Arrows

Relations are interesting and important to us in our everyday lives, in our careers, in school, and in scientific pursuits. We are always trying to establish, explore, and understand relations. In mathematics it is the same; to study mathematics means to study the relations among mathematical objects like numbers or geometric shapes. The tools that we use to understand the everyday world are useful to us in the world of mathematics. Conversely, the tools that we develop to help us think about mathematical things often serve us in nonmathematical situations.

A serious study of anything requires a language for representing the things under investigation. The language of arrows provides an apt language for studying and talking about relations. Arrow diagrams are a handy graphical representation of a relation, somewhat the same way that a blueprint is a handy graphical representation of a house. By means of arrow diagrams, we can represent important facts about a given relation in a simple, suggestive, pictorial way--usually more conveniently than the same information could be presented in words.

The convenience of arrow diagrams has important pedagogical consequences for introducing children to the study of relations in the early grades. A child can "read"--and also draw--an arrow diagram of a relation long before he or she can read or present the same information in words. The difficulty of presenting certain ideas to children lies not in their intellectual inability to grasp the ideas; rather, the limitations are often mechanical, motor problems of writing or reading about the ideas in conventional systems. Arrow diagrams have all the virtues of a good notation: they present information in a clear, natural way; they are attractive, colorful things to look at; they are easy and fun for children to draw. Another educational bonus occurs when arrow diagrams are used imaginatively. A story or problem may be captured by an arrow diagram, and at the same time, the resulting arrow diagram may suggest other, similar situations. This allows students to call upon previous experiences or to expand their creativity.

One of the purposes of this strand is to exploit the power of the language of arrows to help children think logically about relations. Again it must be remembered that our goals concern the thinking process and not the mechanism. We resist making the ability to draw prescribed arrows an objective in itself, just as we resist viewing an arrow diagram as just another format for drill problems in arithmetic.

The general aim of The Languages of Strings and Arrows strand is to suggest situations that are inherently interesting and thought provoking, and to give children modes of thinking and appropriate languages with which they can organize, classify, and analyze. In addition to a varied assortment of lessons concerning sets and relations, this strand includes lessons involving systematic methods for solving combinatorial (counting) problems; methods for collecting, recording, and interpreting data in "real life" situations; and probabilistic thinking.

L1 Permutations and Common Multiples

CAPSULE LESSON SUMMARY

Recall the changing seats game and examine an arrow picture for nine people playing the game. Decide how many rounds of the game need be played before all the players are back in their original chairs. Find that the result is a common multiple of certain numbers which are given by the lengths of cycles in the arrow picture.

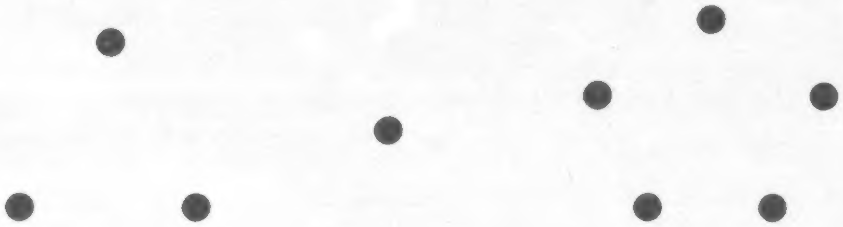
MATERIALS

Teacher: Colored chalk
Student: Paper; pencil

DESCRIPTION OF LESSON

Exercise 1

On the board, draw nine large dots in this arrangement.



T: Do you remember the game of changing seats? You had to change seats according to an arrow picture. We are not going to play the game today — just talk about it.

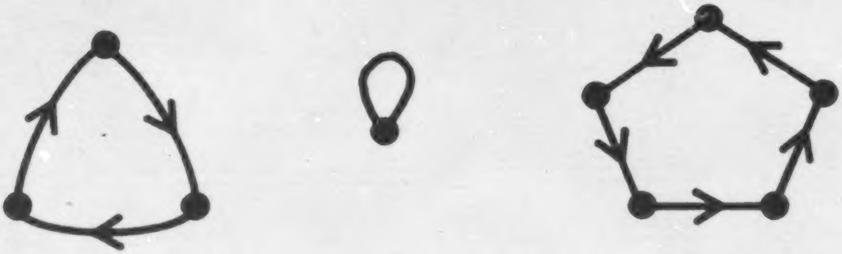
These dots are for the chairs in which some children are sitting. How many dots are there?

S: Nine.

L1

T: I'll draw some arrows to show you how the children in these nine chairs can change seats.

Draw these red arrows in your picture.



T (pointing to one of the dots): Where will the person who starts here go in Round 1?

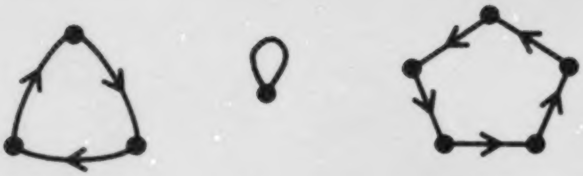
Invite a student to point to the appropriate dot. Repeat this question several times, referring to other dots. Then vary the question slightly as follows.

T (pointing to one of the dots): Imagine the children have played three rounds of the game. Where will the person from this chair be?

Let a student point to the appropriate dot. For the benefit of the class, check the students' response by tracing three arrows in succession starting at the original dot being considered. Repeat this question several times, referring to other dots.

T: After how many rounds will all the children have returned to their own chairs?

Record the students' guesses on the board. The guesses from one of our development classes are given on the right in the next illustration.



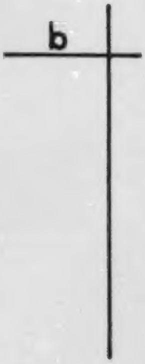
guesses

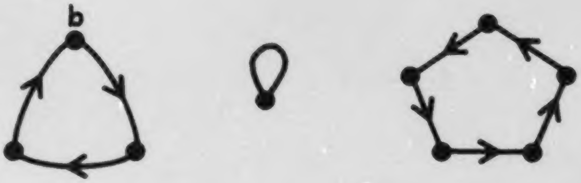
0 rounds
15 rounds
6 rounds
1,005 rounds
17 rounds
8 rounds
10 rounds

NOTE: If no one mentions zero rounds, suggest it yourself.

T: Let's see if we can answer this question completely.

Draw a chart on the board and label one of the dots in the cycle on the left "b".





T (pointing to "b"): The person who starts here will return after how many rounds?

S: Three.

Trace the cycle of three arrows starting at b and returning to b and then record "3" in the chart as you say:

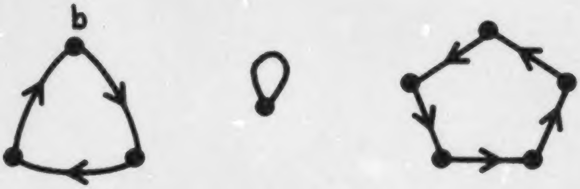
L1

T: Three rounds is correct. Are there any other times?

S: After six rounds.

Trace the cycle of three arrows twice starting at b and returning to b. Record "6" in the chart. Accept many solutions. (Be sure to include zero rounds.)

b	
0	
3	
6	
9	
12	
15	
18	
⋮	

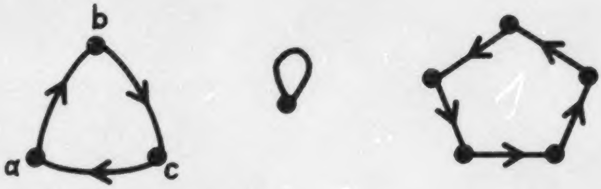


T: What do we call these numbers?

S: Multiples of 3.

In the same manner consider the other two dots in the cycle with b, labeling them "a" and "c". When it is clear that the situation is the same for a, b, and c, add "a" and "c" to your chart heading.

a, b, c	
0	
3	
6	
9	
12	
15	
18	
⋮	



Label the dot which is by itself "d" and extend your chart with another column.

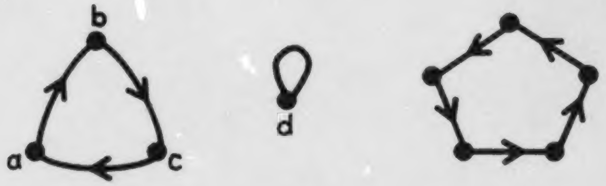
T (pointing to "d"): The person who starts here will return after how many rounds?

S: After zero, one, two, three, four, . . . rounds.

S: Any number of rounds.

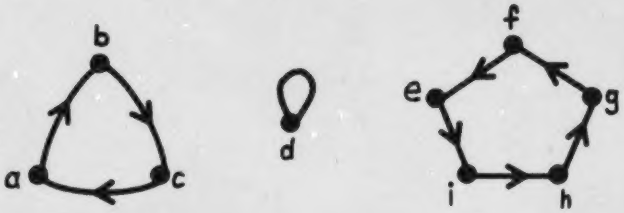
Indicate this in your chart.

a,b,c	d
0	0
3	1
6	2
9	3
12	4
15	5
18	⋮
⋮	⋮



Repeat this activity, considering each dot in the cycle on the right. Conclude that the children starting in those chairs will return after any round numbered with a multiple of 5.

a,b,c	d	e,f,g,h,i
0	0	0
3	1	5
6	2	10
9	3	15
12	4	20
15	5	25
18	⋮	⋮
⋮	⋮	⋮



L 1

Consider each guess recorded earlier in the lesson.

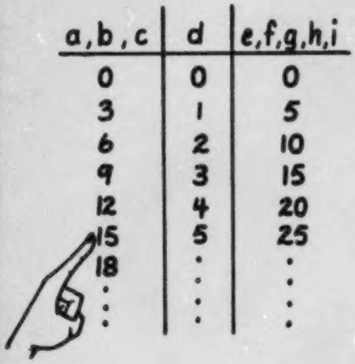
0 rounds
15 rounds
6 rounds
1,005 rounds
17 rounds
8 rounds
10 rounds

T: After how many rounds will all the children have returned to their own chairs? . . . after 0 rounds?

S: Yes.

T: . . . after 15 rounds?

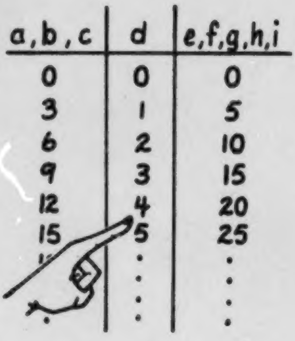
Check 15 in this manner.



a,b,c	d	e,f,g,h,i
0	0	0
3	1	5
6	2	10
9	3	15
12	4	20
15	5	25
18	⋮	⋮
⋮		

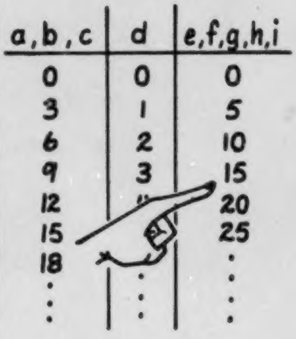
T: Is 15 a multiple of 3?

S: Yes.



a,b,c	d	e,f,g,h,i
0	0	0
3	1	5
6	2	10
9	3	15
12	4	20
15	5	25
18	⋮	⋮
⋮		

T: d is always in the same chair.



a,b,c	d	e,f,g,h,i
0	0	0
3	1	5
6	2	10
9	3	15
12	4	20
15	5	25
18	⋮	⋮
⋮		

T: Is 15 a multiple of 5?

S: Yes.

T: . . . after 6 rounds?

S: No, 6 is not a multiple of 5.

Erase "6" from the list of guesses.

Continue in this manner until each guess has been considered. If a number is not both a multiple of 3 and a multiple of 5, erase it from the list.

T (pointing to the correct guesses): Does anyone know another good answer now?

Check students' answers and record each correct number of rounds in the list. Arrange, and if necessary reorganize, your list so that the numbers can be listed in increasing value. Once a pattern is established, ask the students to continue the sequence for several numbers. You should have a list similar to this one.

0 rounds
15 rounds
30 rounds
45 rounds
60 rounds
75 rounds
80 rounds
105 rounds
120 rounds
⋮
1,005 rounds
⋮

T: What do we call these numbers?

S: Multiples of 15.

L2 A Detective Story

CAPSULE LESSON SUMMARY

Solve a detective story about a secret number where the clues involve (i) finding all the numbers which can be named using the symbols "3", "5", "4", "+", "x", "(", ")", each exactly once; (ii) locating the secret number in a three-string picture where the strings are for multiples of 3, 4, and 5; (iii) observing that the secret number is on the same +5 arrow road as 3.

MATERIALS

Teacher: Colored chalk
Student: Paper; pencil; Worksheet L2* and **

DESCRIPTION OF LESSON

Write "17" on a slip of paper and then fold the paper so that the "17" is hidden.

T: The name of a secret number is written on this paper. I'll give you some clues and we'll see if you are good detectives!

First clue

Write these symbols on the board.

3 5 4

+ × ( )

T: The secret number can be written using each of these symbols exactly once.

Ask the students to find possibilities and to write them on a piece of paper.

Observe the students' work and as soon as you see a correct calculation using only the above symbols, draw it to the class' attention. Suppose a student named "Nicole" has a correct solution.

T: Nicole found a number that the secret number could be.
Nicole used the symbols in this way (write them on the board).

$$(3+4)\times 5$$

T: Who knows which number this names?

S: 35, because 3 + 4 = 7 and 7 × 5 = 35.

$$(3+4)\times 5=35$$

Let the students continue to work individually. Periodically draw the class' attention to a particular students' combination and ask the rest of the class which number the expression names. Continue until all six possibilities have been found and are listed on the board.

$$\begin{aligned}(3+4)\times 5 &= 35\\ 3+(4\times 5) &= 23\\ (3\times 4)+5 &= 17\\ 3\times(4+5) &= 27\\ (3+5)\times 4 &= 32\\ (3\times 5)+4 &= 19\end{aligned}$$

Students may think they have found another combination and suspect it will give a new possibility; for example, if a student suggests . . .

$$(4+3)\times 5$$

when . . .

$$(3+4)\times 5=35$$

has already been given, use the opportunity to comment on commutativity (do not use this word with the students). Students should conclude:

$$(4+3)\times 5=(3+4)\times 5$$

because

$$4+3=3+4$$

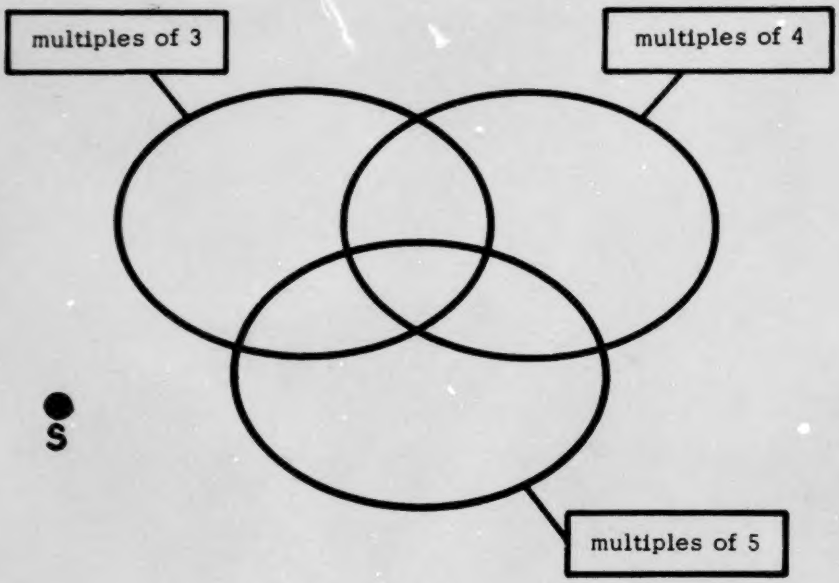
T: The secret number could be 35, 23, 17, 27, 32, or 19.

Erase all but these six numbers on the board.

35
23
17
27
32
19

Second clue

Draw this string picture on the board.



T: The second clue is that the secret number is outside all three of these strings. What are some numbers that the secret number could be?

S: 19.

T: How do you know? Is 19 a multiple of 3 (trace the red string)?

S: No, 15 is a multiple of 3, 18 is the next multiple of 3, and 21 is the next multiple of 3 after 18. 19 is skipped.

T (tracing the blue string): Is 19 a multiple of 4?

S: No, 20 is a multiple of 4 so 19 cannot be. It is only 1 less than 20.

T (tracing the green string): Is 19 a multiple of 5?

S: No, 19 does not end in 0 or in 5.

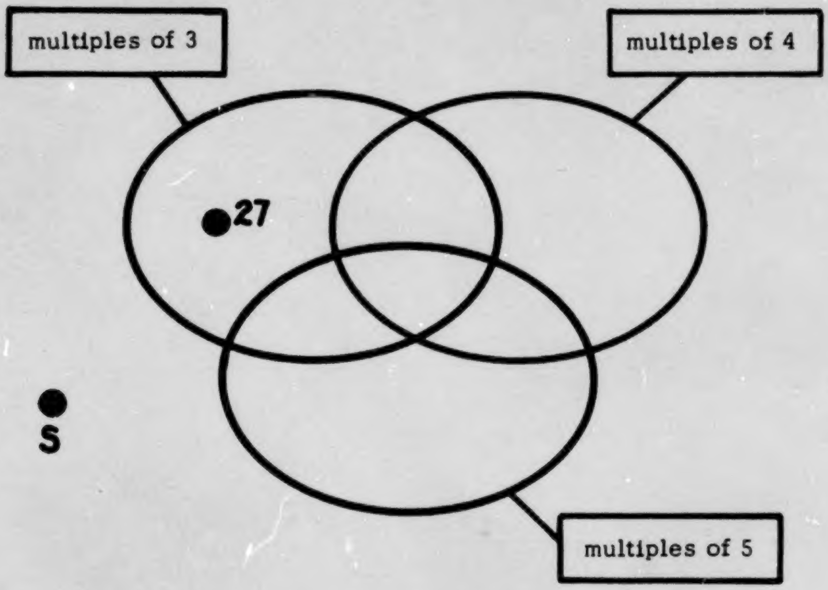
Circle "19" in the list on the board.

35
23
17
27
32
⑲

If a student suggests a number that cannot be the secret number, very likely many students will disagree immediately. Suppose someone suggests 27, for example.

T: Where does 27 belong in this picture?

Ask a student (other than the one who suggested 27) to locate 27 in the picture.


multiples of 3
multiples of 4
27
S
multiples of 5

T (tracing the red string): How do you know that 27 is a multiple of 3?

S: 30 is a multiple of 3 and I counted three backwards from 30.

T (tracing the blue string): How do you know that 27 is not a multiple of 4?

S: I counted by fours from 20: 20, 24, 28. 27 is skipped.

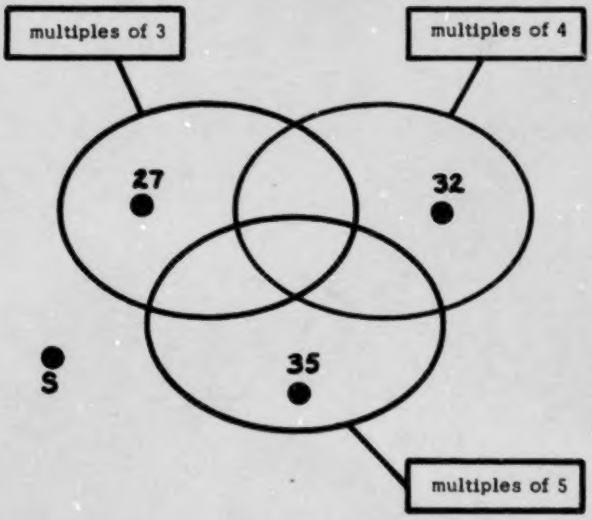
T (tracing the green string): How do you know that 27 is not a multiple of 5?

S: 27 does not end in 0 or in 5.

Eliminate 27 from the list of possibilities.

Continue until the class concludes that the secret number could be 19, 17, or 23. Consider any numbers in the list which have not been circled or crossed out and ask students to locate them in the string picture.

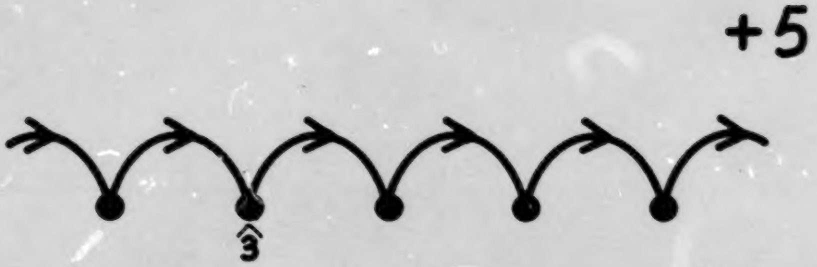
~~35~~
㉓
⑰
~~27~~
~~32~~
⑲



L 2

Third clue

Draw this arrow picture on the board.

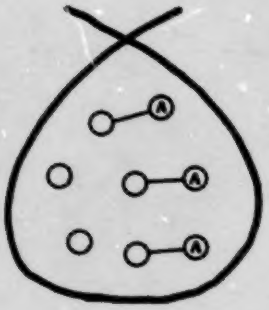


T: The secret number is on a +5 arrow road which meets the number $\hat{3}$. I only drew part of the road because it goes on and on in both directions. Which is the secret number? Is it 23, 17, or 19? Write it on your paper.

Walk around the room acknowledging correct answers.

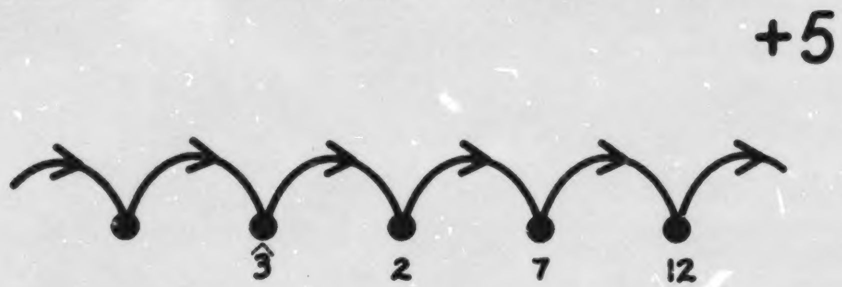
T: Who can convince us that one of these numbers (23, 17, 19) is on this arrow road?

Perhaps a student will ask to label the dots to the right of $\hat{3}$ to show where 17 occurs. Use Eli's bag of peanuts, if necessary, to show that $\hat{3} + 5 = 2$.



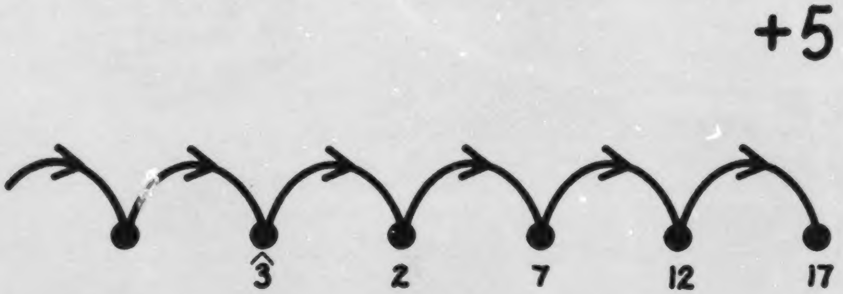
$$\hat{3} + 5 = 2$$

Continue until the three dots to the right of $\hat{3}$ have been labeled.



S: The next number (to the right) is 17.

Finish drawing the arrow and label its ending dot 17.



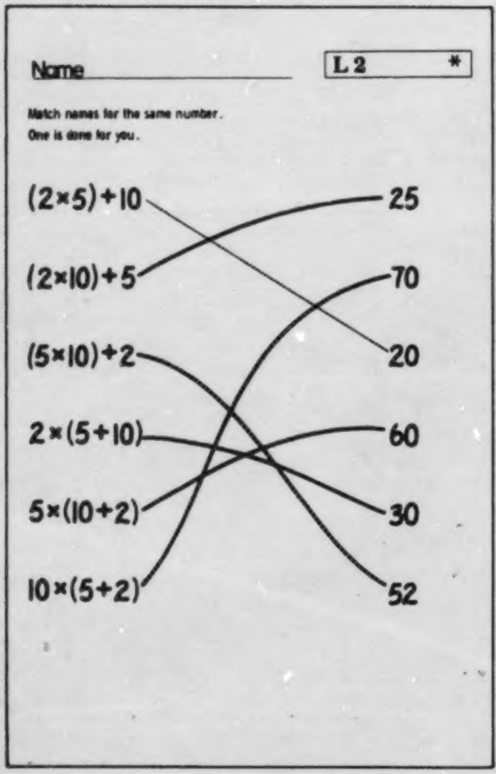
T: How can we be sure that neither 23 nor 19 can be the secret number?

S: The next number we meet is 22, because 17 + 5 = 22. We skip 19.

S: The next number after 22 we meet is 27, because 22 + 5 = 27. We skip 23.

Reveal that "17" is written on the slip of paper.

Worksheets L2* and ** are available for individual work.

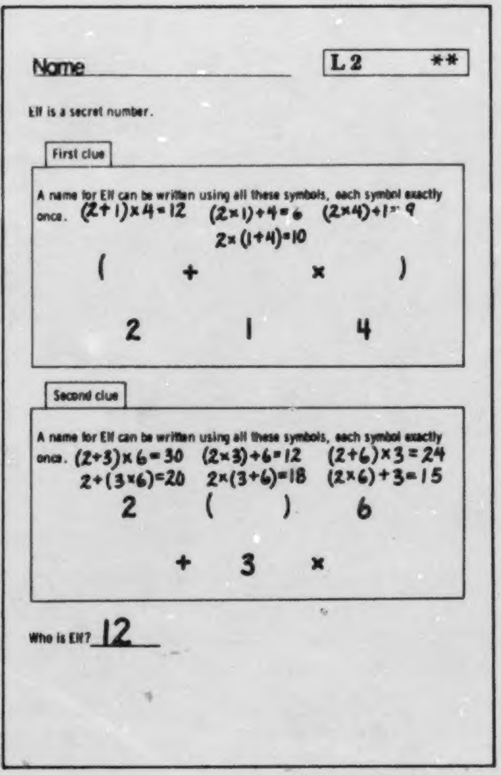


Name

L 2 *

Match names for the same number.
One is done for you.

(2×5)+10	25
(2×10)+5	70
(5×10)+2	20
2×(5+10)	60
5×(10+2)	30
10×(5+2)	52



Name

L 2 **

Elf is a secret number.

First clue

A name for Elf can be written using all these symbols, each symbol exactly once. (2+1)×4=12 (2×1)+4=6 (2×4)+1=9
2×(1+4)=10

(+ ×)

2 1 4

Second clue

A name for Elf can be written using all these symbols, each symbol exactly once. (2+3)×6=30 (2×3)+6=12 (2+6)×3=24
2+(3×6)=20 2×(3+6)=18 (2×6)+3=15

2 ( ) 6

+ 3 ×

Who is Elf? 12

L3 The String Game #1

CAPSULE LESSON SUMMARY

Play the String Game twice, once with two strings and once with three strings.

MATERIALS

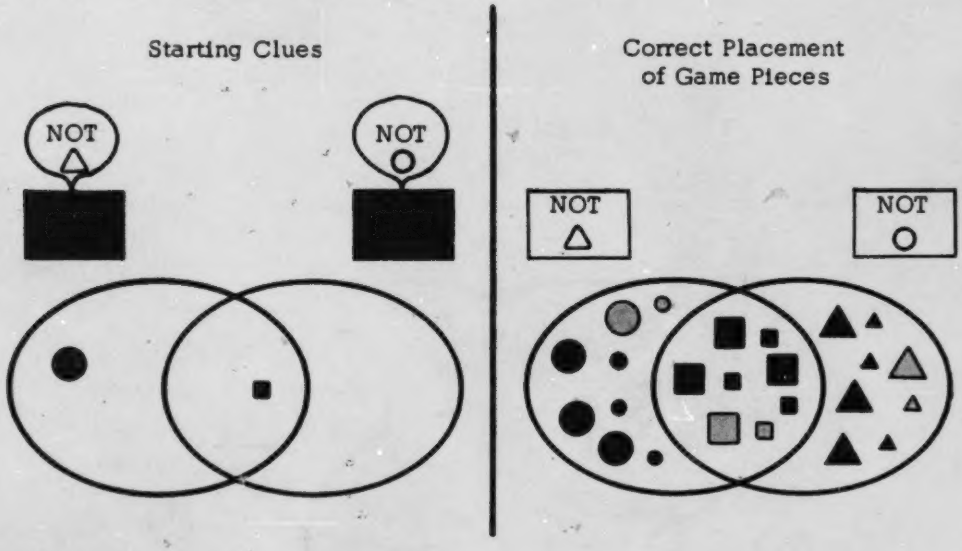
Teacher: String Game with A-Blocks kit; colored chalk
Student: None

DESCRIPTION OF LESSON

Prepare to play the String Game by setting up the team board and taping a list of the sixteen possible string labels above it. Divide the class into two teams and distribute the game pieces on the team board.

Exercise 1

Play a two-string version of the String Game.* A game is suggested below. The illustration on the left shows two possible starting clues. The illustration on the right shows correct placement of all twenty-four A-Block pieces and can be used as a crib-sheet during the game.



Starting Clues

Correct Placement of Game Pieces

---

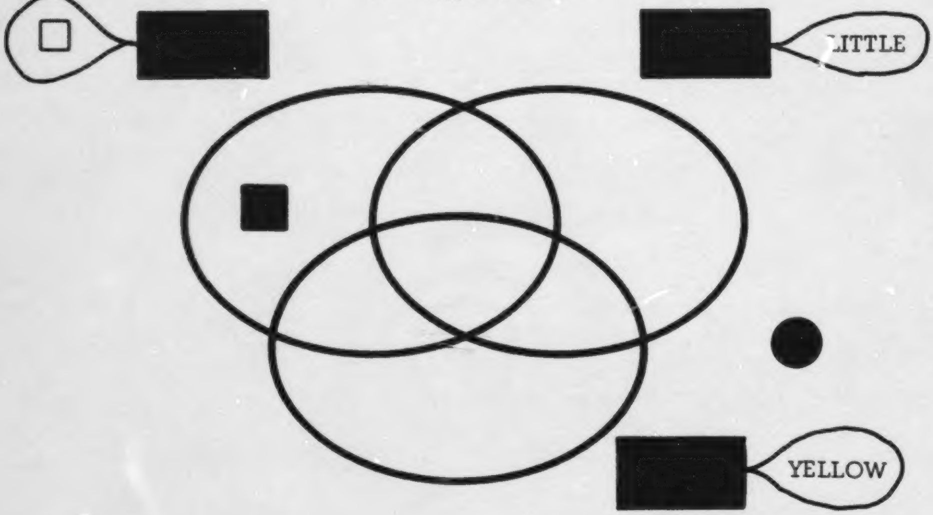
* See the appendix THE STRING GAME for a description of the various versions of the game. This appendix also provides an explanation of how the game is played.

L 3

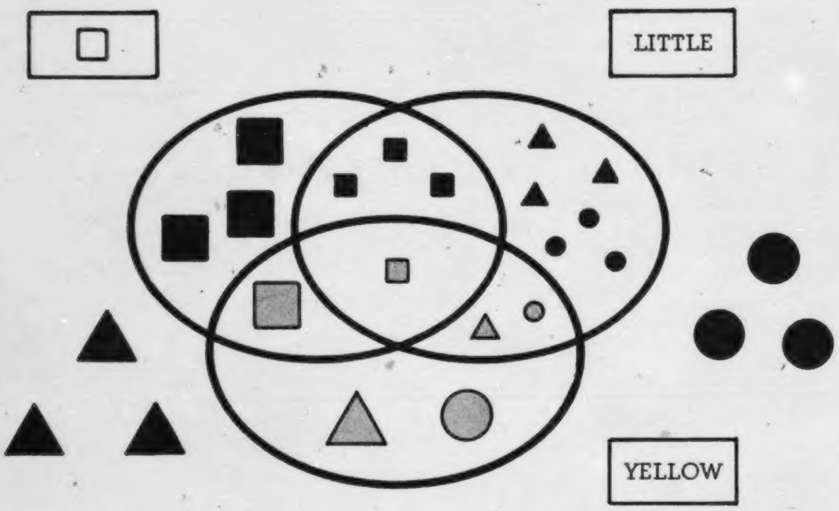
Exercise 2

Play a three-string version of the String Game. A game is suggested below. The first illustration shows two possible starting clues. The second illustration shows correct placement of all the A-Block pieces and can be used as a crib-sheet during the play of the game.

Starting Clues



Correct Placement of Game Pieces



L4 Common Multiples/Composition Games

CAPSULE LESSON SUMMARY

Find common multiples of 2, 4, and 6 by examining a changing seats game with twelve players in which there is a 2-cycle, a 4-cycle, and a 6-cycle. Play a composition game with the rule "blue followed by blue is red."

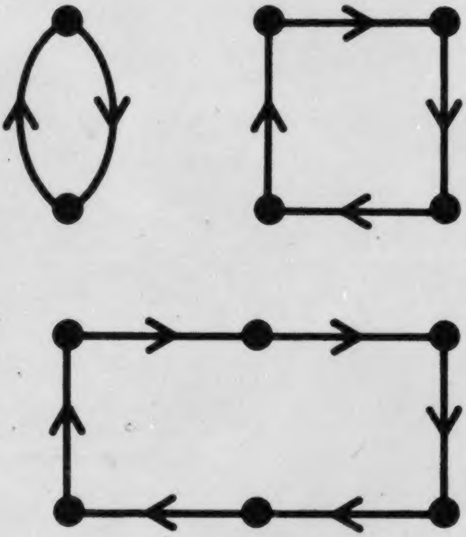
MATERIALS

Teacher: Colored chalk
Student: Worksheets L4* and **; colored pencils

DESCRIPTION OF LESSON

Exercise 1: Changing Seats

Draw this arrow diagram on the board.



T: These dots are for the chairs in which some children are sitting. How many chairs are there?

S: Twelve.

L4

T: The arrows show you how these children play the game of changing seats. After how many rounds will all the children have returned to their own chairs?

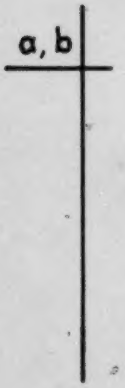
Record the students' guesses on the board. You may get only a couple of different numbers. In one of our development classes only eight and twelve were suggested.

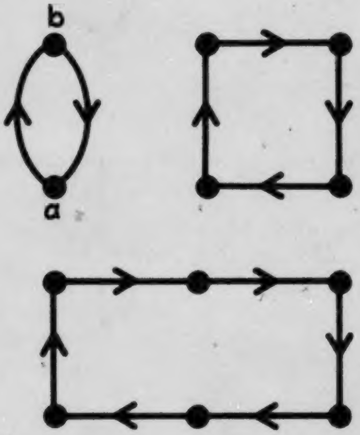
T: Can you explain why you think a certain number is correct?

S: I think that after twelve rounds they will all be back in their seats, because 12 is a multiple of 2, a multiple of 4, and a multiple of 6.

Students also might suggest twelve for the erroneous reason that there are twelve dots. If no one suggests twelve for a correct reason, simply continue with the lesson.

In the cycle in the upper left, label the dots "a" and "b" and draw this chart on the board.

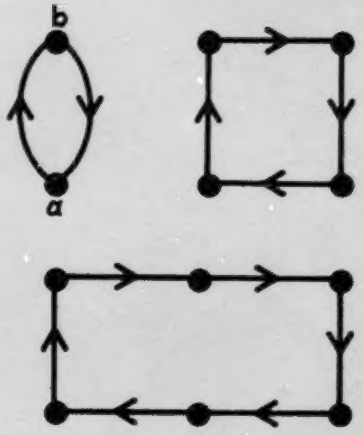




T: The people who start here (point to "a" and then to "b") will return after how many rounds?

As students suggest numbers, record them in the chart in increasing value. If no one mentions zero rounds, tell them that a number is missing at the beginning of the list.

a, b
0
2
4
6
8
10
12
⋮

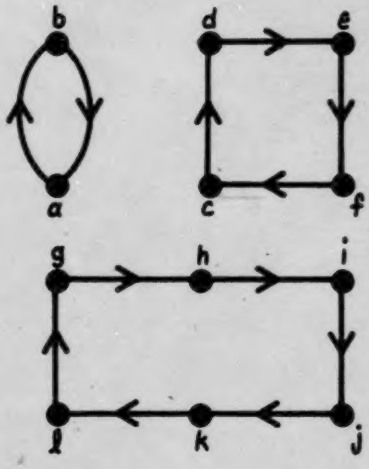


T: What do we call these numbers?

S: Multiples of 2 (even numbers).

Repeat this activity, considering the dots in the other two cycles. Extend the chart appropriately.

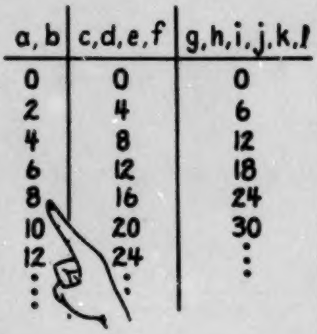
a, b	c, d, e, f	g, h, i, j, k, ℓ
0	0	0
2	4	6
4	8	12
6	12	18
8	16	24
10	20	30
12	24	⋮
⋮	⋮	



T: Let's go back to my original question. After how many rounds will all the children have returned to their own chairs?

L 4

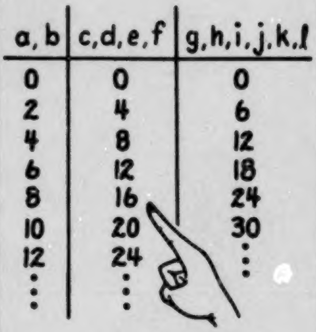
Consider each guess recorded earlier in the lesson. With class participation, rule out numbers which are not common multiples of 2, 4, and 6, i.e. multiples of 12. (Erase them.) For example, suppose "8" is in your list.



a, b	c,d,e,f	g,h,i,j,k,l
0	0	0
2	4	6
4	8	12
6	12	18
8	16	24
10	20	30
12	24	⋮
⋮	⋮	

T: Is 8 an even number?

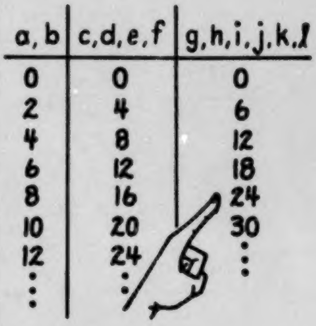
S: Yes.



a, b	c,d,e,f	g,h,i,j,k,l
0	0	0
2	4	6
4	8	12
6	12	18
8	16	24
10	20	30
12	24	⋮
⋮	⋮	

T: Is 8 a multiple of 4?

S: Yes.



a, b	c,d,e,f	g,h,i,j,k,l
0	0	0
2	4	6
4	8	12
6	12	18
8	16	24
10	20	30
12	24	⋮
⋮	⋮	

T: Is 8 a multiple of 6?

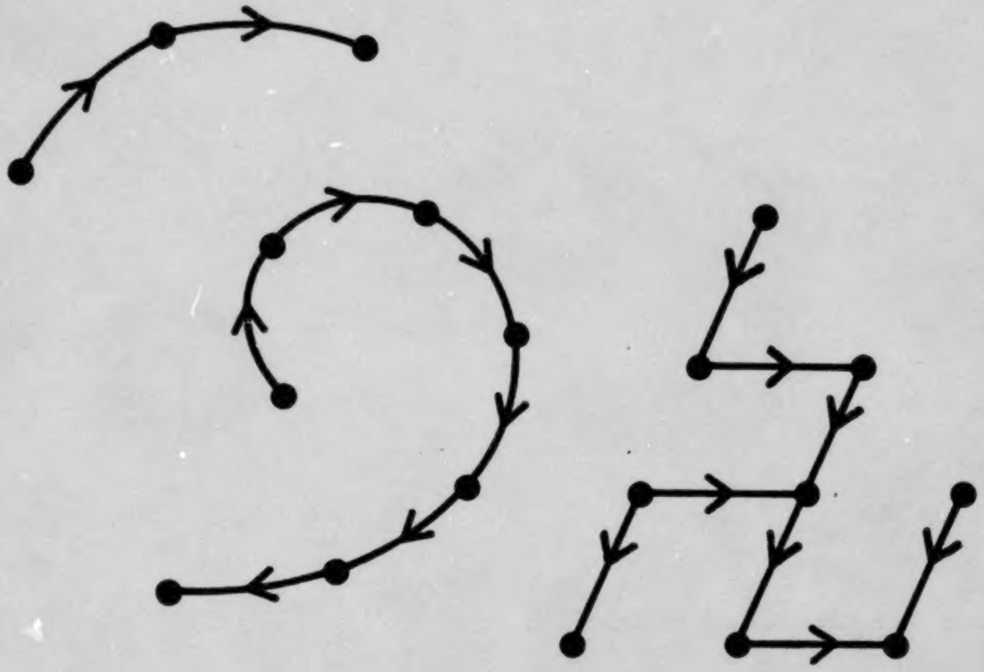
S: No.

Ask the students to expand the list of remaining correct answers. Continue until it is clear that the list contains the (whole number) multiples of 12 and that the students giving correct answers are using that information to find other numbers for the list. Your list should look similar to this one.

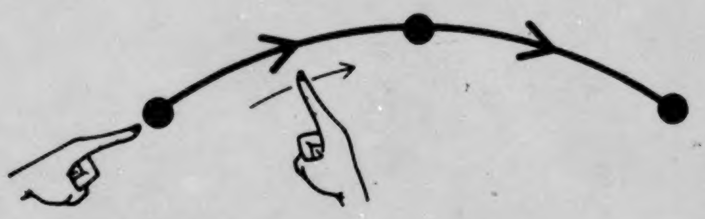
0 rounds
12 rounds
24 rounds
36 rounds
48 rounds
60 rounds
72 rounds
84 rounds
96 rounds
108 rounds
⋮

Exercise 2: A Composition Game

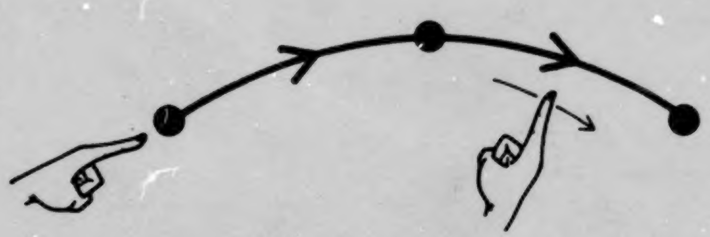
Draw this arrow picture on the board.



T:	We are going to play a game with these blue arrows.	
T:	This is the rule of the game . . .	Place your left forefinger on a dot at which the first blue arrow starts.
	Each time there is a blue arrow . . .	With your left forefinger in place at the dot, follow the arrow with your right forefinger in the direction of the arrowhead.

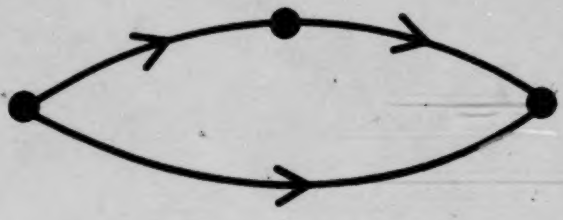


T:	. . . followed by a blue arrow. . .	Stop the motion of your right forefinger at the middle dot, tap the dot, and then follow the next blue arrow.



T:	. . . then we can draw a red arrow from the dot where the first blue arrow starts . . .	Tap this dot several times with your left forefinger.
	. . . to the dot where the second blue arrow ends.	Tap this dot several times with your right forefinger.

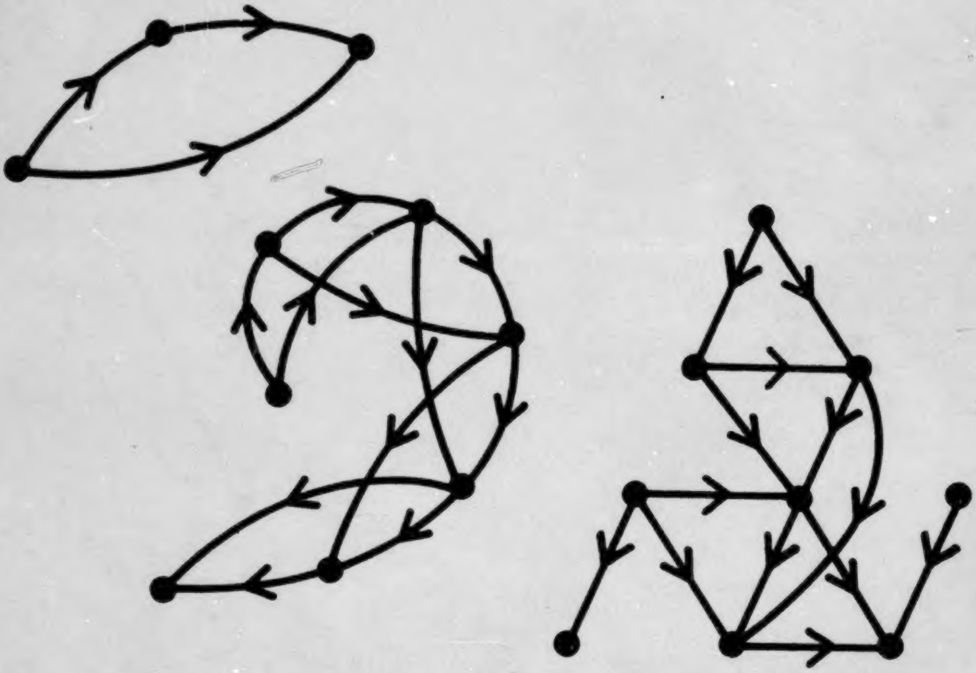
Then draw the indicated red arrow.



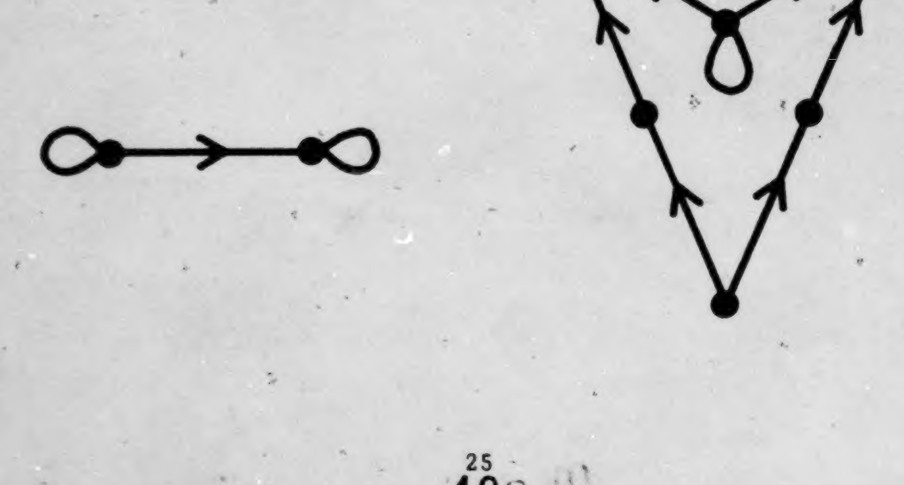
T: Where can we draw another red arrow?

Invite students one at a time to show where other red arrows can be drawn. Ask each student who volunteers to first trace the appropriate red arrow. Stop a student who starts to trace against the direction of an arrow and emphasize that the direction of an arrow must be followed. If a student correctly identifies a red arrow, draw the arrow yourself.

When all the red arrows have been found, your picture should look like this one.

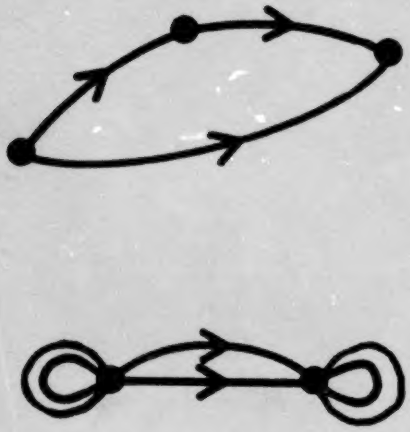


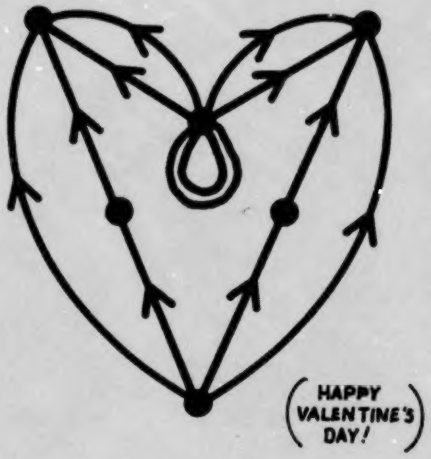
Repeat this activity with this arrow picture.



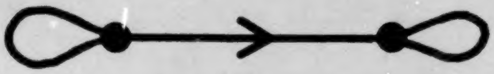
L 4

A completed picture is shown below.

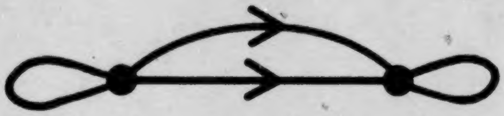




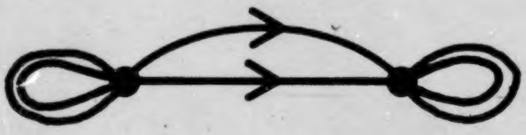
NOTE: For this part of the picture . . .



following the left loop and then the arrow identifies the same red arrow as following the arrow and then the right loop.



Following a blue loop twice around gives a red loop.

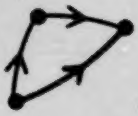


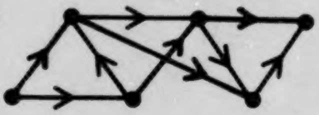
The same is true for the loop in the other part of the picture.

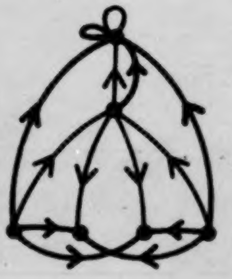
Worksheets L4* and ** are available for individual work.

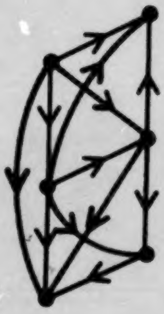
Name ______________ **L 4 ***

Draw all the missing red arrows and loops.



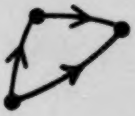


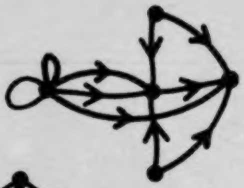


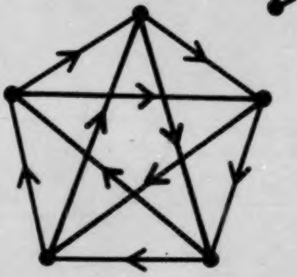


Name ______________ **L 4 ****

Draw all the missing red arrows and loops.







L5 DANCING FRIENDS (Lesson 1)

CAPSULE LESSON SUMMARY

Read the Storybook DANCING FRIENDS and explore a new addition operation $\oplus$ invented for use with just ten whole numbers, 0 through 9.

MATERIALS

Teacher: Storybook DANCING FRIENDS; colored chalk
Student: Storybook DANCING FRIENDS; paper; colored pencils

DESCRIPTION OF LESSON

Distribute copies of the Storybook DANCING FRIENDS and let the students look at them until you are ready to begin. Students will need paper and colored pencils.

Pages 2-3

Read these pages aloud. When you come to the middle of page 3, read the column of number stories on the left and then ask how to extend this list of number stories. For example, the next three number stories might be "8 + 6 = 14"; "9 + 6 = 15"; and "10 + 6 = 16". Do the same with the second column of number stories, asking students to extend the list. For example, "8 x 6 = 48"; "9 x 6 = 54"; and "10 x 6 = 60".

When you finish reading page 3, ask what other games the numbers might play. You could receive a wide variety of suggestions; accept all of them without comment.

Pages 4-5

Read the pages aloud or ask some students to do so. Some of your students may not be able to read "4 ÷ 8 = $\frac{1}{2}$" and will need help with it.

Record these numbers on the board and keep the list on the board for the rest of the lesson.

0 1 2 3 4 5 6 7 8 9

T: Can you imagine a game that just these ten numbers could play?

Pages 6-7

Read the page aloud or ask some students to do so, having each student read only part of the page. Perhaps your students would enjoy reading the number stories in unison. They may express curious surprise at the number stories on page 7.

Pages 8-9

Read page 8 aloud and then ask the students to read page 9 to themselves while you write some problems on the board.

$6 \oplus 9 =$

$8 \oplus 8 =$

$7 \oplus 7 =$

$5 \oplus 5 \oplus 5 \oplus 5 =$

$7 \oplus 7 \oplus 7 =$

$9 \oplus 8 \oplus 6 =$

After most of the students have finished reading page 9, ask:

T: Who can explain 0's game?

S: 0 doesn't add in the usual way.

S: $4 + 7 = 11$, but in 0's game you just keep the last number (the ones' digit) 1.

T (pointing to the problems on the board): What would 0's answers to these problems be?

Ask each student who gives an answer to explain it; for example,

S: $6 \oplus 9 = 5$.[*]

T: How did you get 5?

S: $6 + 9 = 15$ and I kept only the "5" in "15".

Continue until all the number stories are completed.

$$6 \oplus 9 = 5 \qquad 5 \oplus 5 \oplus 5 \oplus 5 = 0$$

$$8 \oplus 8 = 6 \qquad 7 \oplus 7 \oplus 7 = 1$$

$$7 \oplus 7 = 4 \qquad 9 \oplus 8 \oplus 6 = 3$$

Pages 10-11

Read page 10 aloud or ask some students to do so.

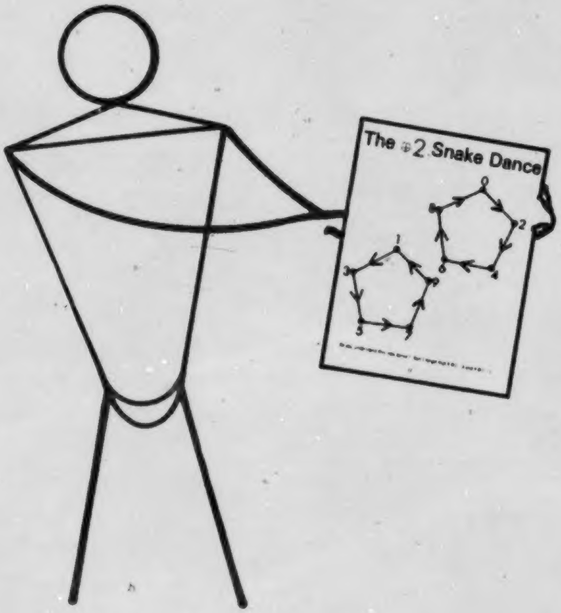
T: Look at the $\oplus 2$ dance. Why is it two pieces?

S: If you add twos starting at 0, you only get even numbers.
If you add twos starting at 1, you only get odd numbers.

---

[*] Read "⊕" as "circle plus."

Hold the page so that you can trace the arrows in the upper piece of the picture and so that the students can see what you are doing. As you trace each arrow, explain the number story told by that arrow and its starting dot and its ending dot.



Point to the lower piece of the arrow picture.

T: But suppose we start the ⊕2 snake dance at 1.

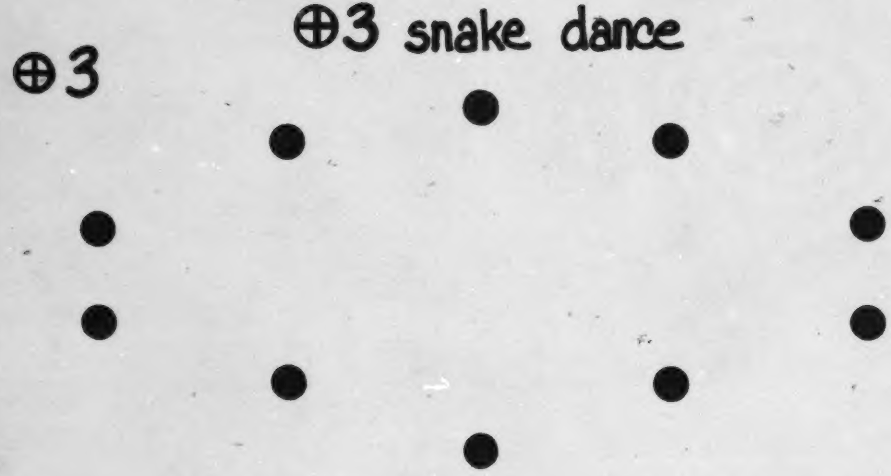
As you did before, trace the arrows in the lower piece of the picture and explain the appropriate number stories.

T: What kind of numbers are in this piece of the ⊕2 dance?

S: The odd numbers from 1 to 9.

Ask the class to think about what a ⊕3 snake dance would look like. If necessary, tell them not to turn the page yet. Allow a minute or two for comments and predictions and then draw this picture on the board.

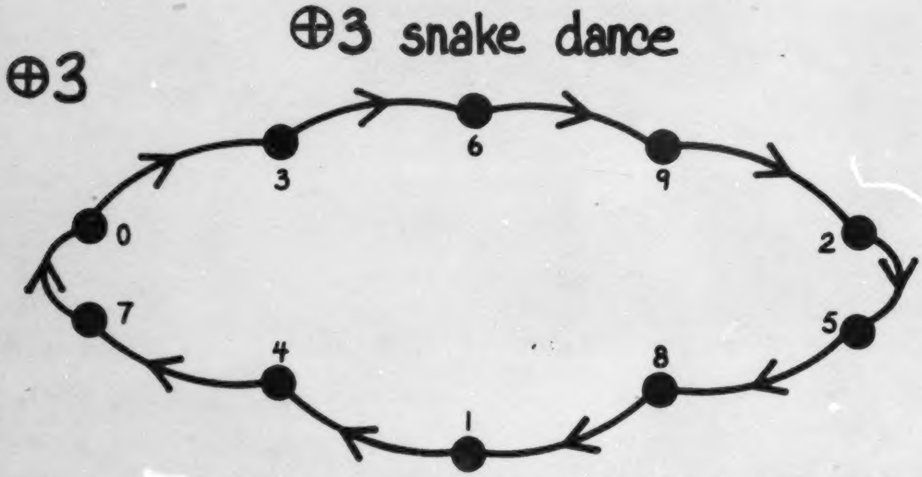
L 5



T: I drew a dot for each of these numbers (point to the list of numbers 0-9 on the board) because they all can dance in the ⊕3 snake dance.

Ask the students to copy this picture on their papers, to label the dots as they choose, and then to complete a picture of the ⊕3 snake dance.

Give individual help to students as they draw ⊕3 arrow pictures. After a few minutes collectively label the dots and draw the ⊕3 arrows.



NOTE: A more complicated looking picture might result if the dots are labeled differently.

Pages 12-13

Students can compare their pictures to the one shown on page 13. If necessary, remind them that their pictures might look different but still be similar. Suggest that they check their pictures by starting at 0 and following the arrows. Then they should meet the ten numbers in this order: 0, 3, 6, 9, 2, 5, 8, 1, 4, 7 and return to 0.

Pages 14-15

After you read page 14 aloud, hold up page 15 so that you can trace the arrows in the $\oplus 5$ dance and so that the students can see what you are doing. As you trace each arrow, explain the number story told by that arrow and its starting dot and its ending dot; for example, while tracing the $\oplus 5$ arrow from 6 to 1 say, "$6 + 5 = 11$, so $6 \oplus 5 = 1$."

T: Before you turn the page, try to draw a $\oplus 7$ dance and a $\oplus 8$ dance on your paper.

When students finish both pictures, let them go on reading the rest of the storybook by themselves. Suggest that they check their pictures against the ones on pages 16 and 18.

L 6

L6 DANCING FRIENDS (Lesson 2)

CAPSULE LESSON SUMMARY

Review the story of DANCING FRIENDS and the "addition with ten friends" operation. Solve some problems involving this operation. Draw arrow pictures for the relations $\oplus 3$, $\oplus 6$, and $\oplus 9$, and observe that return arrows for these relations are, respectively, for $\oplus 7$, $\oplus 4$, and $\oplus 1$.

MATERIALS

Teacher: Colored chalk
Student: Worksheets L6* and **; paper; colored pencils

DESCRIPTION OF LESSON

Exercise 1

Briefly review the story of DANCING FRIENDS.

T: Do you remember the story of DANCING FRIENDS?

S: It was about numbers.

T: And what was happening in the story?

S: 0 made up a dance.

T: Why was 0 making up a dance for numbers?

S: The boy in the story had invited only the numbers 0 through 9, and there weren't enough numbers to play their usual games.

T: . . . games like multiplication and addition.

Write this list of numbers on the board so it can be referred to in various parts of the lesson.

0 1 2 3 4 5 6 7 8 9

T: Do you remember the ⊕ (read: circle-plus) dances? Which number is 6 ⊕ 7?

$$6 \oplus 7$$

A student might add 6 and 7 in the usual way.

S: 13.

T: But remember only these numbers (point to the list) are playing the game.

S: 3.

T: How did you get 3?

S: 6 + 7 = 13 and you just keep the last number (the one's digit) -- that's 3.

Complete the number story on the board and write another calculation under it.

$$6 \oplus 7 = 3$$
$$8 \oplus 1$$

L6

T: Which number is 8 ⊕ 1?

S: 9.

$$6 \oplus 7 = 3$$
$$8 \oplus 1 = 9$$
$$8 \oplus 6$$

T: Which number is 8 ⊕ 6?

S: 4.

T: How did you get 4?

S: 8 ⊕ 6 = 14 and you just keep the "4".

$$6 \oplus 7 = 3$$
$$8 \oplus 1 = 9$$
$$8 \oplus 6 = 4$$

Continue this activity asking your class to complete each of the following number stories. Answers are in the boxes.

$$4 \oplus 4 \oplus 4 \oplus 4 = \boxed{6}$$
$$4 \oplus 4 \oplus 4 \oplus 4 \oplus 4 = \boxed{0}$$
$$7 \oplus 7 \oplus 7 = \boxed{1}$$
$$7 \oplus 7 \oplus 7 \oplus 7 = \boxed{8}$$

Students will need paper and colored pencils for certain parts of the rest of this lesson.

T: I will put some problems on the board. Copy them on your paper and then figure out which of these numbers (point to the list of whole numbers from 0 to 9) can be put in the boxes to make true number stories.

Write these problems on the board. Answers are in the boxes.

$$7 \oplus \boxed{5} = 2$$

$$6 \oplus \boxed{9} = 5$$

$$\boxed{2} \oplus 1 = 3$$

$$\boxed{6} \oplus 5 = 1$$

Collectively fill in the boxes, checking each answer. Then add this open sentence to the list.

$$\square \oplus \square = 2$$

T: The same number must be put in each of these boxes. Which of these numbers (point to the list of whole numbers from 0 to 9) could we put in these boxes to get a true number story?

S: 6.

T: 6 + 6 = 12, so 6 $\oplus$ 6 = 2. That's right.

S: 1.

L 6

T: 1 + 1 = 2, so also 1 ⊕ 1 = 2.

Is there any other number we could put in the boxes?

S: No.

Write this open sentence on the board.

$$\square \oplus \triangle = 5$$

T: These frames have different shapes, so we could put different numbers in them.

S: Put 9 in one of them and 6 in the other.

T: 9 + 6 = 15, so 9 ⊕ 6 = 5.

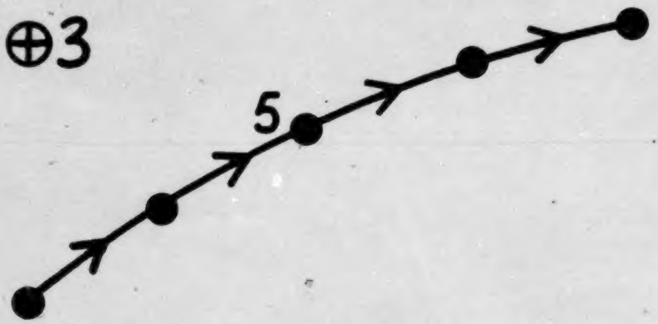
Continue until all of the following pairs have been suggested and have been checked.

$$\square \oplus \triangle = 5$$

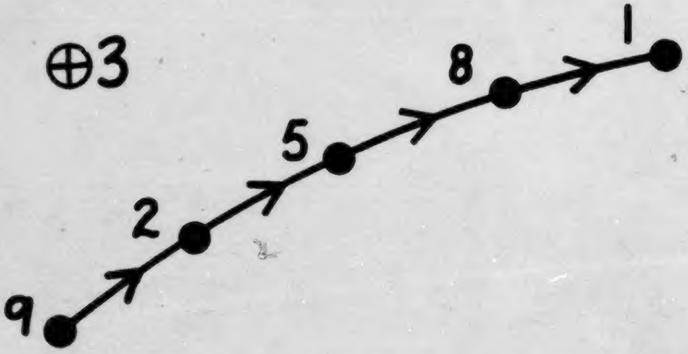
$\square$	$\triangle$
9	6
8	7
5	0
4	1
3	2

Exercise 2

Erase the board except for the list of whole numbers from 0 to 9 and then draw this arrow road.



Collectively label the dots.



Trace imaginary arrows as you ask the following questions. The first such arrow starts at 1 in the arrow road shown above.

T: If we drew more arrows in this ⊕ 3 road, what is the next number we would meet?

S: 4.

T: 1 ⊕ 3 = 4. And the next? S: 7.

T: 4 ⊕ 3 = 7. And the next? S: 0.

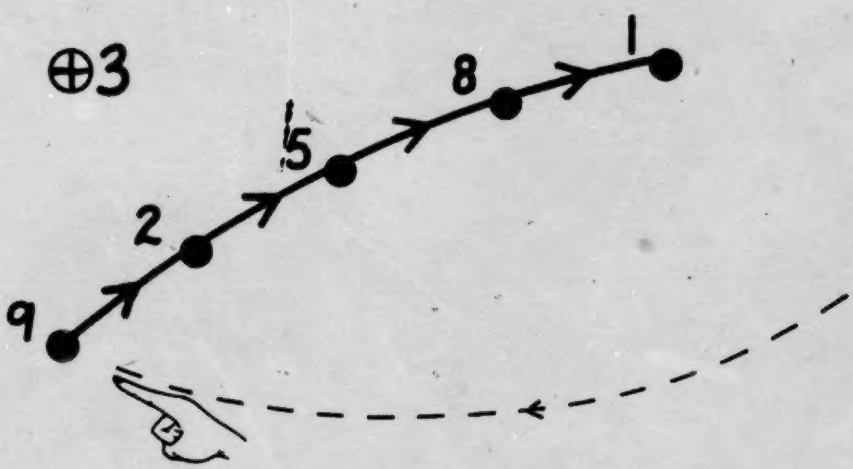
L 6

T: 7 ⊕ 3 = 0. And the next? S: 3.

T: 0 ⊕ 3 = 3. And the next? S: 6.

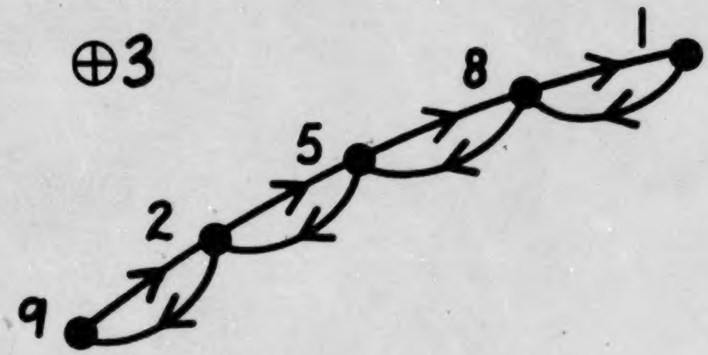
T: 3 ⊕ 3 = 6. And the next? S: 9.

With a sweeping motion indicate that the road returns to 9 as you say:



T: 6 ⊕ 3 = 9.

Draw a return arrow in green for each ⊕ 3 arrow in the road.



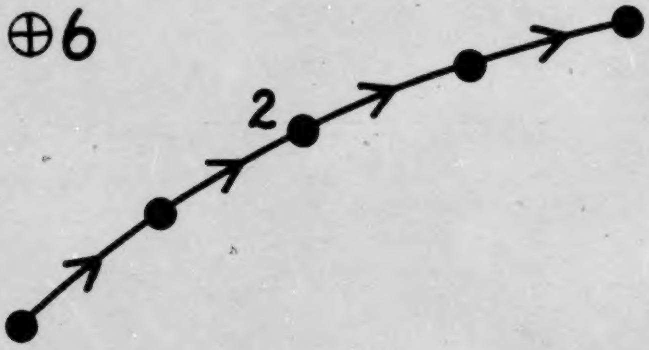
T: What could the green arrows be for?

S: ⊖3.

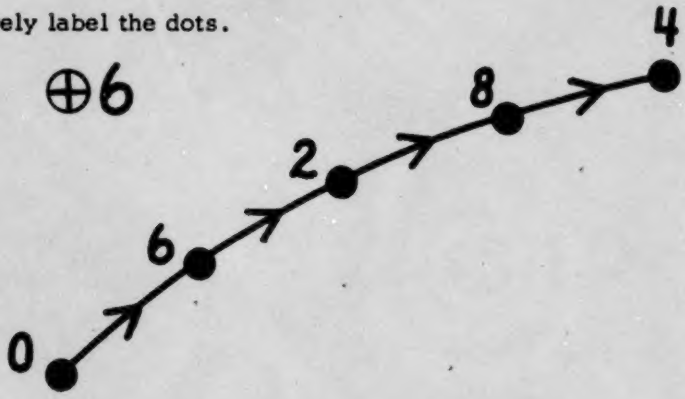
S: ⊕7.

If students do not suggest ⊕7 on their own, write "⊕☐" in green on the board and ask whether the green arrows could be for ⊕ some number. Check that the green arrows could be for ⊕7 be considering each one. Look at the starting number, follow the arrow saying "⊕7", and check that the ending number is correct. Then write "⊕7" in green near the arrow road.

Draw this arrow road next to the ⊕3 arrow road.



Collectively label the dots.

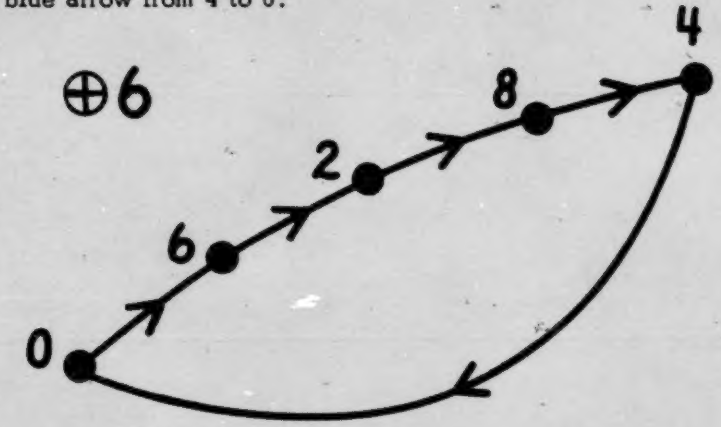


T: If we drew more arrows in this ⊕ 6 road, what is the next number we would meet?

S: 0.

L 6

Draw a blue arrow from 4 to 0.

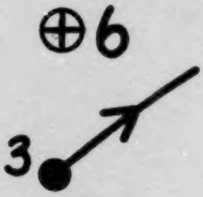


T: Where is the number 3 in the ⊕6 snake dance?

S: Only even numbers are in this piece.

S: You need to start a new part of the ⊕6 dance with 3 in it.

Draw this arrow on the board and label its starting dot "3".



Trace arrows (all but the first one will be imaginary) starting at 3 as you ask the following questions.

T: What is 3 ⊕ 6? S: 9.

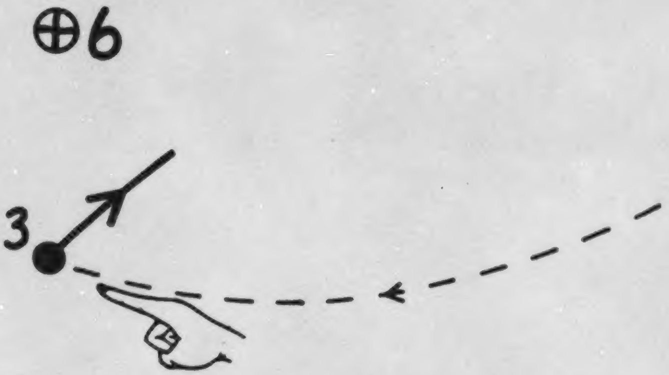
T: What is 9 ⊕ 6? S: 5.

T: What is 5 ⊕ 6? S: 1.

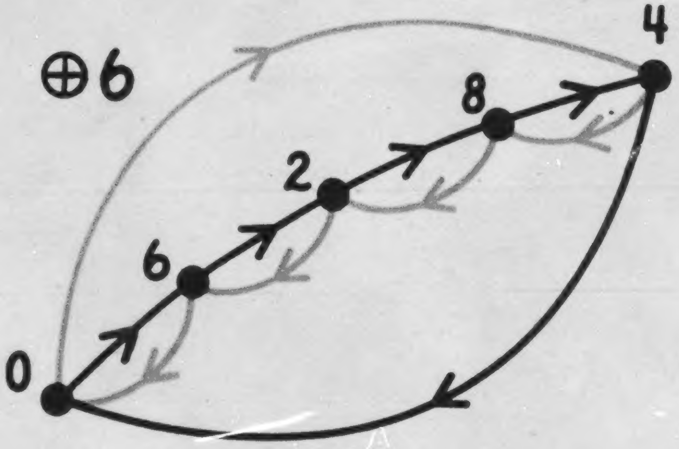
T: What is 1 ⊕ 6? S: 7.

T: What is 7 ⊕ 6? S: 3.

With a sweeping motion indicate that the road returns to 3.



Erase the arrow, the dot, and "3" shown in the preceding illustration. Then draw the return arrows in this part of the ⊕6 snake dance.



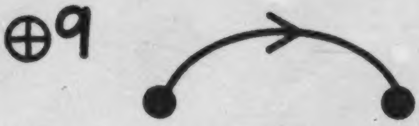
T: What could the yellow arrows be for? Try to label them circle-plus some number.

S: ⊕4.

Check that the yellow arrows could be for ⊕4 by considering each one. Then write "⊕4" in yellow near the arrow picture.

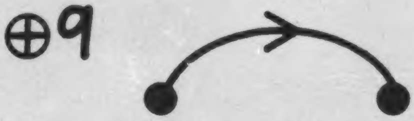
Exercise 3

Erase the board and then draw this arrow picture.



T: How can we label these dots?

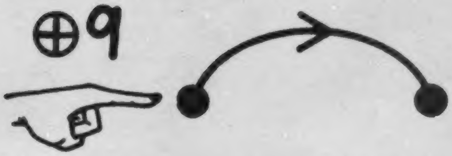
Let students make suggestions. Record correct number pairs in a chart. The chart shown below indicates the order in which one of our development classes found the number pairs.



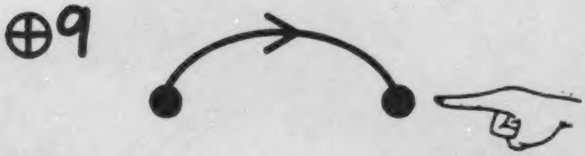
8	7
5	4
4	3
2	1
1	0
7	6
0	9
6	5
3	2
9	8

If your students have not found all the possible number pairs, you can help by calling attention to the left column of the list and asking whether any whole number from 0 to 9 is not included. Then ask if a ⊕9 arrow could start at that number. For example, suppose that the pair 2, 1 is not in the list.

T: 2 is not on this side (trace down the left column of the chart). Could this ⊕9 arrow start at 2? (Yes)



T: If so, what number would be here?

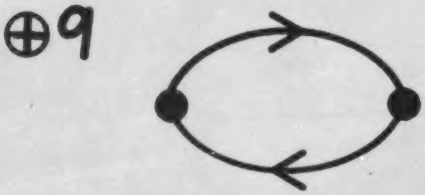


S: 1.

T: 2 + 9 = 11, so 2 ⊕ 9 = 1.

Repeat this procedure if necessary until all ten possible number pairs are listed in the chart.

Draw the return arrow in the picture.



L 6

T: What could the red arrow be for? Could it be for circle-plus some number?

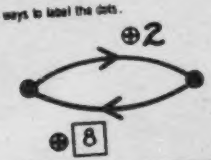
S: ⊕1.

Using the entries in the chart, check if the return arrow could be for ⊕1. (You will need to go from right to left in the chart to check the return arrow.) Then write "⊕1" in red near the arrow picture.

Worksheets L6* and ** are available for individual work.

Name ____________________ **L 6** *

Fill in the chart with ways to label the dots.

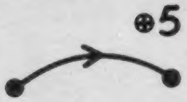


5	7
6	8
9	1
1	3
4	6
7	9
8	0
2	4
3	5

What could the blue arrow be for? Fill in the box.

Name ____________________ **L 6** **

Fill in the chart with ways to label the dots.



0	5
1	6
2	7
3	8
7	2
4	9
5	0
6	1
8	3
9	4

L7

L7 Three Boxes #1

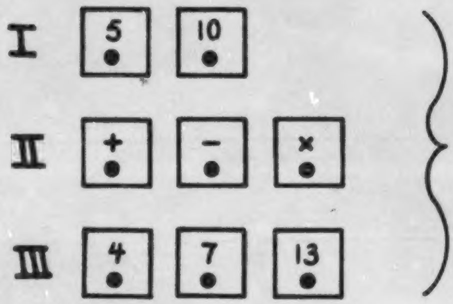
CAPSULE LESSON SUMMARY

Examine numbers that can be named by selecting in order one symbol from each of three boxes where the first box contains two numbers, the second box contains three operations, and the third box contains three numbers. Determine how many possibilities there are in this situation and discuss the chances of getting a number less than 13. Make operation tables for the three operations in the second box, thereby finding all the possibilities. Discuss how to make a fair game in which a team receives points when the number formed is more (less) than a specified number.

MATERIALS

Teacher:	Three boxes; eight pieces of paper (about 10 cm by 10 cm); colored chalk
Student:	Worksheet L7; pencil

NOTE: Prior to the presentation of this lesson, prepare three small boxes with labels "I", "II", and "III". Into each box put pieces of paper marked as indicated below and folded so as to hide what is written on them.

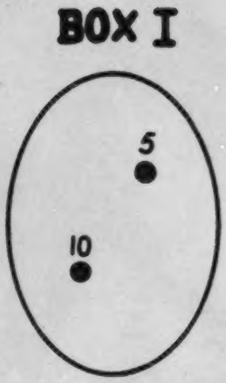


Dots indicate the bottom of the pieces of paper to avoid possible confusion.

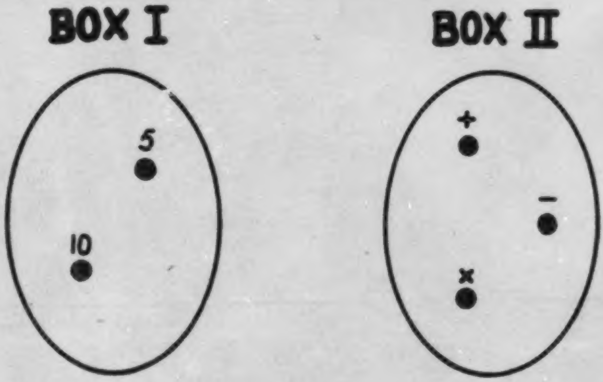
DESCRIPTION OF LESSON

Draw a string picture for each box as you reveal its contents to the class.

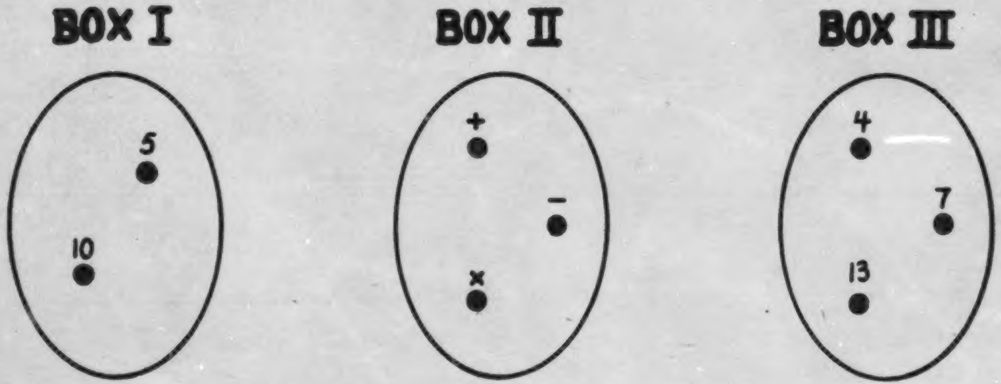
T: I have three boxes with pieces of paper in them. Each piece of paper has something written on it. In the first box there are two pieces of paper; one with "5" written on it and the other with "10".



T: In the second box there are three pieces of paper; one with "+", one with "-", and one with "x".


BOX I
5
10
BOX II
+
−
×

T: And in the third box there are three pieces of paper; one with "4", one with "7", and one with "13".


BOX I
5
10
BOX II
+
−
×
BOX III
4
7
13

Exercise 1

T: The numbers were chosen in such a way that every combination of something from the first box, something from the second box, and something from the third box names a different number.

Ask a student to draw a paper out of each box. For each of the three pieces of paper, the student should read aloud what is written on it, fold the paper, and then return it to the box from which it was taken. Write the combination on the board. A sample dialogue is given below; the student's name is "Derrick".

S: 10.

10

S: -.

10 -

S: 7.

10 - 7

T: What is Derrick's number? What number is 10 - 7?

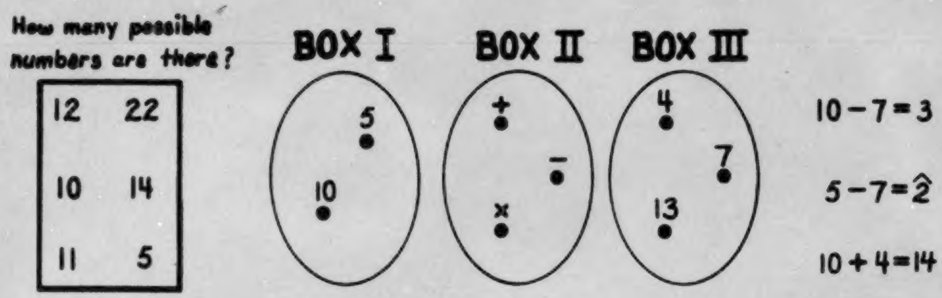
S: 3.

10 - 7 = 3

Repeat this activity, letting some other students take a turn at drawing papers from the boxes and naming the resulting number. Record appropriate number stories on the board.

T: How many numbers do you think that we can get this way?

Record the students' predictions on the board. Your chalkboard might look similar to this (see the next illustration).


How many possible numbers are there?
12
22
10
14
11
5
BOX I
5
10
BOX II
+
−
×
BOX III
4
7
13
10−7=3
5−7=?
10+4=14

T: Suppose we are playing a game and we have a Red Team and a Blue Team. In the game, the Red Team gets a point for each combination which names a number larger than 13 and the Blue Team gets a point for each combination which names a number less than 13. What are some numbers that we could get which are larger than 13?

S: 70; $10 \times 7 = 70$ and $70 > 13$.

S: 14; $10 + 4 = 14$ and $14 > 13$.

T: What are some numbers that we could get which are less than 13?

S: 9; $5 + 4 = 9$ and $9 < 13$.

S: 3; $10 - 7 = 3$ and $3 < 13$.

T: Do you think this is a fair game?

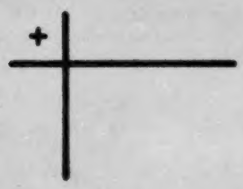
Let the students discuss the situation. Some students might want to play the game, although playing a few games[#] would not decide whether the game is fair or not. Commend anyone who notices that combinations which include "-"

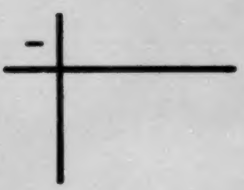
[#] From the results of a large number of games, a person could conclude empirically whether or not the game is fair.

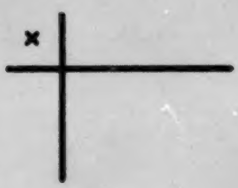
always give points to the Blue Team and that combinations which include "x " always give points to the Red Team, but do not offer this observation yourself.

T: Let's find out which numbers we could get and see how many give the Red Team a point and how many give the Blue Team a point. In the game, we can combine numbers through addition, subtraction, or multiplication.

Indicate the formation of three tables.



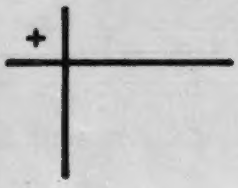


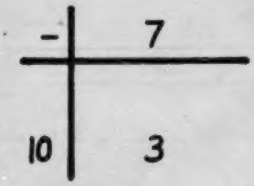


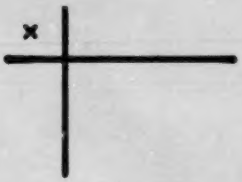
Choose one of the number stories written on the board when students were drawing papers from three boxes. "10 - 7 = 3" will be used in the following dialogue.

T: Derrick got the number 3 when he drew "10", "-", and "7". Let's show that in the subtraction table.

Say "10 - 7 = 3" as you show that information in the table.







T: But we could draw "5" from the first box . . .

+	

−	7
5	
10	3

×	

. . . and "4" or "13" from the second box.

+	

−	4	7	13
5			
10		3	

×	

Distribute copies of Worksheet L7.

T (holding a copy of Worksheet L7): On this paper there is a table for +, for −, and for ×. Fill in the tables to find all of the numbers which are possible to get in the game.

While the students are working, set up the tables on the board in the same way as they are on the worksheet.

+	4	7	13
5			
10			

−	4	7	13
5			
10			

×	4	7	13
5			
10			

L7

Observe students filling in the tables. Help any students who do not know which number to subtract from the other when they are considering pairs of numbers in the subtraction table. Refer to the three strings on the board as a reminder of which numbers are in the first box and which are in the third box. Choose three students with correct answers to fill in the tables on the board.

+	4	7	13
5	9	12	18
10	14	17	23

−	4	7	13
5	1	$\hat{2}$	$\hat{8}$
10	6	3	$\hat{3}$

×	4	7	13
5	20	35	65
10	40	70	130

T: Now we have found all the numbers which are possible to get in the game. How many are there?

S: Eighteen.

NOTE: A student who suggests that there are twenty-three or thirty-three possible numbers is including the numbers 5, 10, 4, 7, and 13, one or three times respectively. Remind the class that you are only counting the numbers that you can get in the game.

Compare 18 to the predictions recorded on the board earlier in the lesson. Decide which number was closest to the correct number of possibilities and then erase the list and write "18".

T (pointing to the three tables): For which of these numbers would the Red Team gain a point? Which of these numbers is larger than 13?

Put red checkmarks (✓) by such numbers as they are suggested.

+	4	7	13
5	9	12	18✓
10	14✓	17✓	23✓

−	4	7	13
5	1	$\hat{2}$	$\hat{8}$
10	6	3	$\hat{3}$

×	4	7	13
5	20✓	35✓	65✓
10	40✓	70✓	130✓

T: How many of the possible numbers are larger than 13?

S: Ten.

Write this fraction on the board as you say:

Red $\frac{10}{18}$

T: So, the Red Team has 10 chances out of 18 of getting a point. How many of the possible numbers are less than 13?

S: Eight.

T: So the Blue Team has 8 chances out of 18 of getting a point.

Red $\frac{10}{18}$ Blue $\frac{8}{18}$

T: Is this a fair game?

S: No.

T: Who has the advantage — the Red Team or the Blue Team?

S: The Red Team.

Erase the checkmarks from the tables.

T: How could we change the game to make it a fair game?

In answer to this question you might get very few suggestions or you might get a wide variety of suggestions ranging from simple alterations to very complex ones which may well be correct. But a simple way to make the game fair is to change the number that determines which team gets a point; that number was 13 in the game just described. These four ways to make the game fair are essentially the same.

- Give one team points for numbers less than or equal to 14 and give the other team points for numbers larger than 14.
- Give one team points for numbers less than 15 and give the other team points for numbers larger than 15.
- Give one team points for numbers less than 16 and give the other team points for numbers larger than 16.
- Give one team points for numbers less than 17 and give the other team points for numbers larger than or equal to 17.

This is a sample dialogue in which one of these games will be discussed in detail.

S: We could change the numbers in the strings.

T: We could do that, but would need to be very careful. Could we make a fair game without changing the numbers in the strings?

S: Instead of giving points for numbers larger or smaller than 13, we could choose some other number.

T: Such as?

S: 14.

T: 10 + 4 = 14. Who would get the point for 14?

S: I don't know.

S: Let's try 15.

T: Which of the possible numbers are less than 15?

Put blue checkmarks (✓) by such numbers as they are suggested.

+	4	7	13
5	9 ✓	12 ✓	18
10	14 ✓	17	23

−	4	7	13
5	1 ✓	$\widehat{2}$ ✓	$\widehat{8}$ ✓
10	6 ✓	3 ✓	$\widehat{3}$ ✓

×	4	7	13
5	20	35	65
10	40	70	130

Conclude with the class that there are nine possible numbers which are less than 15 and also nine possible numbers larger than 15.

Red $\frac{9}{18}$ Blue $\frac{9}{18}$

T: Is the game fair?

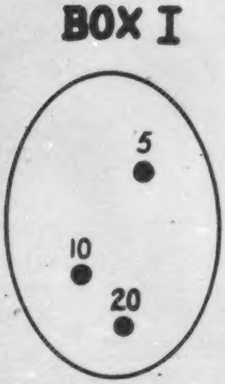
S: Yes!

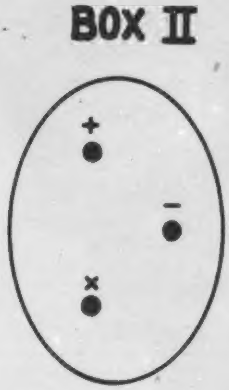
Erase the checkmarks in the tables.

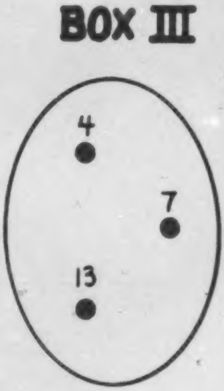
Exercise 2

T: Suppose we put a piece of paper with "20" written on it on the first box together with the other two papers in that box.

Draw a dot in the first string and label it "20".







T: How many more numbers could we get now?

S: We could get three more with "+", three more with "-", and three more with "x".

Students also might make erroneous suggestions; for example, "Eighteen more; I just doubled what we had before." Express doubt and ask for another suggestion.

If no one indicated the correct answer (see preceding student dialogue), draw a new row in each table as in the next illustration and ask the question again.

+	4	7	13
5	9	12	18
10	14	17	23
20			

−	4	7	13
5	1	$\widehat{2}$	$\widehat{8}$
10	6	3	$\widehat{3}$
20			

×	4	7	13
5	20	35	65
10	40	70	130
20			

Instruct the students to extend the tables on their worksheets to include 20 as a choice from the first box and to complete the tables. Give individual help to students who do not know where to write "20" in the tables.

T: There were eighteen possible numbers before. How many are there now?

S: Twenty-seven.

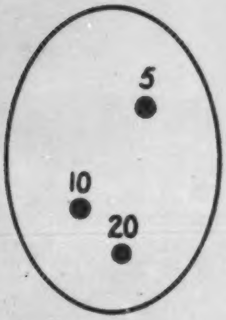
T: How much more is 27 than 18?

S: 9.

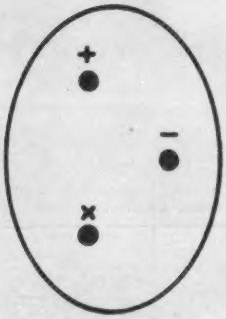
Put "21" in the third string and repeat the activity.

L7

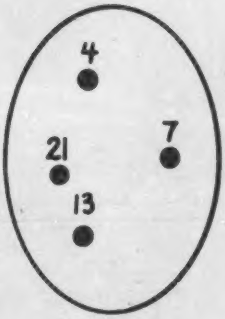
BOX I



BOX II



BOX III



+	4	7	13	21
5	9	12	18	26
10	14	17	23	31
20	24	27	33	41

−	4	7	13	21
5	1	$\widehat{2}$	$\widehat{8}$	$\widehat{16}$
10	6	3	$\widehat{3}$	$\widehat{11}$
20	16	13	7	$\widehat{1}$

×	4	7	13	21
5	20	35	65	105
10	40	70	130	210
20	80	140	260	420

T: Now there are thirty-six possible numbers we could get in this three boxes game. How could we decide to give points to the Red Team and the Blue Team so the game would be fair with these possibilities?

Let students discuss the situation. There are many solutions, but one that might become evident is that the Red Team gets a point if the number is more than 20 and the Blue Team gets a point if the number is 20 or less.

L8 Basketball Bar Graphs

CAPSULE LESSON SUMMARY

Make bar graphs for the season's records (scores of games) for two basketball teams. Compare these graphs to the basketball bar graph your class has been making during the past three months.

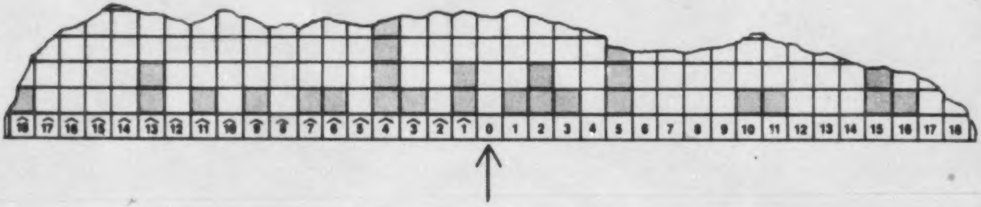
MATERIALS

Teacher: Classroom basketball charts (see Lesson L14 in UPG-III); Language Poster #1 for UPG-III/IV; felt-tip marker
Student: Basketball data and charts; pencil

NOTE: Cut out the two pieces of the demonstration chart from the poster and tape them together so that the numbers read from $\hat{36}$ to 36 (left to right).

DESCRIPTION OF LESSON

Teach this lesson when the basketball season is over. On the board display the basketball charts on which you have been recording data throughout the season. Draw an arrow pointing to the "0" column of each chart as in the next illustration.



Discuss the charts with your class. Some questions you might ask during the discussion are:

- What do the shaded squares to the right of the "0" column represent? (Wins) What do the shaded squares to the left of the "0" column represent? (Losses) Why are there no squares shaded in the "0" column? (There are no tie games in basketball)

- Did Team A (B)[*] usually win or usually lose?
- Did Team A (B) have some close games?
- Which team had the better season?
- Which team had the best win?
- Which team had the worst loss?

Tape the demonstration chart from Language Poster #3 to the board with the other two charts. Distribute copies of the sheet with Basketball data and charts. Explain to the class that these are the scores from the basketball games St. Louis University played in the 1977-1978 season. Ask the students to find the information for the St. Louis University — Illinois State game in the data. Record the same information on the board.

ST. LOUIS UNIVERSITY	ILLINOIS STATE
83	90

T: Did St. Louis University win or lose that game?

S: They lost it.

T: By how many points . . . ?

S: 7.

T: Note that this information is recorded on your sheet. I'll also record it on the board.

ST. LOUIS UNIVERSITY	ILLINOIS STATE	Win or loss for ST. LOUIS UNIVERSI?
83	90	lost by 7

---

[*] Use the names of the teams on which your class collected basketball data.

T: Where should we shade a square in the chart?

S: Above "$\hat{7}$".

T: Shade a square above "$\hat{7}$" in your chart and I will do the same in the chart on the board.

Let the students individually complete their charts. While the students are working, ask them one at a time to record the Saint Louis University's wins and losses in the demonstration chart. For example, a student would shade a square above "$\hat{16}$" for St. Louis University's loss to Indiana; then another student would shade a square above "$\hat{1}$" for St. Louis University's loss to the University of Missouri at St. Louis; and so on.

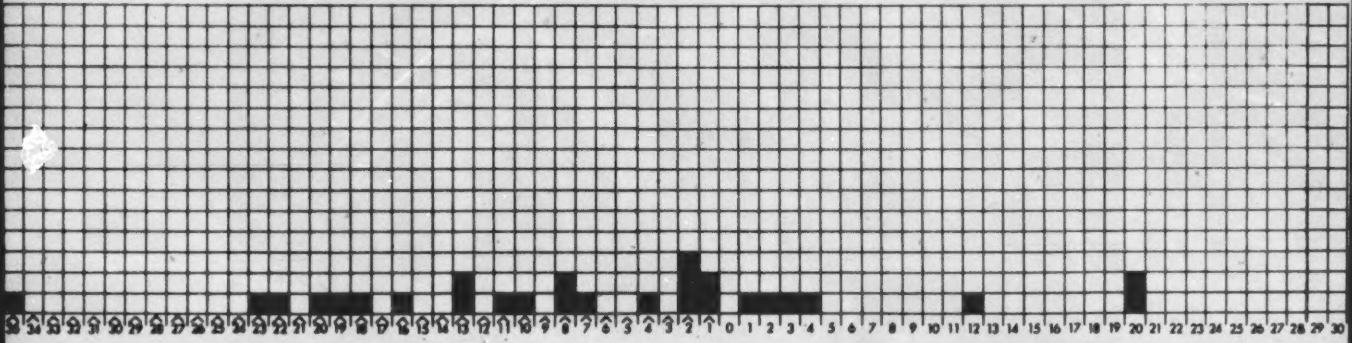
After about 20 to 25 minutes, most students should have completed their charts, and there should be completed charts displayed on the board. (See page 64). Discuss St. Louis University's basketball season in 1977-1978 and compare it to the seasons of the two teams on which your class collected data. Ask questions similar to those asked in the initial discussion.

Record of the 1977-78 season

St. Louis University Basketball Team

St. Louis University	Opposing Team	Win or Loss for St. Louis U.
St. Louis U. 83	Illinois State 90	Lost by 7
St. Louis U. 68	Indiana State 84	Lost by 16
St. Louis U. 88	U. of Missouri 89 (St. Louis)	Lost by 1
St. Louis U. 61	Detroit U. 69	Lost by 8
St. Louis U. 66	Oklahoma City 89	Lost by 23
St. Louis U. 65	Kansas 85	Lost by 20
St. Louis U. 80	Tulsa 77	Won by 3
St. Louis U. 74	U. of Missouri 54 (St. Louis)	Won by 20
St. Louis U. 78	Columbia U. 66	Won by 12
St. Louis U. 83	Southern Illinois 63 (Edwardsville)	Won by 20
St. Louis U. 55	Cincinnati 66	Lost by 11
St. Louis U. 54	Marquette 56	Lost by 2
St. Louis U. 79	Georgia Tech. 92	Lost by 13
St. Louis U. 81	De Paul 100	Lost by 19
St. Louis U. 75	Florida State 88	Lost by 13
St. Louis U. 49	Georgia Tech. 57	Lost by 8
St. Louis U. 85	Memphis State 86	Lost by 1
St. Louis U. 68	Florida State 72	Lost by 4
St. Louis U. 65	Winconsin- Milwaukee 64	Won by 1
St. Louis U. 58	Southern Illinois 60 (Carbondale)	Lost by 2
St. Louis U. 61	Louisville 63	Lost by 2
St. Louis U. 62	Cincinnati 80	Lost by 18
St. Louis U. 83	Tulane 79	Won by 4
St. Louis U. 82	Memphis State 80	Won by 2
St. Louis U. 75	Tulane 85	Lost by 10
St. Louis U. 59	Louisville 94	Lost by 35
St. Louis U. 57	Memphis State 79	Lost by 22

BASKETBALL CHART



A A

L9 The String Game #2

CAPSULE LESSON SUMMARY

Review not-cards as string labels. Analyze the information obtained from individual clues (correctly placed pieces) in a String Game until the string labels are determined. Play a 3-string version of the String Game.

(Optional) Play a 3-string game involving multiples and order.

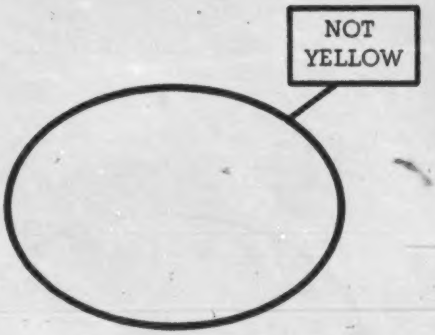
MATERIALS

Teacher: A-Blocks String Game kit; colored chalk
Student : None

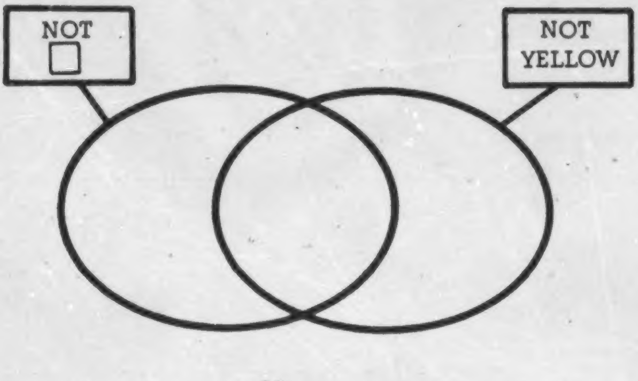
DESCRIPTION OF LESSON

Exercise 1

Begin the lesson with a brief review of not-cards. First, place several pieces in a 1-string picture . . .

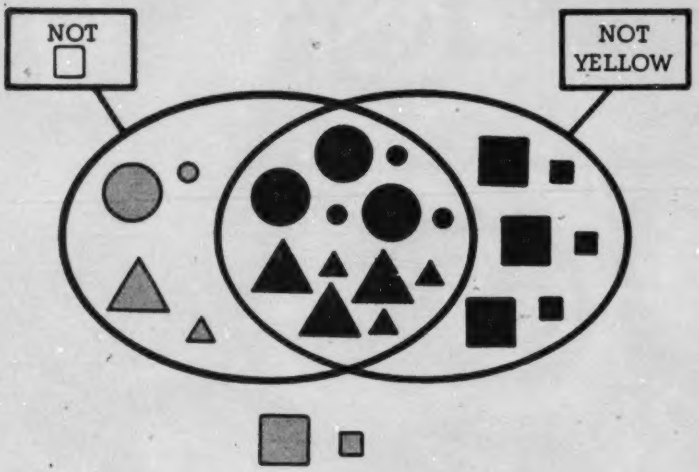


. . . and then add a second string and place several pieces in the resulting 2-string picture.



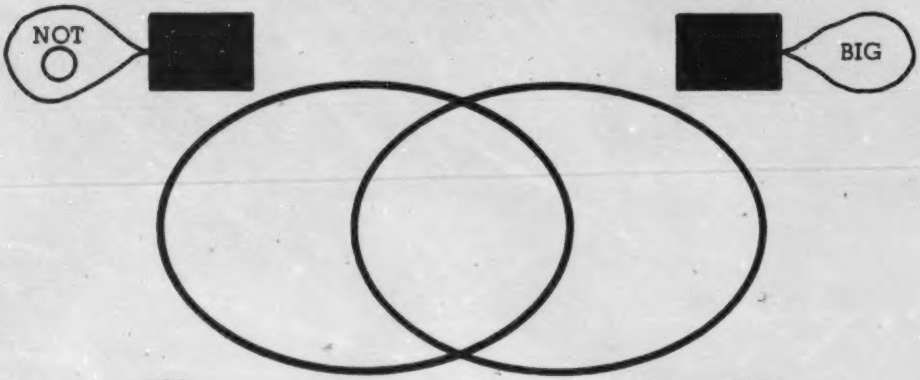
L 9

The correct placement of all A-Block pieces is shown below for your reference.



Exercise 2

Set up your board for the String Game as illustrated below. The bubbles show what is on the hidden tags. Tape two A-Block String Game Posters (Version B) to the board, one for the red string and one for the blue string.



RED

RED	YELLOW	GREEN	BLUE
NOT RED	NOT YELLOW	NOT GREEN	NOT BLUE
○	△	□	BIG
NOT ○	NOT △	NOT □	LITTLE

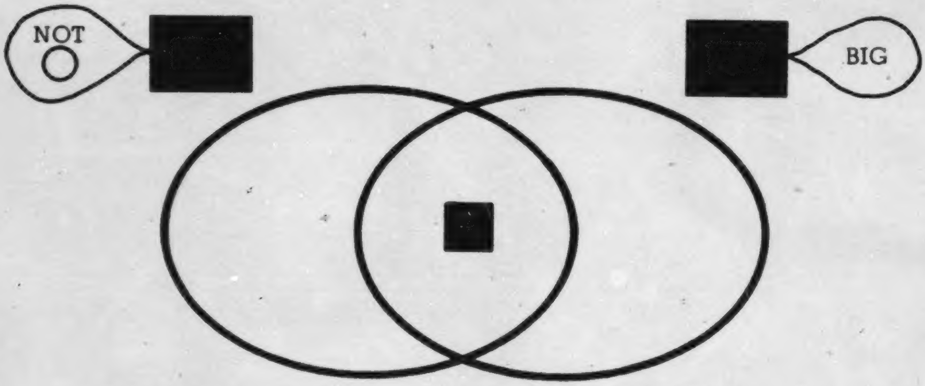
BLUE

RED	YELLOW	GREEN	BLUE
NOT RED	NOT YELLOW	NOT GREEN	NOT BLUE
○	△	□	BIG
NOT ○	NOT △	NOT □	LITTLE

T: We are going to play the String Game today, but first we are going to look at what information we get from knowing where some of the game pieces belong in the picture. Finding out what the strings are for will be like solving a detective story; each piece placed will be a clue.

First clue

T: The big blue square is in the middle region, in both the red and the blue strings.



T: What does this tell us about the strings? Is there any of these labels (point to one of the lists) that the strings cannot have?

S: NOT BLUE.

T: We can cross off NOT BLUE for which string?

S: Both.

T: If one of the strings were for NOT BLUE, where would this piece belong?

S: Outside that string.

L 9

Use red and blue crayons or felt-tip markers to cross off NOT BLUE from both lists.

RED

RED	YELLOW	GREEN	BLUE
NOT RED	NOT YELLOW	NOT GREEN	~~NOT BLUE~~
○	△	□	BIG
NOT ○	NOT △	NOT □	LITTLE

BLUE

RED	YELLOW	GREEN	BLUE
NOT RED	NOT YELLOW	NOT GREEN	~~NOT BLUE~~
○	△	□	BIG
NOT ○	NOT △	NOT □	LITTLE

In the same manner, let students eliminate as many labels as they can from both lists. Each time they cross off a label on one list because the corresponding string cannot have that label, they should see that the same label also should be crossed off on the other list. A piece in the center region gives the same information about both strings.

A student may suggest incorrectly that some label be crossed off the lists; for example,

S: Cross off NOT RED.

T: But this piece (point to the big blue square) is not red.

When your class has exhausted this clue of information, they should find that there are eight remaining possibilities for each string.

RED

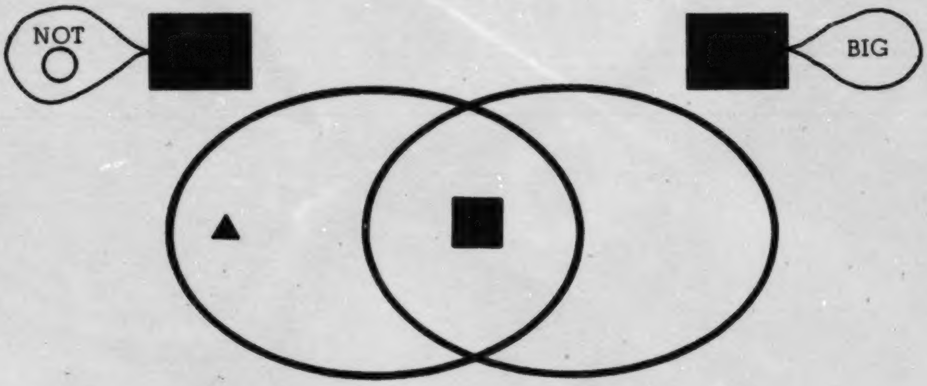
~~RED~~	~~YELLOW~~	~~GREEN~~	BLUE
NOT RED	NOT YELLOW	NOT GREEN	~~NOT BLUE~~
~~○~~	~~△~~	□	BIG
NOT ○	NOT △	~~NOT □~~	~~LITTLE~~

BLUE

~~RED~~	~~YELLOW~~	~~GREEN~~	BLUE
NOT RED	NOT YELLOW	NOT GREEN	~~NOT BLUE~~
~~○~~	~~△~~	□	BIG
NOT ○	NOT △	~~NOT □~~	~~LITTLE~~

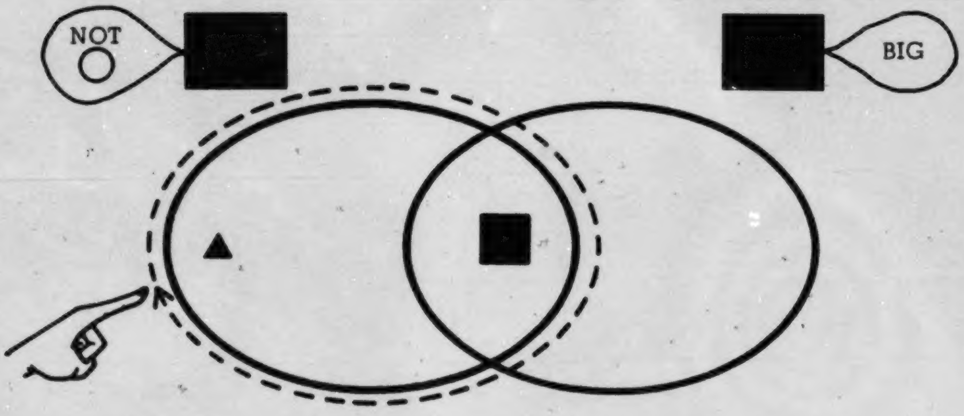
Second clue

T: I'll give you another clue. The little red triangle is in the red string but not in the blue string.



Consider the labels on the Red list which remain as possibilities for the red string after the first clue. Ask if the red string could be for any of them. The following dialogue gives a discussion of two of the right remaining labels.

Point to "BLUE" on the Red list and then trace the red string as you say:



T: Could the red string be for BLUE?

S: No, the triangle isn't blue — it's red!

Cross off BLUE from the Red list.

Point to "NOT ○" on the Red list and then trace the red string as you say:

T: Could the red string be for NOT ○?

S: Yes, both pieces in it are not circles.

Do not cross off NOT ○.

Continue until each of the eight labels have been discussed for the red string.

RED

~~RED~~	~~YELLOW~~	~~GREEN~~	~~BLUE~~
~~NOT RED~~	NOT YELLOW	NOT GREEN	~~NOT BLUE~~
~~○~~	~~△~~	~~□~~	~~BIG~~
NOT ○	~~NOT △~~	~~NOT □~~	~~LITTLE~~

BLUE

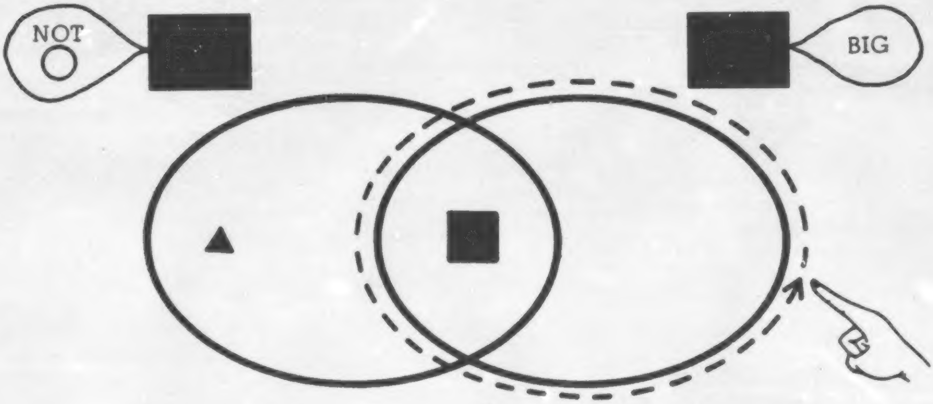
~~RED~~	~~YELLOW~~	~~GREEN~~	BLUE
NOT RED	NOT YELLOW	NOT GREEN	~~NOT BLUE~~
~~○~~	~~△~~	□	BIG
NOT ○	NOT △	~~NOT □~~	~~LITTLE~~

Consider the blue string. The analysis involved is slightly different for this string because the small red triangle is outside the blue string. Because of the position of this piece in the picture,

- whenever a label is eliminated as a possibility for the red string, it remains as a possibility for the blue string;
- whenever a label remains as a possibility for the red string, it is eliminated as a possibility for the blue string.

The following dialogue gives a discussion of two of the eight labels which remain after the first clue.

Point to "BLUE" on the Blue list and then trace the blue string as you say:



T: Could the blue string be for BLUE?

S: Yes, the small red triangle is outside the blue string.

Point to "NOT ○" on the Blue list and then trace the blue string as you say:

T: Could the blue string be for NOT ○?

S: No, because the little red triangle is not a circle and it is outside the blue string.

If no one responds, ask where the little red triangle would belong in the picture if the blue string were for NOT ○.

L 9

After each of the eight possibilities for the blue string have been discussed, your lists should look like these.

RED

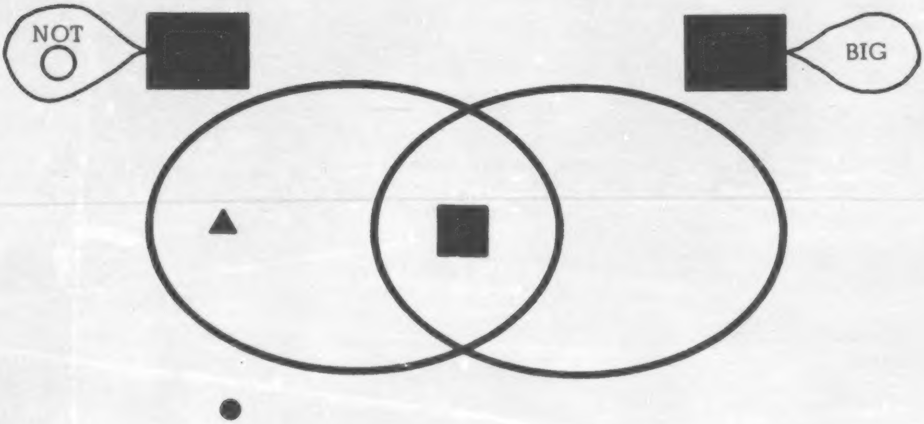
~~RED~~	~~YELLOW~~	~~GREEN~~	~~BLUE~~
~~NOT RED~~	NOT YELLOW	NOT GREEN	~~NOT BLUE~~
~~○~~	~~△~~	~~□~~	~~BIG~~
NOT ○	~~NOT △~~	~~NOT □~~	~~LITTLE~~

BLUE

~~RED~~	~~YELLOW~~	~~GREEN~~	BLUE
NOT RED	~~NOT YELLOW~~	~~NOT GREEN~~	~~NOT BLUE~~
~~○~~	~~△~~	□	BIG
~~NOT ○~~	NOT △	~~NOT □~~	~~LITTLE~~

Third clue

T: The next clue is that the little green circle is outside both strings.



As before, cross out appropriate labels on the two lists. Since the little green circle is outside both strings, the analysis for both strings is the same as that used to consider the blue string in the second clue (small red triangle outside the blue string).

After considering all the remaining labels (from the third clue) on the lists, you should have only one possibility for the red string and two possibilities for the blue string.

RED

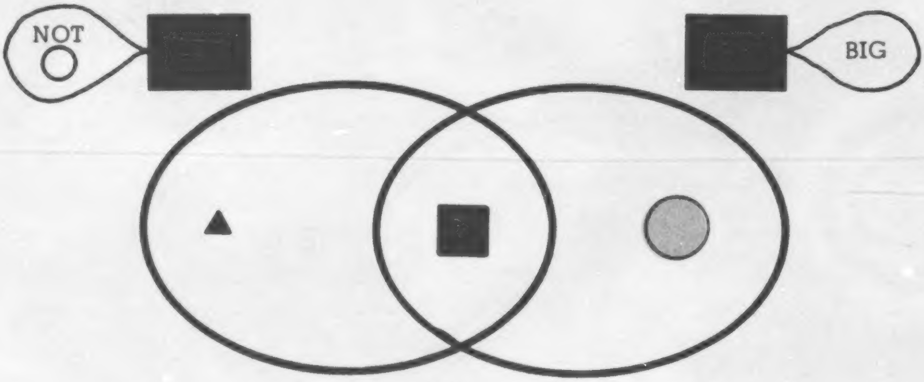
~~RED~~	~~YELLOW~~	~~GREEN~~	~~BLUE~~
~~NOT RED~~	~~NOT YELLOW~~	NOT GREEN	~~NOT BLUE~~
~~○~~	~~△~~	~~□~~	~~BIG~~
NOT ○	~~NOT △~~	~~NOT □~~	~~LITTLE~~

BLUE

~~RED~~	~~YELLOW~~	~~GREEN~~	~~BLUE~~
~~NOT RED~~	~~NOT YELLOW~~	~~NOT GREEN~~	~~NOT BLUE~~
~~○~~	~~△~~	□	BIG
~~NOT ○~~	~~NOT △~~	~~NOT □~~	~~LITTLE~~

Fourth clue

T: The big yellow circle is inside the blue string but outside the red string.



From this clue, your class should determine the strings.

- The red string must be for NOT ○. (The larger yellow circle is not green, so NOT GREEN can be crossed off from the Red list.)

L 9

- The blue string must be for BIG. (The large yellow circle is not blue and is not a square, so BLUE and □ can be crossed off from the Blue list.)

RED

~~RED~~	~~YELLOW~~	~~GREEN~~	~~BLUE~~
~~NOT RED~~	~~NOT YELLOW~~	~~NOT GREEN~~	~~NOT BLUE~~
~~○~~	~~△~~	~~□~~	~~BIG~~
[✓ NOT ○]	~~NOT △~~	~~NOT □~~	~~LITTLE~~

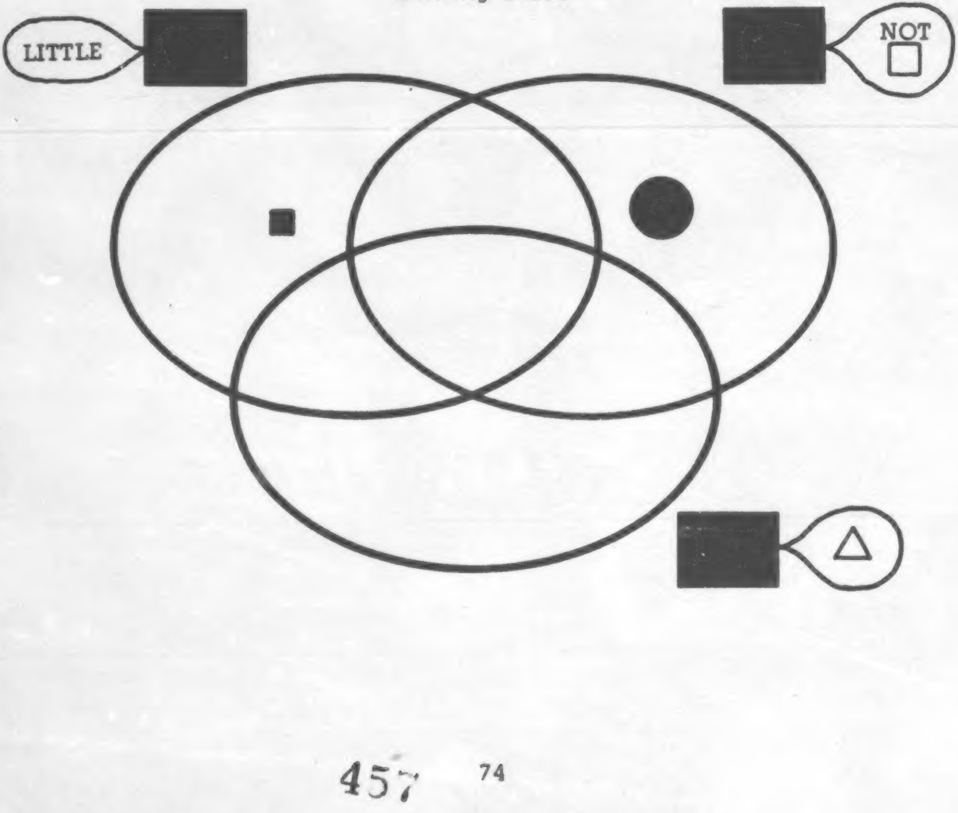
BLUE

~~RED~~	~~YELLOW~~	~~GREEN~~	~~BLUE~~
~~NOT RED~~	~~NOT YELLOW~~	~~NOT GREEN~~	~~NOT BLUE~~
~~○~~	~~△~~	~~□~~	[✓ BIG]
~~NOT ○~~	~~NOT △~~	~~NOT □~~	~~LITTLE~~

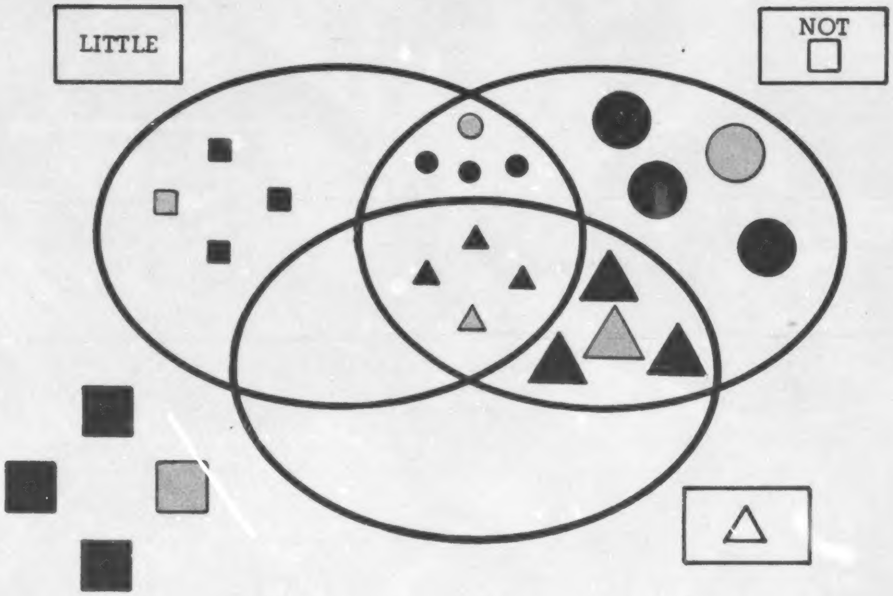
Exercise 3

Play the String Game with these strings in the usual way. Possible string labels and appropriate starting clues for a game are suggested below. A crib-sheet is provided on the next page.

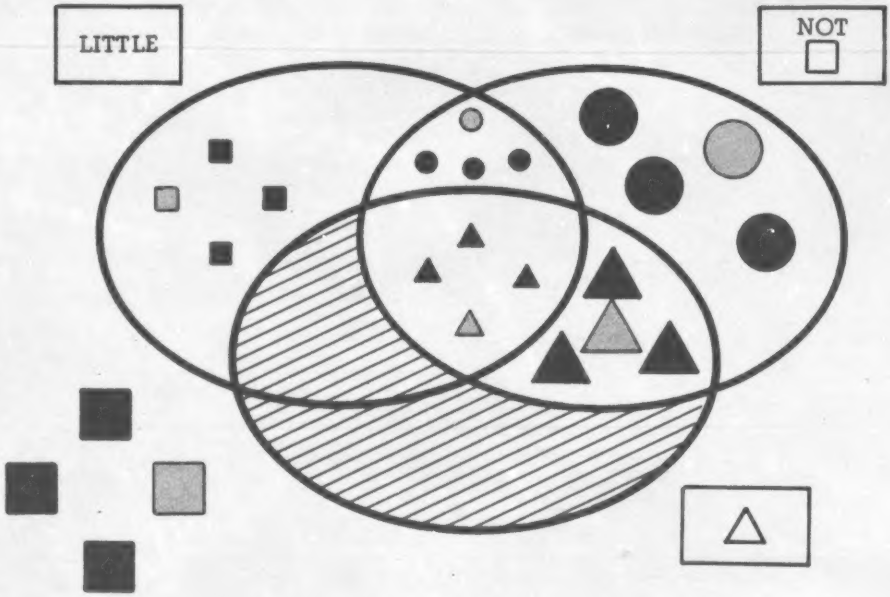
Starting Clues



Correct Placement of Pieces



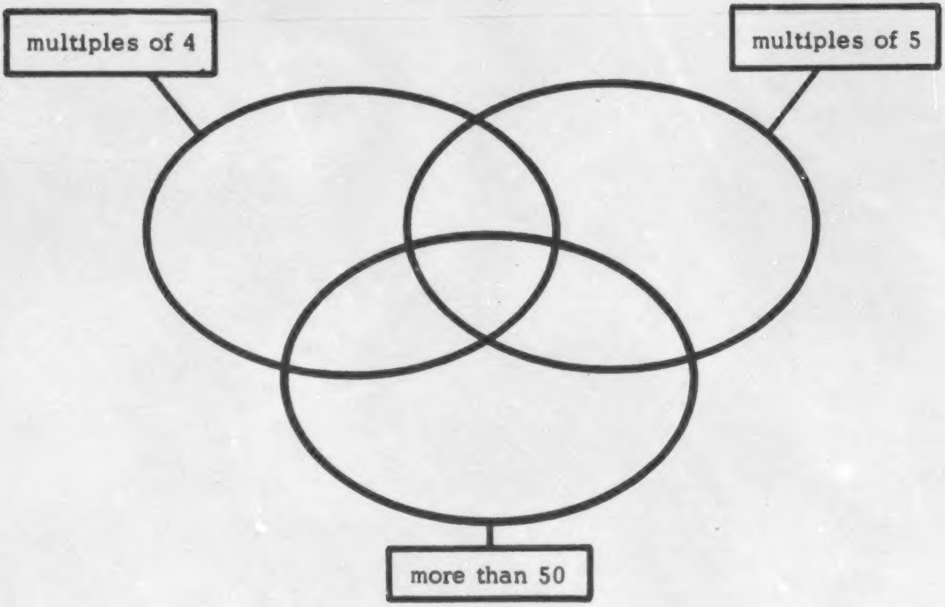
At the end of the game, decide with your class that two of the regions in the picture are empty and hatch them.



Exercise 3 (Optional)

Do this exercise if there is time and student interest.

Draw this string picture and put these lists of numbers on the board.

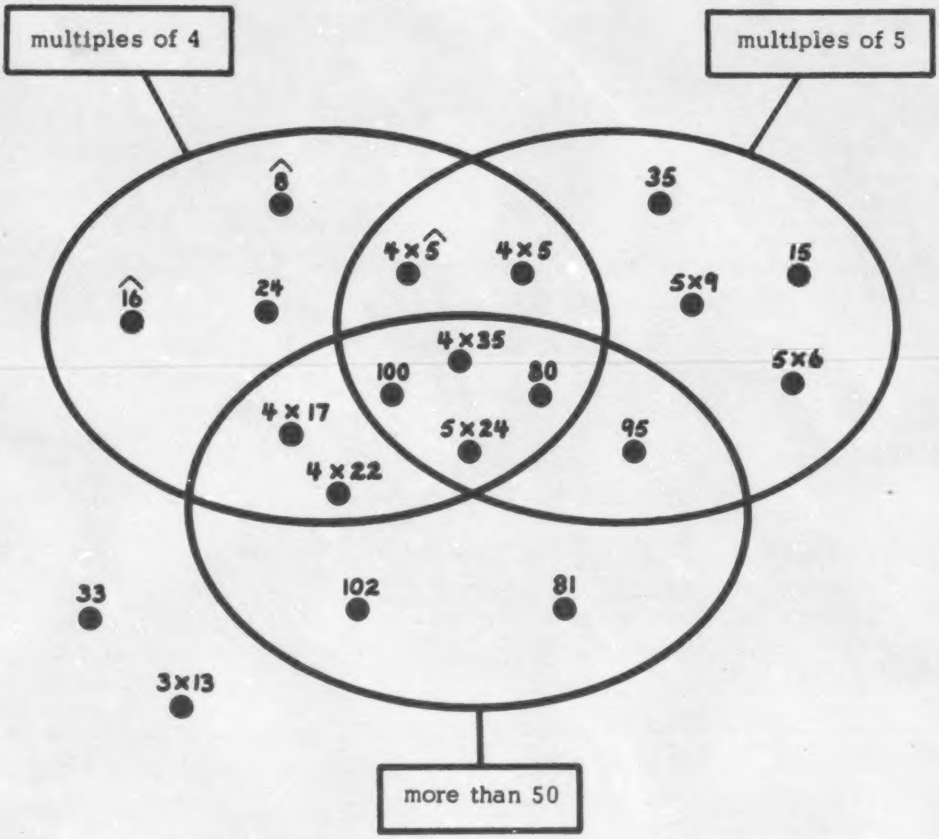

multiples of 4
multiples of 5
more than 50

TEAM A	TEAM B
$\widehat{16}$	5×24
81	80
4×22	24
15	4×17
5×6	35
100	5×9
$4 \times \hat{5}$	$\hat{8}$
33	102
4×35	4×5
3×13	95

T: We are going to play a game with numbers. During a turn, you will have only one chance to put a number in the string picture. The first team to put all the numbers from its list where they belong in the picture wins.

Alternating teams, let students locate numbers in the picture. If a number is located correctly, leave the number in the string picture and erase it from the list in which it occurs; if located incorrectly, erase it from the picture.

The next illustration shows the correct placement of the numbers in both lists.


multiples of 4
multiples of 5
8
16
24
4×5
4×5
35
15
5×9
5×6
4×35
100
80
4×17
5×24
4×22
95
33
102
81
3×13
more than 50

L 10

L10 I AM NOT MY NAME

CAPSULE LESSON SUMMARY

Read and discuss the Storybook I AM NOT MY NAME. Find many different names for 0 and for $\frac{1}{2}$, and perhaps for other numbers such as 1.

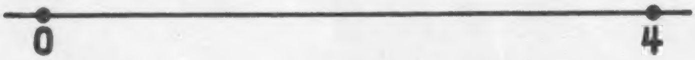
MATERIALS

Teacher: Storybook I AM NOT MY NAME; colored chalk
Student : Storybook I AM NOT MY NAME

DESCRIPTION OF LESSON

Distribute copies of the Storybook I AM NOT MY NAME and let the students look at them until you are ready to begin the lesson.

On the board draw a line segment and two dots on it, one for the number 0 and one for the number 4.



Pages 2-7

Read these pages aloud or ask some students to do so.

T: Can you imagine a dance that the numbers 16, 8, 4, 2, 1, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, and so on could do?

S: An "is larger than" dance.

S: A 2 x dance.

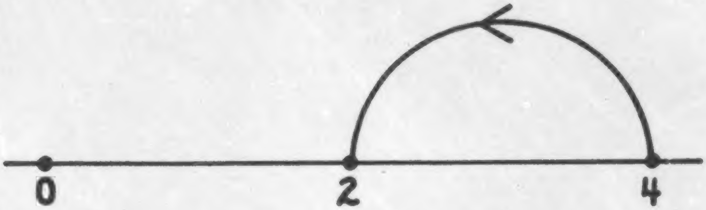
S: A $\frac{1}{2}$ x dance.

Pages 8-9

Read these pages aloud or ask some students to do so.

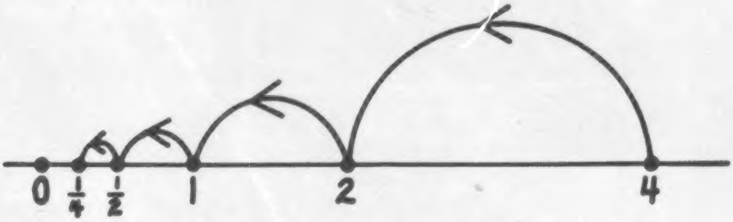
T (pointing to the number line on the board): Where is 2 on this number line?

When a student indicates the point halfway between "0" and "4", draw a dot there and label it "2". Emphasize that 2 is halfway between 0 and 4 on a number line. Draw a blue arrow from 4 to 2.



Continue in the same manner asking these questions.

- Where is 1 on this number line?
- Where is $\frac{1}{2}$ on this number line?
- Where is $\frac{1}{4}$ on this number line?



Put your finger halfway between "0" and "$\frac{1}{4}$" on the number line as you ask:

T: What number is here?

S: $\frac{1}{8}$.

T: On the number line, what number is halfway between 0 and $\frac{1}{8}$?

S: $\frac{1}{16}$.

T: On the number line, what number is halfway between 0 and $\frac{1}{16}$?

S: $\frac{1}{32}$.

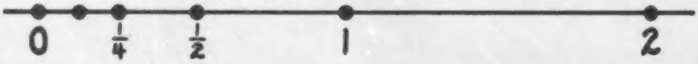
T: What could the blue arrows be for?

S: $\frac{1}{2}x$.

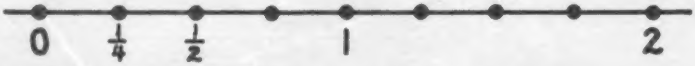
Pages 10-13

Read these pages aloud or ask some students to do so.

Erase the arrows in the picture. Rescale the number line on the board by repositioning the numbers $\frac{1}{4}$, $\frac{1}{2}$, 1, and 2 each one dot to the right. (4 is excluded from the part of the number line drawn.

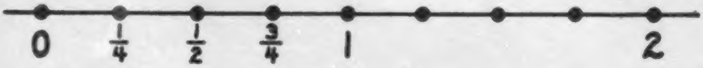


Erase the unlabeled dot next to "0". Draw additional dots so that there are seven equally spaced dots between "0" and "2".



T: Where is $\frac{3}{4}$ on this number line?

When a student suggests the dot halfway between $\frac{1}{2}$ and 1, label that dot "$\frac{3}{4}$".


0
$\frac{1}{4}$
$\frac{1}{2}$
$\frac{3}{4}$
1
2

Pages 14-15

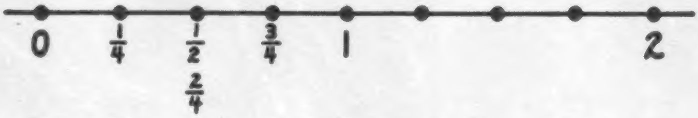
Read these pages aloud.

T: Why did $\frac{1}{2}$ begin to cry?

S: The dot for $\frac{1}{2}$ is also the dot for $\frac{2}{4}$.

Point to the appropriate dots as you say:

T: This dot is for $\frac{1}{4}$ and this dot is for $\frac{3}{4}$; so this dot (labeled "$\frac{1}{2}$") must be for $\frac{2}{4}$.


0
$\frac{1}{4}$
$\frac{1}{2}$
$\frac{2}{4}$
$\frac{3}{4}$
1
2

Pages 16-21

Read these pages aloud.

Pages 22-23

Read these pages aloud. Discuss why "175 x 45 x 0 x 45 x 728 x 75" is a name for 0.

T: What are some other names for 0?

L 10

These are some names for 0 offered by one of our development classes.

$1,000,000 - 1,000,000$	$10 - 10 + 2 - 2$
$(5 \times 50) - 250$	$100 - 1 - 99$
$16 + 16 - 32$	$50 + 50 - 50 - 50$
$1 + \hat{1}$	$0 \div 2$*

T: How could the boy in the story have more than one name?

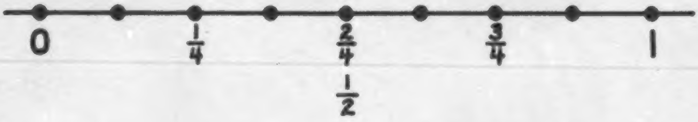
S: Maybe he has a nickname as well as his real name.

You might ask several students to give their full names.

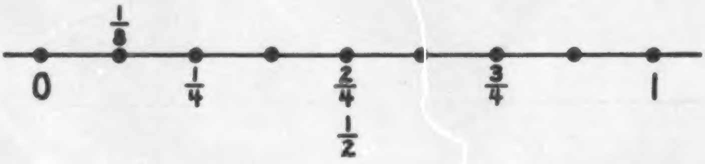
Pages 24-25

Read these pages aloud.

Rescale the number line on the board by repositioning the numbers $\frac{1}{4}$, $\frac{1}{2}$ $(\frac{2}{4})$, $\frac{3}{4}$, and 1 as in the next illustration. (2 is excluded from the part of the number line drawn.)



Ask someone to locate and label the dot for $\frac{1}{8}$.



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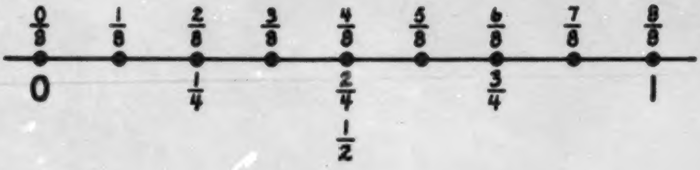
* $0 = 0 \div 2$, but $0 \neq 2 \div 0$.

Label as many of the other dots in terms of eighth's as long as there is student interest. It may be helpful to the students to consider $\frac{1}{2}$ first.


$\frac{0}{8}$
$\frac{1}{8}$
$\frac{2}{8}$
$\frac{3}{8}$
$\frac{4}{8}$
$\frac{5}{8}$
$\frac{6}{8}$
$\frac{7}{8}$
$\frac{8}{8}$
0
$\frac{1}{4}$
$\frac{2}{4}$
$\frac{1}{2}$
$\frac{3}{4}$
1

Pages 26-27

Read these pages aloud. With class assistance, draw and label dots for $\frac{9}{8}$ and $\frac{10}{8}$ on the number line.

Pages 28-31

Read these pages aloud. Ask questions such as the following:

T: What are some other names for $\frac{1}{2}$?

What is a name for $\frac{1}{2}$ which is 2 × some number? ("2 × $\frac{1}{4}$")

What is a name for $\frac{1}{2}$ which is 4 × some number? ("4 × $\frac{1}{8}$")

Do you remember a name for $\frac{1}{2}$ we used when we were working with money? ("0.50" or "0.5")

L 11

L11 Multiplication Modulo 10 #1

CAPSULE LESSON SUMMARY

Review the addition with ten friends operation, ⊕, introduced in DANCING FRIENDS and introduce a multiplication operation, ⊗, for the ten friends. Draw an arrow picture for 2⊗ with all ten numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 in it.

MATERIALS

Teacher: Colored chalk
Student : Worksheets L11* and **; pencil

DESCRIPTION OF LESSON

Exercise 1

List the whole numbers from 0 to 9 on the board. Keep the list on the board for quick reference during the lesson.

0 1 2 3 4 5 6 7 8 9

Briefly review the operation ⊕ (read: circle-plus) that the number 0 invented in the story DANCING FRIENDS. Write these problems on the board and do them as a collective activity; answers are in boxes.

$9 \oplus 9 = \boxed{8}$ $4 \oplus 7 = \boxed{1}$

$3 \oplus 6 = \boxed{9}$ $6 \oplus 6 = \boxed{2}$

$7 \oplus 5 = \boxed{2}$ $5 \oplus 8 = \boxed{3}$

Ask which numbers can be put in the frames in each of the following number sentences. A table of the solutions for the second one is given below it. (The □ and the △ numbers can, of course, be reversed.)

$$\boxed{7} \oplus 3 = 0 \qquad\qquad \square \oplus \triangle = 1$$

□	△
1	0
2	9
3	8
4	7
5	6

T: So the number friends 0 to 9 can do ⊕ dances. Now let's look at similar dances they can do using multiplication.

What number do you think 2 ⊗ 4 (read: two circle-times four) is?

S: 8.

Record this on the board.

$$2 \otimes 4 = \boxed{8}$$

T: What number do you think 3 ⊗ 4 is?

S: 2.

T: How did you get 2?

S: 3 × 4 = 12, and we keep just the last number — just 2.

$$2 \otimes 4 = \boxed{8}$$

$$3 \otimes 4 = \boxed{2}$$

Make sure the class understands this last example and then continue with other number stories. Answers are in boxes.

$$2 \otimes 4 = \boxed{8}$$

$$3 \otimes 4 = \boxed{2}$$

$$4 \otimes 4 = \boxed{6}$$

$$4 \otimes 5 = \boxed{0}$$

$$2 \otimes 9 = \boxed{8}$$

$$3 \otimes 7 = \boxed{1}$$

$$5 \otimes 3 = \boxed{5}$$

$$4 \otimes 7 = \boxed{8}$$

$$6 \otimes 4 = \boxed{4}$$

Write this open sentence on the board.

$$3 \otimes \square = 4$$

T: 3 ⊗ one of these numbers (point to the list 0 to 9) is 4. Which number could be in the box?

S: 8, because 3 × 8 = 24 which ends in "4".

T: If I change this number (4) to 7 . . .

$$3 \otimes \square = 7$$

which number could be in the box?

S: 9, because 3 × 9 = 27 which ends in "7".

T: If I change this number (7) to 1 . . .

$$3 \otimes \square = 1$$

which number could be in the box?

S: 7, because 3 × 7 = 21 which ends in "1".

Erase the number sentence and then write this one.

$$2 \otimes \square = 8$$

T: Which number could be in the box?

S: 4, because 2 × 4 = 8.

S: 9, because 2 × 9 = 18 which ends in "8".

T: If I change this number (8) to 6 . . .

$$2 \otimes \square = 6$$

which number could be in the box?

S: 3, because 2 × 3 = 6.

S: 8, because 2 × 8 = 16 which ends in "6".

L 11

Erase the number sentence and then write this one.

$$5 \otimes \square = 0$$

T: Which number could be in the box?

Students should suggest that 0, 2, 4, 6, or 8 could be in the box and should notice that these are all of the even numbers in the list 0 to 9.

T: When we take $5 \otimes$ some number, we could get 0 or we could get . . . ?

S: 5.

Erase "0" in the number sentence and write "5" in place of it.

$$5 \otimes \square = 5$$

T: Now which number could be in the box?

Students should suggest that 1, 3, 5, 7, or 9 could be in the box and should notice that these are all the odd numbers in the list 0 to 9.

Erase the number sentence and then write this one.

$$\square \otimes \triangle = 4$$

T: The number in the box may be different from or the same as the number in the triangle.

Students should be able to find all the solutions. List them in a chart as they are suggested. (The □ and the △ numbers can, of course, be reversed.)

$$\square \otimes \triangle = 4$$

1	4
2	2
2	7
3	8
4	6
6	9
8	8

Exercise 2

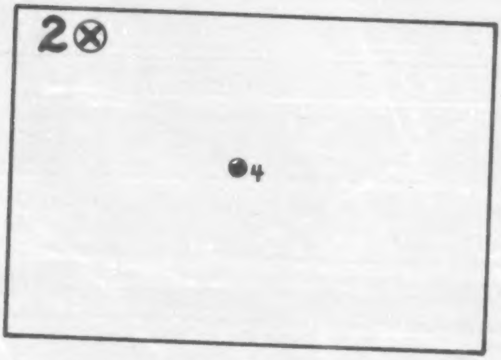
Erase the board and then write "2⊗".

T: Let's show the ten number friends in a 2⊗ arrow picture. What number would you like to start with? Remember the ten dancing friends are the numbers 0 to 9.

We do not know which number a student of yours will suggest; all we can provide is a sample dialogue. The illustrations on the right show how the picture on the board progressively develops. Careful location of the numbers in the picture will yield an attractive final drawing (see the last illustration in the sequence).

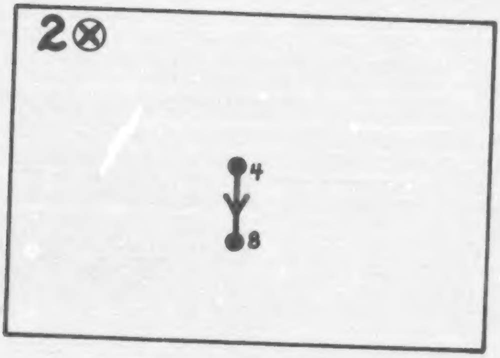
L 11

S: 4.



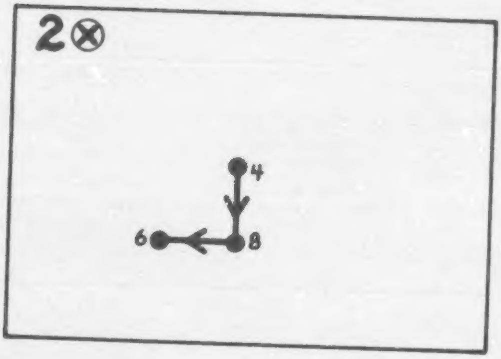
T: 2 ⊗ 4 = . . . ?

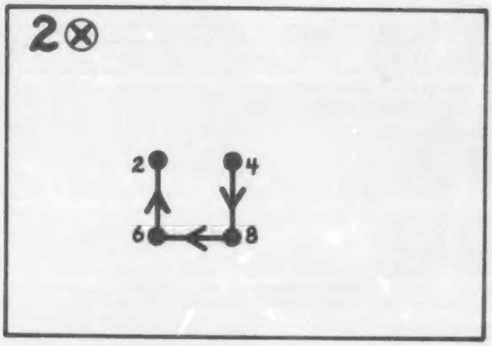
S: 8.



T: 2 ⊗ 8 = . . . ?

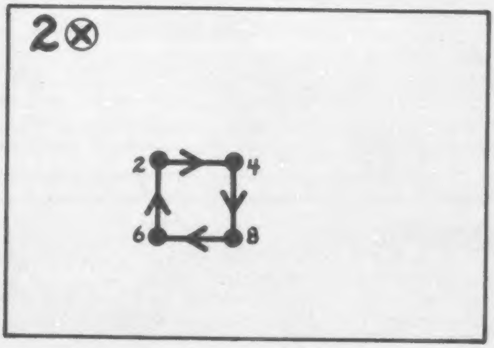
S: 6.





T: 2 ⊗ 6 = . . . ?

S: 2.

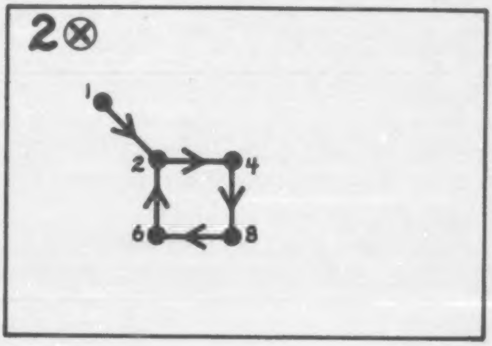


T: 2 ⊗ 2 = . . . ?

S: 4.

T: We already have a dot for 4.

T: Where should we put 1 in this picture?



S: Outside the blue square near "2".

T: 2 ⊗ 1 = . . . ?

S: 2.

T: 2 is already in our picture.

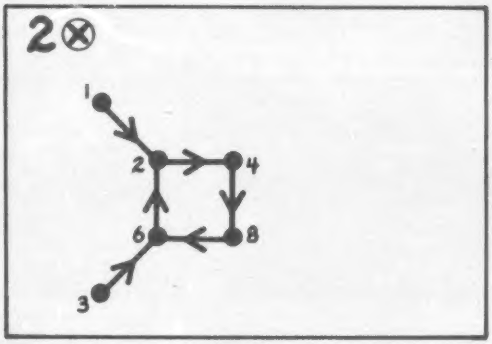
L 11

T: Where should we put 3?

S: Near 6.

T: Why there?

S: Because 2 ⊗ 3 = 6.



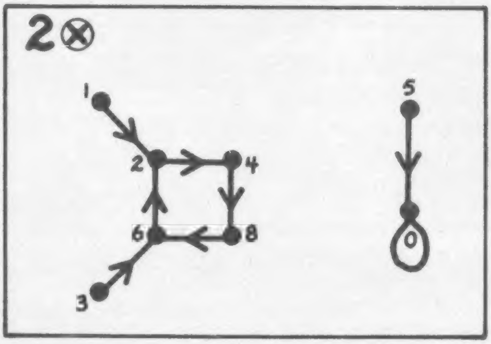
T: Where should we put 5?

S: To the side because 5 doesn't hook on to this picture.

T: 2 ⊗ 5 = . . . ?

S: 0.

S: Draw a loop at 0 because 2 ⊗ 0 = 0.

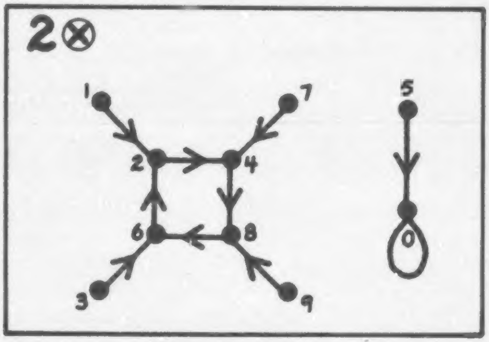


T: Where should we put 7?

S: Near 4, because 2 ⊗ 7 = 4.

T: Where should be put 9?

S: Near 8, because 2 ⊗ 9 = 8.



Worksheets L11 * and ** are available for individual work.

Name ____________________ **L 11 ***

Complete these number stories about the ten friends' multiplication game.

$1 \otimes 6 = \boxed{6}$ $9 \otimes 1 = \boxed{9}$

$2 \otimes 6 = \boxed{2}$ $2 \otimes 8 = \boxed{6}$

$3 \otimes 6 = \boxed{8}$ $3 \otimes 8 = \boxed{4}$

$4 \otimes 6 = \boxed{4}$ $5 \otimes 7 = \boxed{5}$

$5 \otimes 6 = \boxed{0}$ $4 \otimes 9 = \boxed{6}$

$6 \otimes 6 = \boxed{6}$ $7 \otimes 3 = \boxed{1}$

Name ____________________ **L 11 ****

Find several solutions to this number story.
One is done for you.

□ ⊗ △ = 2

1	2	1⊗2=2
3	4	3⊗4=2
6	2	6⊗2=2
7	6	7⊗6=2
4	8	4⊗8=2
8	9	8⊗9=2

The numbers for □ and △ may be reversed to find other solutions.

L12 Talkative Numbers #1

CAPSULE LESSON SUMMARY

Present a story situation to introduce a multiples relation; that is, two whole numbers are related if and only if one number is a multiple of the other. Construct some pictures and label dots in pictures involving this relation.

MATERIALS

Teacher: Colored chalk
Student : Worksheets L12* and **; colored pencils

DESCRIPTION OF LESSON

Distribute paper and red pencils. Start the lesson by telling this story to your class. If necessary, recall that the whole numbers are 0, 1, 2, 3, 4, and so on.

T: For an entire month it has been raining continuously. In the Whole Numbers School, the numbers are not able to play outside because of the rain so they are getting noisier and noisier. Each day they think of new tricks to play on one another.

The number 0 is the principal of the school and the number 1 is the assistant principal. They have a meeting to try and figure out what to do to make the situation better and decide upon a rule. Then they call an assembly of the whole school. 0 tells the numbers that a rule is going to be imposed because they have been so noisy. "From now on," 0 announces, "two numbers may talk to each other if and only if one of them is a multiple of the other."

Write the rule on the board.

Two numbers may talk to each other
if and only if
one of them is a multiple of the other.

T: The numbers are confused. They are not sure that they understand this rule.

Do you think that very many numbers may talk to one another with this rule in effect? Can you think of an example of two numbers who may talk to one another?

Suppose a student suggests 4 and 8.

T: Why can 4 and 8 speak to one another?

S: Because 8 is a multiple of 4.

Draw two dots and label them "4" and "8".

T: So 8 can talk to 4 and 4 can talk to 8. Instead of drawing two red arrows, I'll draw one red cord between them.

Suppose a student suggests an incorrect pair of numbers, for example 9 and 12.

T: Is 9 a multiple of 12?

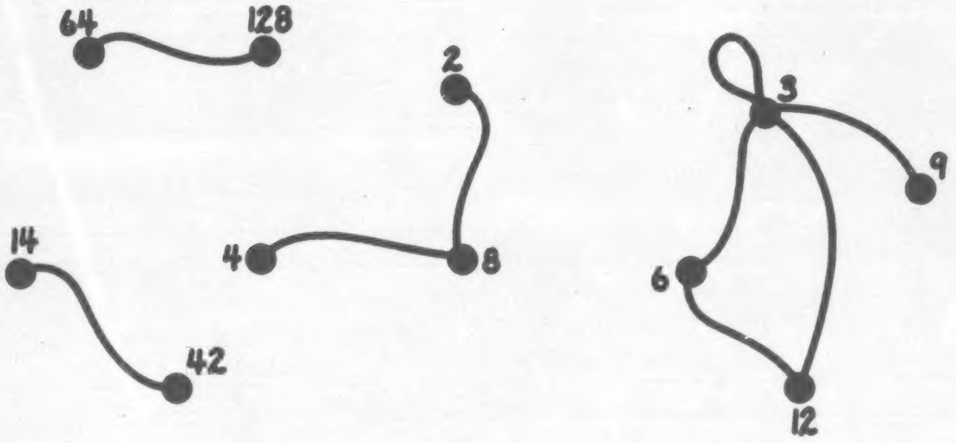
S: No.

T: Is 12 a multiple of 9?

S: No.

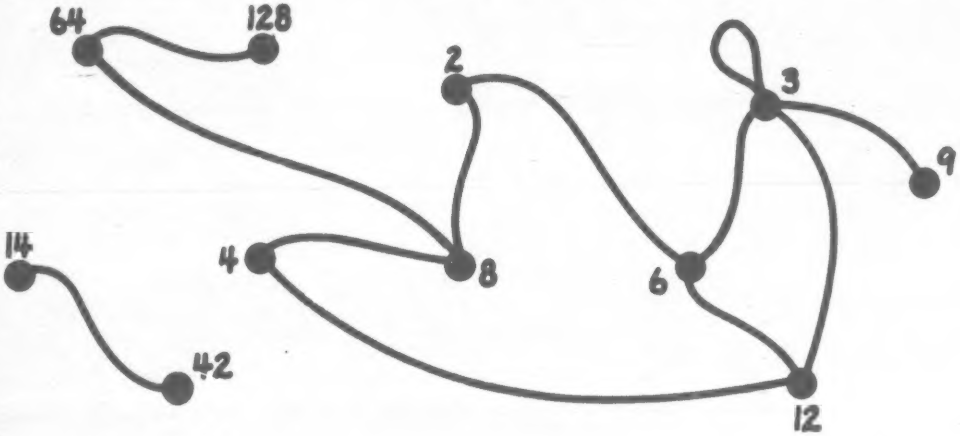
T: So 9 and 12 may not talk to one another.

Continue until several pairs of numbers have been suggested. We cannot predict which numbers your students will give, so we are providing an illustration of the picture that developed in one of our development classes.



T: Looking at the picture, do you see where we could draw other cords between these numbers?

Very likely there will be many cords which could be added to your picture; accept only a few suggestions. The next illustration shows three of the many possible cords which could be added to the picture in the preceding illustration.



479

Erase the board.

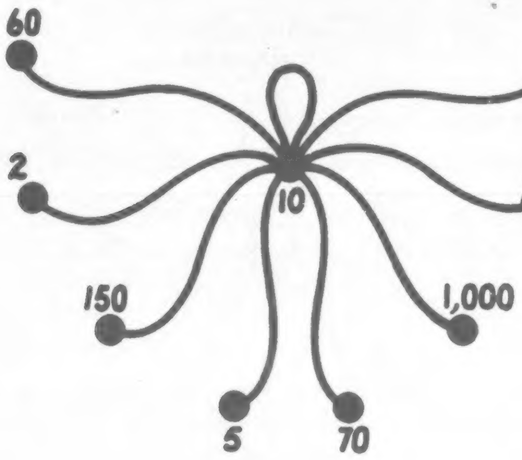
T: This time let's start with the number 10.

Draw a dot on the board and label it "10".

Ask students for some of the numbers which 10 may talk to and record them in the picture. If no one suggests numbers smaller than 10, ask the students whether or not 10 may talk to any whole numbers smaller than itself. (Yes, 10 may talk to 0, 1, 2, and 5)

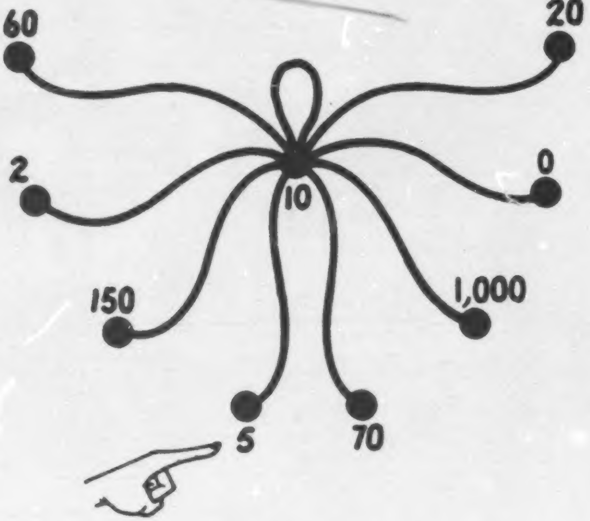
NOTE: 0 is a multiple of any whole number. When someone suggests that 0 and 10 may talk to one another, ask if 0 is a multiple of 10 or if 10 is a multiple of 0. Also, every whole number is a multiple of 1. When someone suggests that 1 and 10 may talk to one another, ask if 1 is a multiple of 10 or if 10 is a multiple of 1.

Your picture might look similar to this one.



L 12

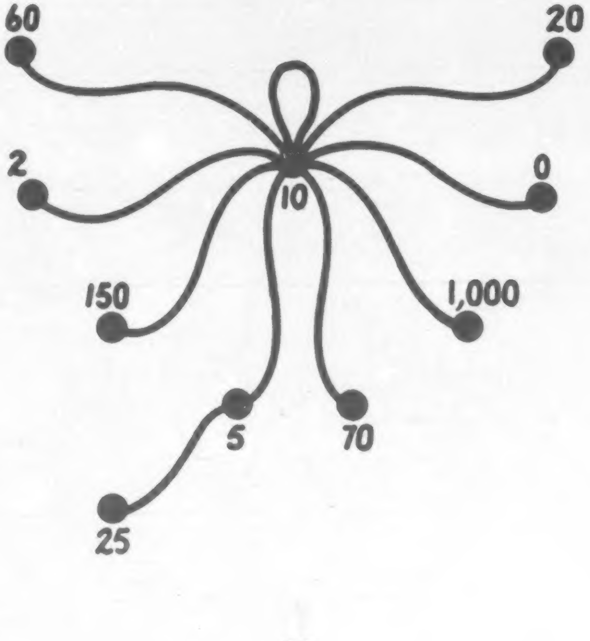
Point to one of the numbers in the picture other than 10.


60
20
2
0
10
150
1,000
5
70

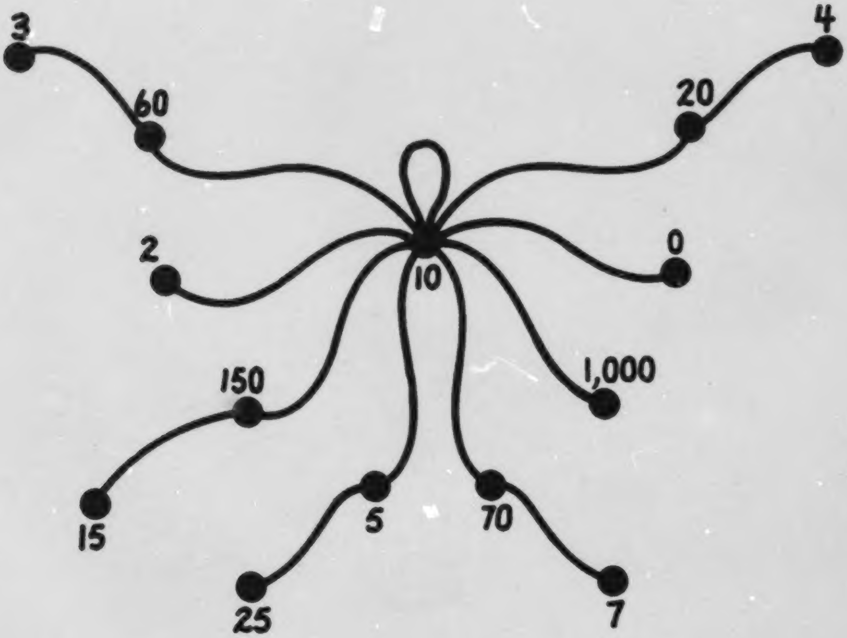
T: What is a number 5 may talk to which is not already in our picture?

S: 25.

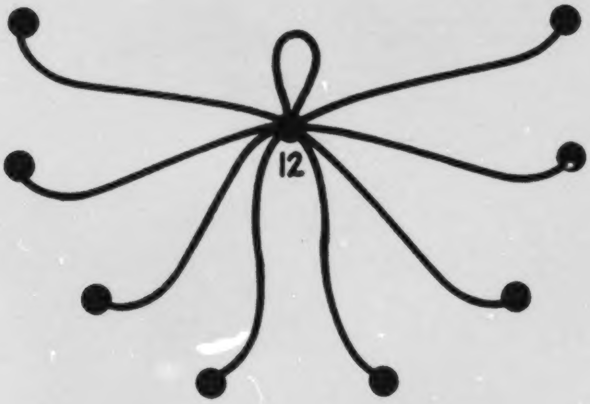
Draw a dot for the number suggested and the appropriate cord.


60
20
2
0
10
150
1,000
5
70
25

In a similar manner extend your picture several times; for example,


3
60
4
20
2
10
0
150
1,000
15
5
70
25
7

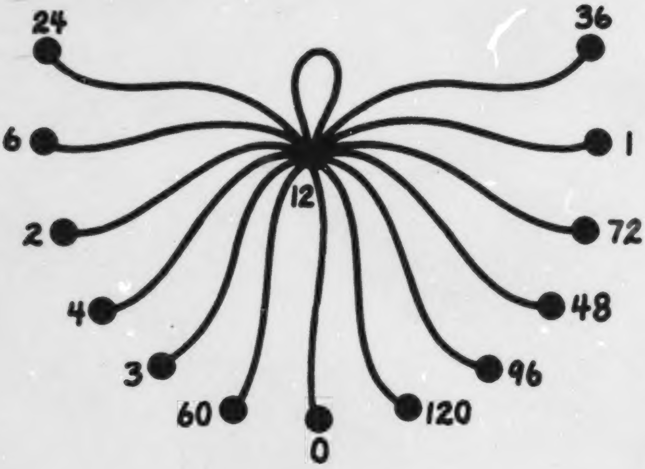
Erase the board and then draw this picture on the board.



T: Draw a picture on your paper showing some numbers 12 may talk with.

As students are working, walk around the room and observe which numbers they are putting in their pictures. Ask students, one at a time, to put particular numbers in the picture on the board; choose a wide range of numbers from their papers.

The following illustration shows the numbers one of our development classes put in the picture.



Check that each number in the picture has permission to talk with 12; for example,

T: Why may 12 and 6 talk to one another?

S: 12 is a multiple of 6.

T: Why may 12 and 24 talk to one another?

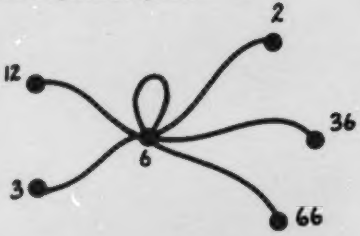
S: 24 is a multiple of 12.

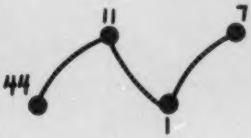
Worksheets L12* and ** are available for individual work.

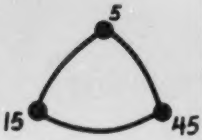
Name ____________________ L 12 *

Two numbers may talk to each other
if and only if
one number is a multiple of the other.

Label the dots. Many solutions are possible.





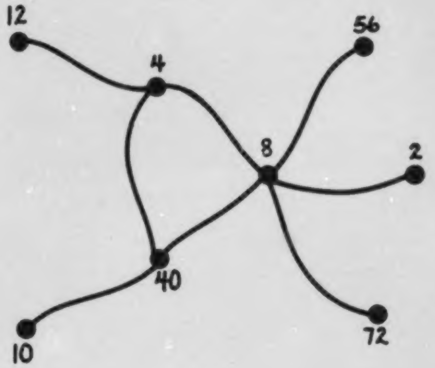


Other solutions are possible.

Name ____________________ L 12 **

Two numbers may talk to each other
if and only if
one number is a multiple of the other.

Label the dots. Many solutions are possible.


12
56
4
8
2
40
10
72

Other solutions are possible.

L 13

L13 Who is Max?

CAPSULE LESSON SUMMARY

Solve a detective story about a secret number called "Max". One clue involves finding all the numbers which can be named using, in order, one symbol from each of three given sets. A second clue involves finding which numbers can be obtained by moving exactly one checker of a given configuration on the Minicomputer. The final clue involves Max's location in a 3-string picture. Play a game with strings for multiples of 10 and multiples of 4.

MATERIALS

Teacher: Minicomputer set; colored chalk
Student : None

DESCRIPTION OF LESSON

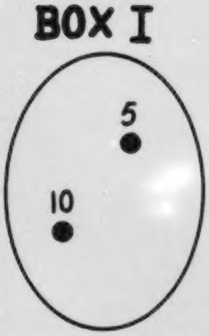
Exercise 1

T: I have a secret number which I'm going to call "Max". See if you can figure out which number Max is.

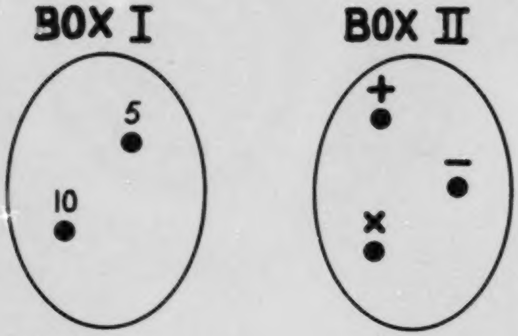
First clue

Briefly recall the situation from Lesson L7 Three Boxes #1. Draw string pictures for three imaginary boxes as you reveal their contents.

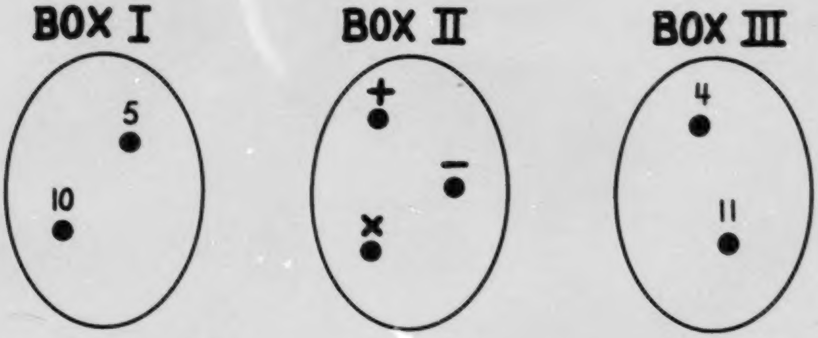
T: Today you must pretend that I have three boxes. In the first box there are two pieces of paper; one has "5" written on it and the other "10".



T: In the second box there are three pieces of paper; one with "+", one with "-", and one with "x".



T: And in the third box there are two pieces of paper; one with "4" and one with "11".



T: A combination of something from the first box, something from the second box, and something from the third box gives a name for Max. What are some numbers Max could be?

Let two or three students suggest possibilities and record them on the board.

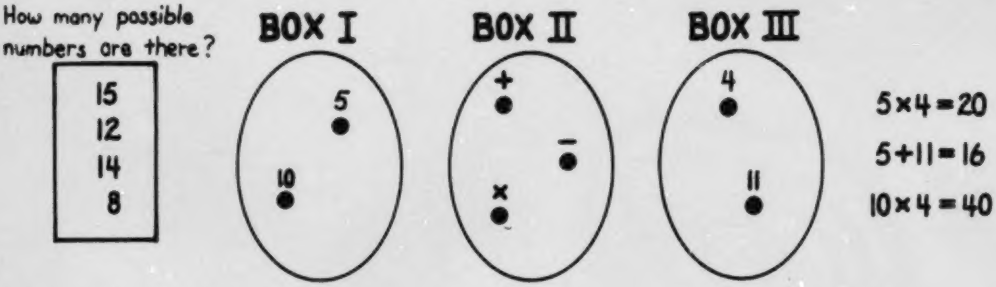
S: 20, because 5 × 4 = 20.

S: 16; 5 + 11 = 16.

S: 40; 10 x 4 = 40.

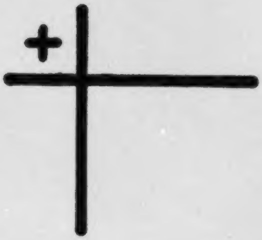
T: How many numbers do you think we could get in this way?

Record the students' predictions on the board. Your chalkboard might look similar to this.

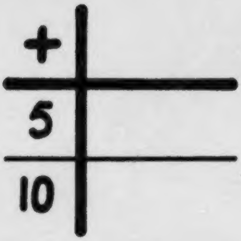

How many possible numbers are there?
15
12
14
8
BOX I
5
10
BOX II
+
−
×
BOX III
4
11
5×4=20
5+11=16
10×4=40

T: How can we find all the possibilities? Do you remember what we did before?

If no one suggests making tables for each of the three operations in the second string, suggest this yourself. Draw a table for addition on the board.



T: This table is just for addition. I'll put the numbers from the first box here . . .



. . . and the numbers from the third box here.

+	4	11
5		
10		

Let the students fill in the table on the board.

+	4	11
5	9	16
10	14	21

T: So far we have four possibilities Max could be. What other tables do we need?

S: One for "-" and one for "x".

Draw tables for each of these operations and invite students to fill them in with the appropriate numbers. At some point, students should realize that this clue gives twelve possibilities for Max. Compare 12 to the predictions made earlier in the lesson.

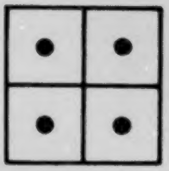
+	4	11
5	9	16
10	14	21

−	4	11
5	1	$\hat{6}$
10	6	$\hat{1}$

x	4	11
5	20	55
10	40	110

Second clue

Display one Minicomputer board and put this configuration on it.

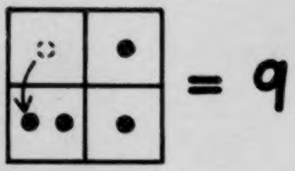


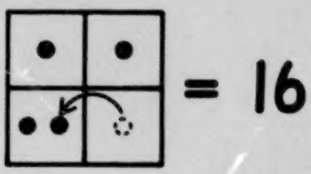
T: What number is on the Minicomputer?

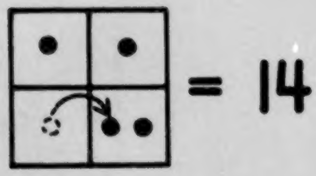
S: 15.

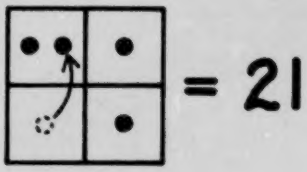
T: You can get Max on the Minicomputer by moving exactly one of these checkers. Which of these numbers (point to the three tables) could Max be?

Whenever a student suggests a number from the tables, ask the student to convince the class that the number is a possibility for Max by moving one of the checkers in the configuration on the Minicomputer. Since only the numbers in the addition table can be put on the Minicomputer by moving exactly one checker, they very likely will be the first numbers considered. Circle them in the table as they are offered and checked.









+	4	11
5	⑨	⑯
10	⑭	㉑

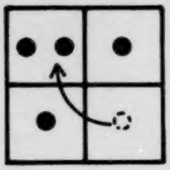
−	4	11
5	1	$\hat{6}$
10	6	$\hat{1}$

×	4	11
5	20	55
10	40	110

T: What is the largest number you can get by moving one of these checkers?

S: 22.

Ask a student to move one of the checkers so that 22 is on the Minicomputer.



T (pointing to the multiplication table): So which of these numbers do we need to check?

S: 20.

T: Could we get 20 on the Minicomputer by moving one of these checkers?

Let students have a few minutes to think about this. By trial and error most will arrive at the conclusion that it cannot. Perhaps a student will notice that to go from 15 to 20 is an increase of 5, which is an odd number. Moving a checker from a square whose value is an even number to another square whose value is an even number will increase or decrease a number by an even number. Therefore, the checker on the 1-square must be moved. Since it would have to be moved to a square of value 6 in order to increase the number by 5, the student would conclude 20 cannot be put on the Minicomputer by moving exactly one checker.

+	4	11
5	⑨	⑯
10	⑭	㉑

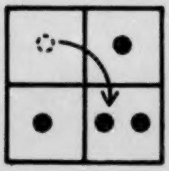
−	4	11
5	1	$\hat{6}$
10	6	$\hat{1}$

~~×~~	~~4~~	~~11~~
~~5~~	~~20~~	~~55~~
~~10~~	~~40~~	~~110~~

T: What is the smallest number we could get by moving exactly one of these checkers?

S: 8.

Ask a student to move one of these checkers so that 8 is on the Minicomputer.



T (pointing to the table for subtraction): If 8 is the smallest number we can get in this way, can we get any of these numbers? (No)

Cross out the table for subtraction.

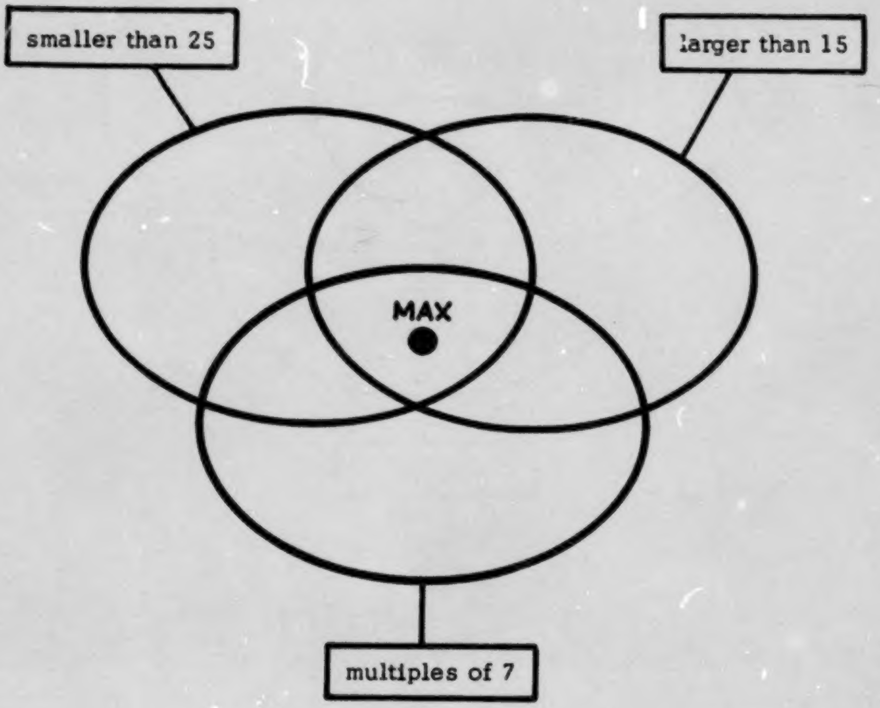
+	4	11
5	⑨	⑯
10	⑭	㉑

~~−~~	~~4~~	~~11~~
~~5~~	~~1~~	~~$\hat{6}$~~
~~10~~	~~6~~	~~$\hat{1}$~~

~~×~~	~~4~~	~~11~~
~~5~~	~~20~~	~~55~~
~~10~~	~~40~~	~~110~~

Third clue

Draw this string picture on the board.

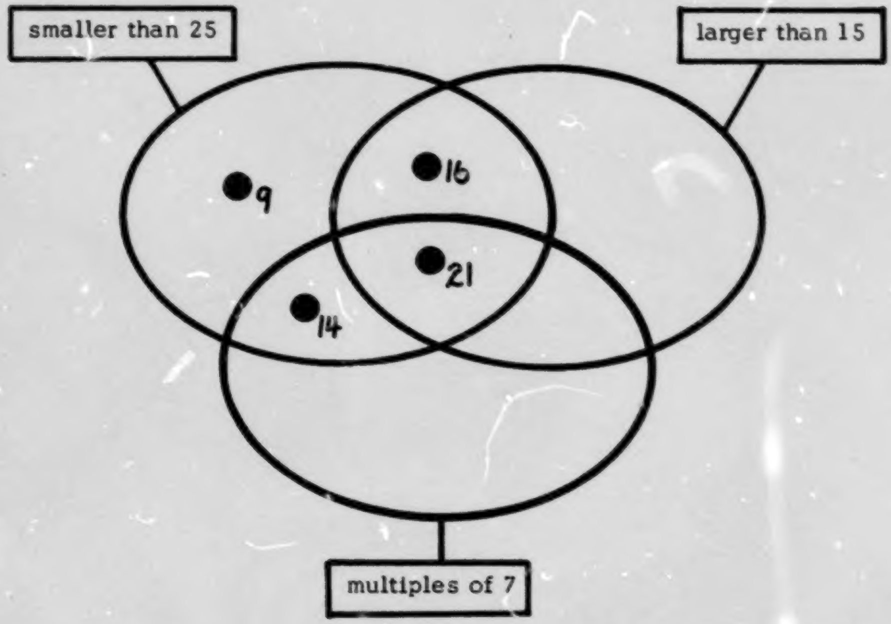

smaller than 25
larger than 15
MAX
multiples of 7

T: Max is one of these four numbers (point to the addition table) and I've shown you where Max is in this string picture. Which number is Max?

Take whispers and then let someone explain why Max is 21.

S: 21 is smaller than 25; 21 is larger than 15; and 21 is a multiple of 7.

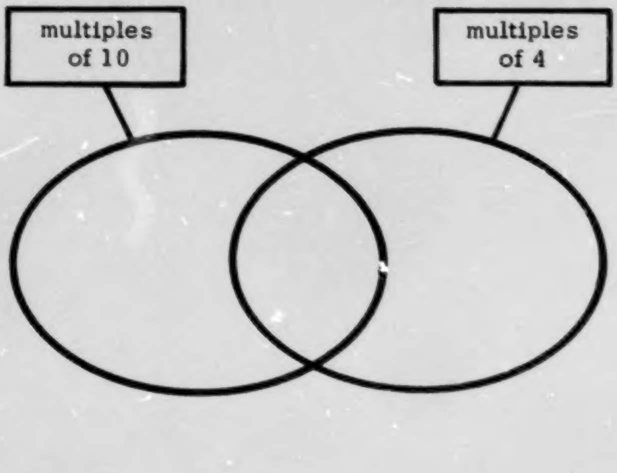
Ask students to locate the other three numbers (9, 14, and 16) in the picture.


smaller than 25
larger than 15
9
16
21
14
multiples of 7

Exercise 2

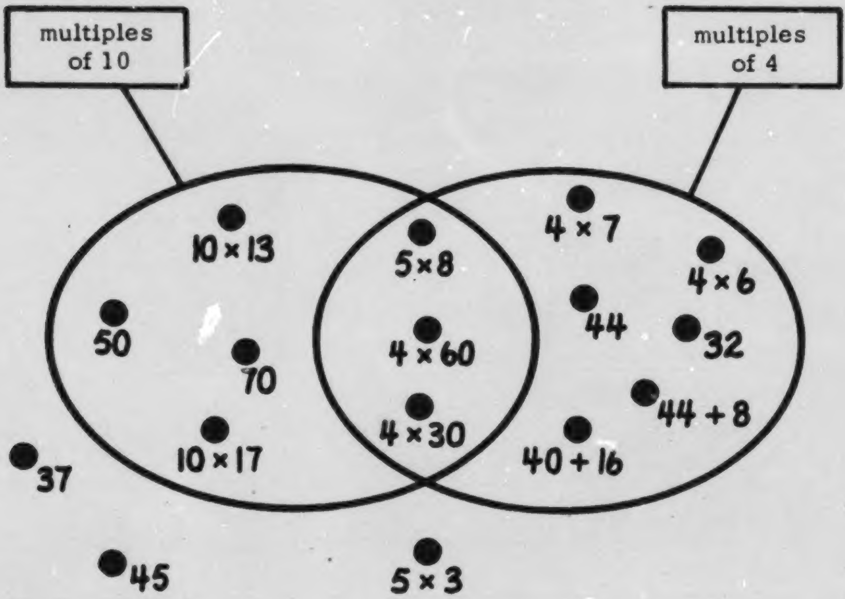
Erase the board and then draw two different colored, overlapping strings. Label one string "multiples of 10" and the other "multiples of 4". Make two lists (as shown below) of numbers near the strings. Divide the class into two teams, Team A and Team B.

TEAM A	TEAM B
37	70
44+8	4×7
4×6	32
44	4×60
4×30	45
50	40+16
5×8	5×3
10×13	10×17


multiples of 10
multiples of 4

Alternating teams, let students locate numbers in the picture. If a number is located correctly, erase it from the list; if a number is located incorrectly, erase it from the string picture. Continue until one team wins by placing all of its numbers in the picture. If you want, let students locate any remaining numbers in the picture.

Correct placement of all the numbers is given for your convenience. You may wish to use this picture as a crib-sheet during the play of the game.


multiples of 10
multiples of 4
10 × 13
50
70
10 × 17
5 × 8
4 × 60
4 × 30
4 × 7
4 × 6
44
32
44 + 8
40 + 16
37
45
5 × 3

L14 Three Boxes #2

CAPSULE LESSON SUMMARY

Introduce the idea of two numbers being on the same +5 (+10) arrow road. Examine sentences that can be formed by selecting in order a number from Box I, a "is on the same + □ arrow road" relation from Box II, and a number from Box III. Find that some of these sentences are true and some are false. Make tables thereby looking at all the possible sentences and determine the chances of getting a true (false) sentence. Discuss how to change the situation so that you always get a true sentence, always get a false sentence, or get a true sentence about half the time.

MATERIALS

Teacher: Three boxes; eleven pieces of paper (about 10 cm by 10 cm)

Student: Worksheet L14; pencil

NOTE: Prior to presenting this lesson, prepare three small boxes with labels "I", "II", and "III". Into the respective boxes put pieces of paper marked as indicated below and folded so as to hide what is written.

I: $\boxed{\hat{8}}$ $\boxed{\hat{2}}$ $\boxed{7}$ $\boxed{33}$

II: | is on the same +5 arrow road | | is on the same +10 arrow road |

III: $\boxed{13}$ $\boxed{17}$ $\boxed{28}$ $\boxed{32}$ $\boxed{103}$

DESCRIPTION OF LESSON

Exercise 1

Write this sentence on the board.

□ is on the same +5 arrow road as △.

Ask someone to read the sentence.

T: What numbers could we put in the box and in the triangle to make a true number story?

S: 55 and 60.

S: 110 and 125.

When someone suggests two numbers that are on a +5 arrow road which does not meet the number 0, pursue the choice with the student. For example,

S: 1 and 101.

T: How do you know that 1 and 101 are on the same +5 arrow road?

S: 0 and 100 are on the same +5 arrow road; I added 1 to each.

Continue until several pairs of numbers have been suggested. Then write "7" in the triangle.

☐ is on the same **+5** arrow road as 7⃤.

T: What number could we put in the box to make a true number story?

S: 2.

S: $\hat{3}$, because $\hat{3} + 5 = 2$ and $2 + 5 = 7$.

If no one suggests a negative number, ask specifically for a number less than 0.

T: What large number could we put in the box?

S: 7,007.

S: 7,012.

If students suggest numbers which do not end in "7" or in "2", ask them how they decided upon those numbers; for example,

S: 7,024.

T: How did you decide that 7,024 is on the same +5 arrow road as 7?

S: I added fives to 7,012.

T: Let's check. 7,012 + 5 = . . . ? (7,017)
7,017 + 5 = . . . ? (7,022)
7,022 + 5 = . . . ? (7,027)
Can 7,024 be in the box? (No)

Change "+5" to "+10" in the sentence on the board and erase "7" in the triangle. Point out the change to the students.

□ is on the same +10 arrow road as △.

T: What numbers could we put in the box and in the triangle to make a true number story?

S: 10 and 100.

S: 110 and 120.

S: 3 and 103.

S: 85 and 105.

Write "28" in the box.

$\boxed{28}$ is on the same +10 arrow road as △

T: What number could we put in the triangle to make a true number story?

Elicit several possibilities, large and small. Any positive number ending in "8" is correct and any negative number ending in "2" is correct.

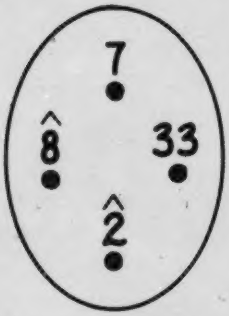
Exercise 2

Display the three boxes that you prepared for the lesson.

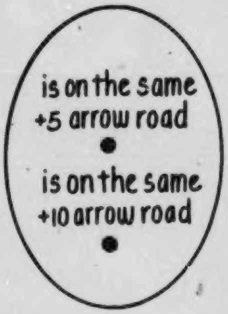
T: In these three boxes there are pieces of paper. Each piece of paper has something written on it.

Draw a string picture for each box as you reveal its contents to the class.

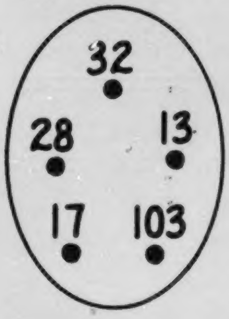
BOX I



BOX II



BOX III



Invite a student to draw a piece of paper out of each box. For each of the three pieces of paper, the student should read aloud what is written on it, fold the paper, and then return it to the box from which it was taken.

Write the resulting number story on the board. A sample dialogue is given.

S: 33 . . . is on the same +5 arrow road . . . as 17.

33 is on the same +5 arrow road 17

T: Is that a true or a false number story?

S: False. 17 + 5 = 22, 22 + 5 = 27, 27 + 5 = 32, and 32 + 5 = 37. We can get to 32 but not to 33.

Indicate that this number story is false.

33 is on the same +5 arrow road 17 Ⓕ

In the same manner let students perform two or three more trials of the experiment with the boxes and record the results. Then ask another student to try the experiment only this time interrupt the activity before the student draws from the third box. Suppose the student's name is "Stanley".

Stanley: $\hat{2}$ . . . is on the same +10 arrow road . . .

T: Wait a moment before you select from the third box. Do you (address the class) think Stanley could get a true number story?

S: If he gets "28", it will be true.

Permit the student to select from the third box and write the result on the board; for example,

$\hat{2}$ is on the same +10 arrow road 13 Ⓕ

You should have a list of number stories on the board. This is a sample list.

33	is on the same +5 arrow road	17	Ⓕ
$\hat{8}$	is on the same +10 arrow road	32	Ⓣ
7	is on the same +5 arrow road	32	Ⓣ
$\hat{2}$	is on the same +10 arrow road	13	Ⓕ

T: How many different number stories do you think we can get? How could we find out?

S: Make tables.

Perhaps someone will suggest another systematic way of counting the possible number stories; but for the benefit of the class, make the tables.

Begin two tables on the board. Explain that the numbers in the first box (string) are listed vertically under "is on the same +5 arrow road" and under "is on the same +10 arrow road" and that the numbers in the third box (string) are listed horizontally to the right of "is on the same +5 arrow road" and to the right of "is on the same +10 arrow road."

is on the same +5 arrow road	13	17	28	32	103
$\hat{8}$					
$\hat{2}$					
7					
33					

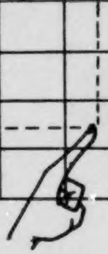
is on the same +10 arrow road	13	17	28	32	103
$\hat{8}$					
$\hat{2}$					
7					
33					

L 14

Fill in one of the squares of the table for being on the same + 5 arrow road.

T: 7 is on the same + 5 arrow road as 32.
Is that true or false?

is on the same + 5 arrow road	13	17	28	32	103
$\hat{8}$					
$\hat{2}$					
7					
33					



S: True.

is on the same +5 arrow road	13	17	28	32	103
$\hat{8}$					
$\hat{2}$					
7				T	
33					

T: How many different number stories are represented in this table for being on the same + 5 arrow road?

Trace each row and count by fives with the students: "5, 10, 15, 20."

T: We also have a table for being on the same + 10 arrow road. So how many number stories altogether can we get using these boxes?

S: Forty, because 20 + 20 = 40.

Comment if anyone had predicted 40 earlier.

In the tables enter the true and false values of the number stories listed on the board earlier in the lesson.

is on the same +5 arrow road	13	17	28	32	103
$\hat{8}$					
$\hat{2}$					
7				T	
33		F			

is on the same +10 arrow road	13	17	28	32	103
$\hat{8}$				T	
$\hat{2}$	F				
7					
33					

T: There are forty number stories altogether. Do you think there are more true or more false or the same number of each? If we play the game with the boxes and one team receives a point for every true number story and the other team receives a point for every false number story, would the game be fair?
How can we find out?

S: We could play the game over and again.

T: But we'd need to play a long time.

S: We could fill in the tables.

Distribute copies of Worksheet L14. Students should fill in the tables on their worksheets while you invite students to complete the tables at the board. Let each student who comes to the board fill in an entire row. When a student fills in a square, read the corresponding number story and announce that it is true or false as the case may be.

The next illustration shows the completed tables.

is on the same +5 arrow road	13	17	28	32	103
$\hat{8}$	F	T	F	T	F
$\hat{2}$	T	F	T	F	T
7	F	T	F	T	F
33	T	F	T	F	T

is on the same +10 arrow road	13	17	28	32	103
$\hat{8}$	F	F	F	T	F
$\hat{2}$	F	F	T	F	F
7	F	T	F	F	F
33	T	F	F	F	T

T: How many true number stories are there? (Fifteen)

How many false . . . ? (Twenty-five)

Record this information on the board.

$$\begin{array}{rl} 15 & \text{true} \\ +\,25 & \text{false} \\ \hline 40 & \end{array}$$

T: In the game do we have a better chance of getting a true number story or a false number story?

S: We have a better chance of getting a false number story.

T: Right! We have 25 chances out of 40 of getting a false number story, but only 15 chances out of 40 of getting a true number story.

Let the students openly discuss some or all of the following questions.

- Could we change the situation so that we would always get true number stories?

  (Put numbers which end in the same number in both the first string and the third string; for example, put positive numbers which all end in "2".)

- Could we change the situation so that we could always get false number stories?

  (One way would be to put numbers which end in "0" in the first string and to put numbers which end in "1" in the third string.)

- Could we make the game fair?

  (One way would be to put numbers which end in "5" in the first string and to put numbers which end in "0" in the third string.)

L 15

L15 The String Game #3

CAPSULE LESSON SUMMARY

Analyze the information obtained from three starting clues (correctly placed pieces) in a String Game and find that one string is determined. Determine the other string from one additional clue. Play a 3-string version of the String Game.

(Optional) Play a 3-string game involving multiples and order.

MATERIALS

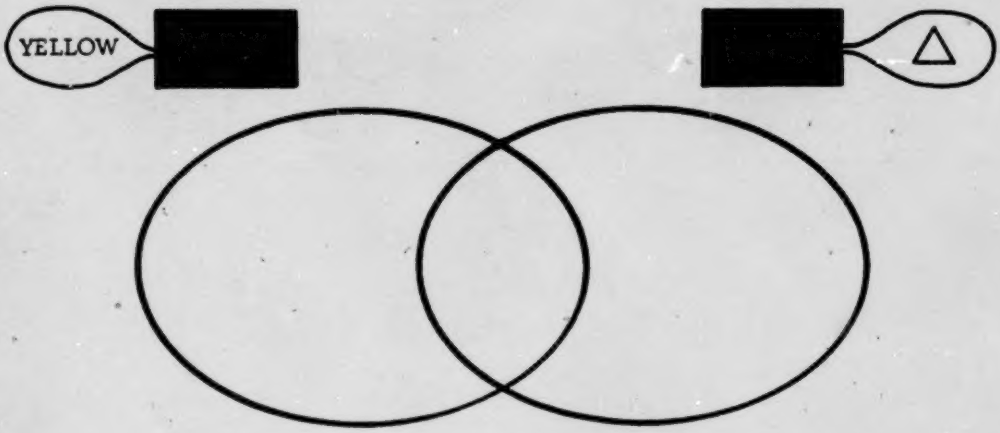
Teacher: A-Blocks String Game kit; colored chalk

Student : None

DESCRIPTION OF LESSON

Exercise 1

Before the lesson begins, set up your board for the String Game as illustrated below. The bubbles show what is on the hidden tags. Tape two A-Blocks String Game Posters (Version B) to the board.



RED

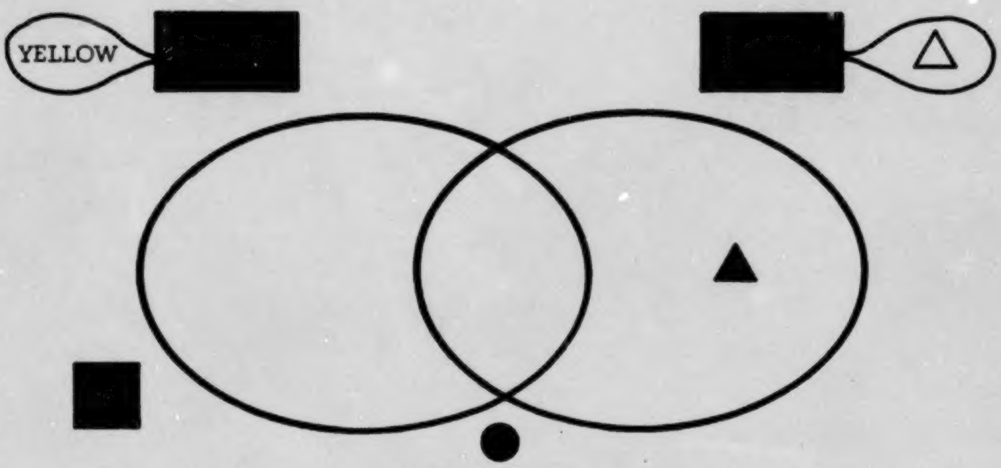
RED	YELLOW	GREEN	BLUE
NOT RED	NOT YELLOW	NOT GREEN	NOT BLUE
○	△	□	BIG
NOT ○	NOT △	NOT □	LITTLE

BLUE

RED	YELLOW	GREEN	BLUE
NOT RED	NOT YELLOW	NOT GREEN	NOT BLUE
○	△	□	BIG
NOT ○	NOT △	NOT □	LITTLE

T: We are going to play the String Game today, but first we are going to look at what information we get from knowing where some of the game pieces belong in the picture. It's like a detective story in which we are trying to figure out what the strings are for.

Put the small green triangle, the small blue circle, and the large red square in the picture.



T: What does this information tell us about the strings? Is there any of these labels (point to one of the lists) that the strings could not have?

When explaining why a label can be crossed off, a student should identify which of the three pieces gives that information and why. If no one responds, choose one of the labels and ask if the red string could have that label; for example, ask if the red string could be for □.

The following sample dialogue indicates how some labels are eliminated from the lists.

S: The red string cannot be for GREEN, because the little green triangle is outside the red string.

T: Could the blue string be for GREEN?

S: Yes.

Cross off GREEN from the Red list.

S: Neither string could be for LITTLE, because the little blue circle is outside both strings.

Cross off LITTLE from both lists.

S: The blue string cannot be for NOT △, because there is a triangle in it.

T: Could the red string be for NOT △?

S: No, because the large red square and the small blue circle are outside of it. They are not triangles.

Cross off NOT △ from both lists.

Continue in this way until the red string is determined (YELLOW) and there are only two remaining possibilities for the blue string (GREEN and △).

RED

~~RED~~	YELLOW	~~GREEN~~	~~BLUE~~
~~NOT RED~~	~~NOT YELLOW~~	~~NOT GREEN~~	~~NOT BLUE~~
~~○~~	~~△~~	~~□~~	~~BIG~~
~~NOT ○~~	~~NOT △~~	~~NOT □~~	~~LITTLE~~

BLUE

~~RED~~	~~YELLOW~~	GREEN	~~BLUE~~
~~NOT RED~~	~~NOT YELLOW~~	~~NOT GREEN~~	~~NOT BLUE~~
~~○~~	△	~~□~~	~~BIG~~
~~NOT ○~~	~~NOT △~~	~~NOT □~~	~~LITTLE~~

Your students might be surprised that the red string is determined and yet no pieces were placed inside the red string as clues.

T: The red string is for YELLOW, but we don't know yet whether the blue string is for GREEN or for △. I'll give you another clue.

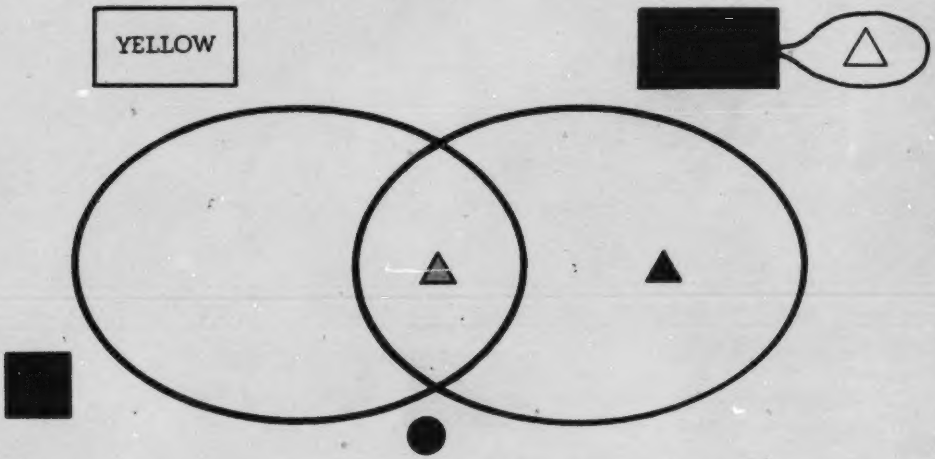
Hold up the small yellow triangle as you ask:

T: Do we know anything about where this piece belongs in the picture?

S: It must go in the red string because it is yellow.

If no one suggests this, ask whether the small yellow triangle belongs inside or outside the red string.

T: I'll show you where it belongs.



T: Now, do we know what the blue string is for?

S: △.

T: Why can we cross off GREEN from the Blue list?

S: Because there is a yellow piece in the blue string.

RED

~~RED~~	[✓ YELLOW]	~~GREEN~~	~~BLUE~~
~~NOT RED~~	~~NOT YELLOW~~	~~NOT GREEN~~	~~NOT BLUE~~
~~○~~	~~△~~	~~□~~	~~BIG~~
~~NOT ○~~	~~NOT △~~	~~NOT □~~	~~LITTLE~~

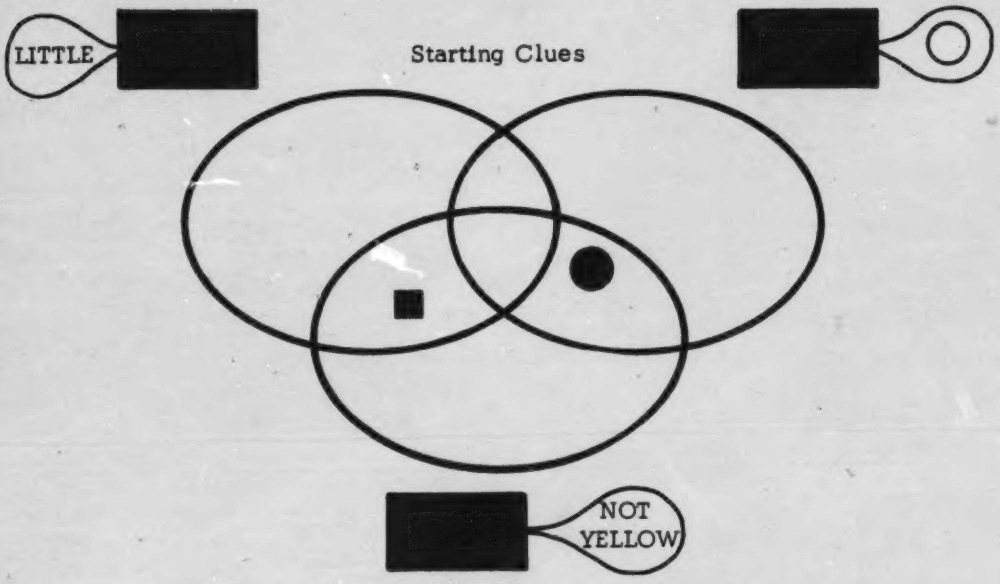
BLUE

~~RED~~	~~YELLOW~~	~~GREEN~~	~~BLUE~~
~~NOT RED~~	~~NOT YELLOW~~	~~NOT GREEN~~	~~NOT BLUE~~
~~○~~	[✓ △]	~~□~~	~~BIG~~
~~NOT ○~~	~~NOT △~~	~~NOT □~~	~~LITTLE~~

L 15

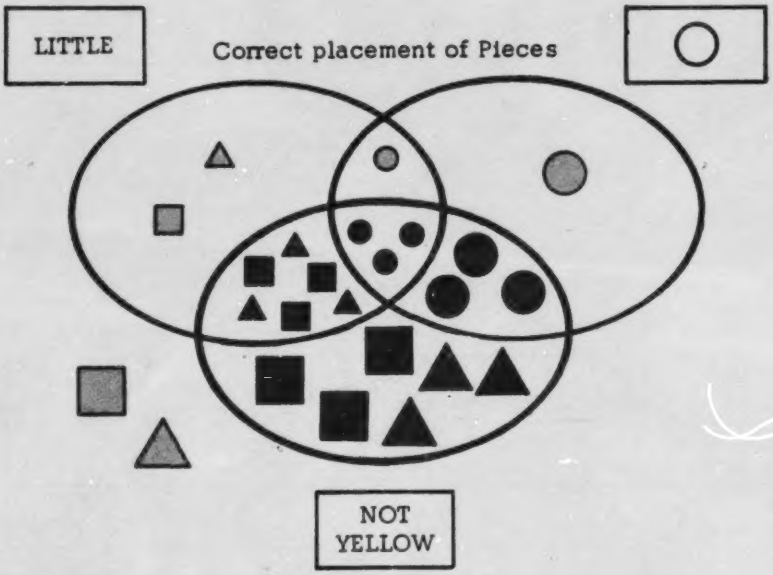
Exercise 2

Play the String Game with three strings in the usual way. A possible game is suggested below.



Starting Clues

LITTLE
NOT YELLOW



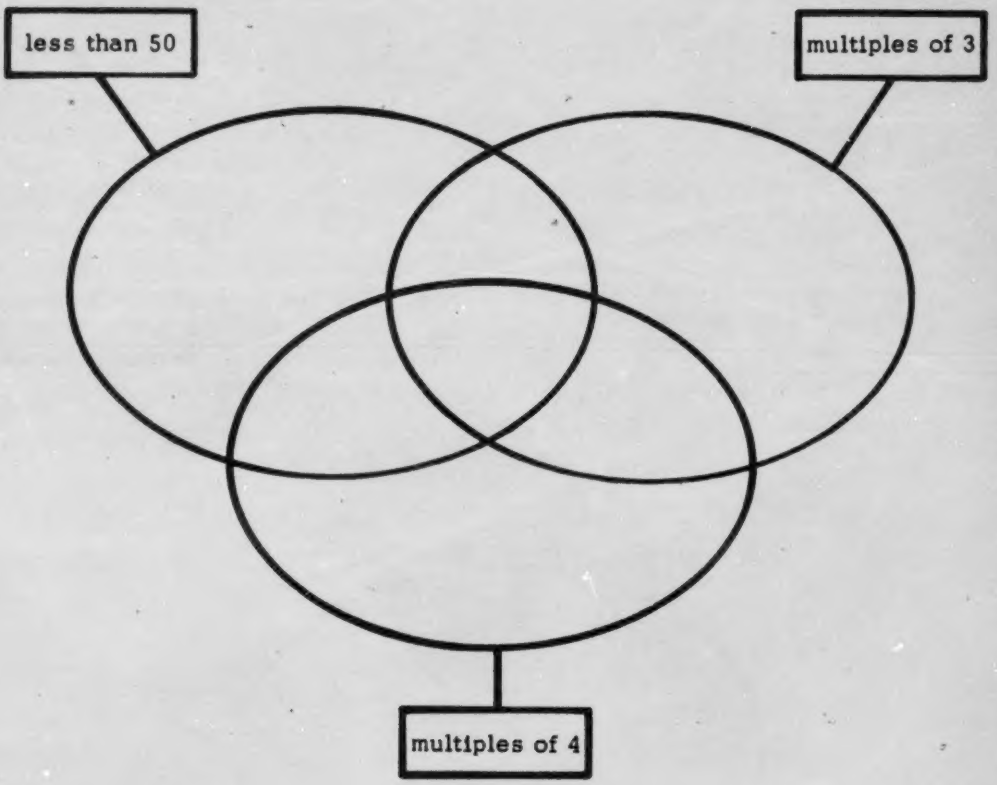
Correct placement of Pieces

LITTLE
NOT YELLOW

Exercise 3 (Optional)

Do this exercise if there is sufficient time and student interest.

Draw this string picture and put these two lists of numbers on the board.


less than 50
multiples of 3
multiples of 4

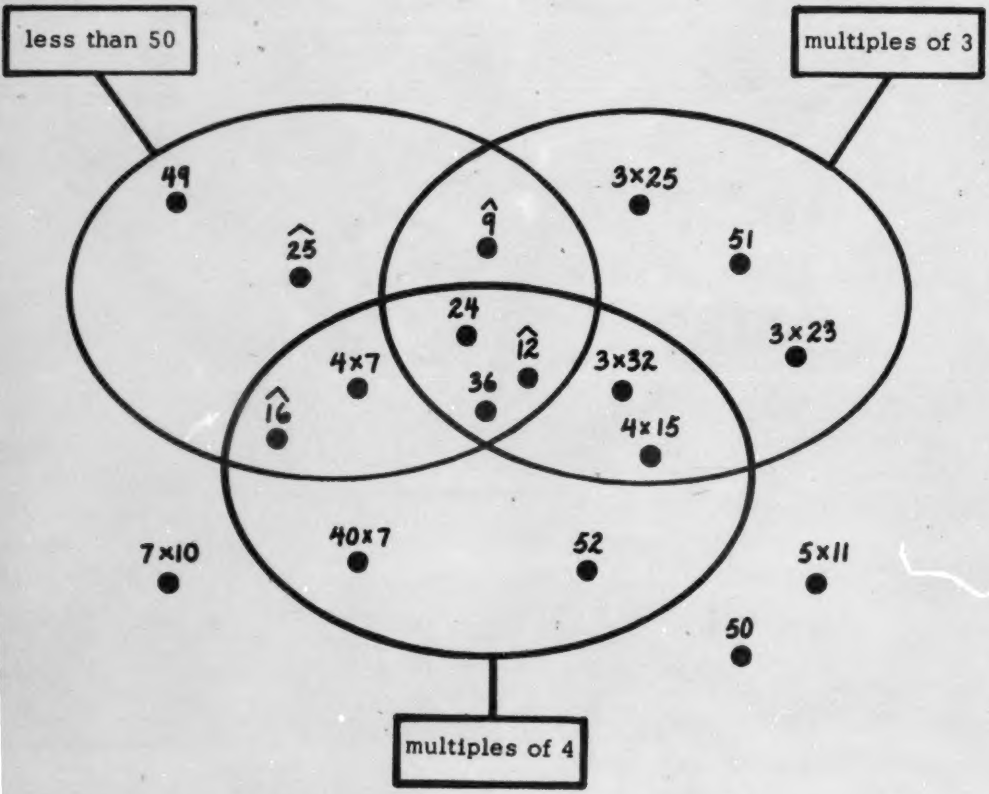
TEAM A		TEAM B	
$\widehat{12}$	4×7	4×15	36
3×25	24	$\widehat{9}$	50
49	51	3×23	52
$\widehat{16}$	3×32	$\widehat{25}$	40×7
5×11		7×10	

L 15

T: We are going to play a game with numbers. During a turn, you will have only one chance to put a number in the string picture. The first team to put all the numbers in its list where they belong in the picture wins.

Alternating teams, let students locate numbers in the picture. If a number is located correctly, erase it from the list in which it occurs; if located incorrectly, erase it from the picture.

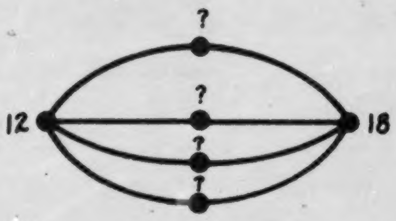
The next illustration shows the correct placement of all the numbers in both lists.

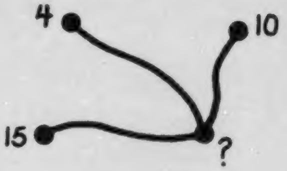

less than 50
multiples of 3
multiples of 4
49
25
9
3×25
51
24
12
3×23
4×7
3×32
36
16
4×15
7×10
40×7
52
5×11
50

L16 Talkative Numbers #2

CAPSULE LESSON SUMMARY

Review the story from Lesson L12 and the multiples relation.
Create pictures and label dots in pictures involving this relation.
Find numbers which could be in pictures such as:





MATERIALS

Teacher: Colored chalk
Student : Paper; colored pencils

DESCRIPTION OF LESSON

Exercise 1

Briefly review the story from Lesson L12.

T: Do you remember the story I told you about a school for whole numbers?

S: 0 is the principal and 1 is the assistant principal.

S: The numbers were getting too noisy and were playing tricks on one another.

S: 0 and 1 made a rule that two numbers may talk to one another if and only if one of them is a multiple of the other.

L 16

Write the rule [illegible] the board.

> **Two numbers may talk to each other**
> **if and only if**
> **one of them is a multiple of the other.**

T: What is an example of two numbers that may talk to one another?

S: 5 and 10.

T: Which number is a multiple of the other?

S: 10 is a multiple of 5.

S: 20 and 60.

T: Which number is a multiple of the other?

S: 60 is a multiple of 20.

S: 0 can talk to everybody.

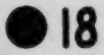
T: Why is that?

S: 0 is a multiple of every number.

T: That's right because 0 × any number is 0.

Continue until ten to twelve examples have been suggested.

Draw a large dot on the board and label it "18".

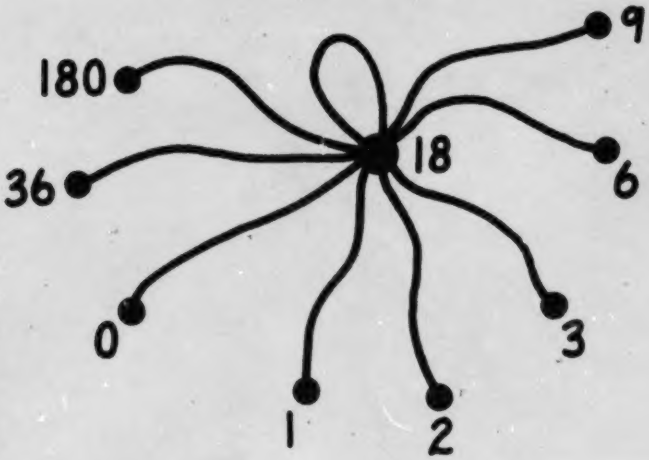


T: How did we show in a picture that two numbers may talk to each other?

S: We drew a red line between their dots.

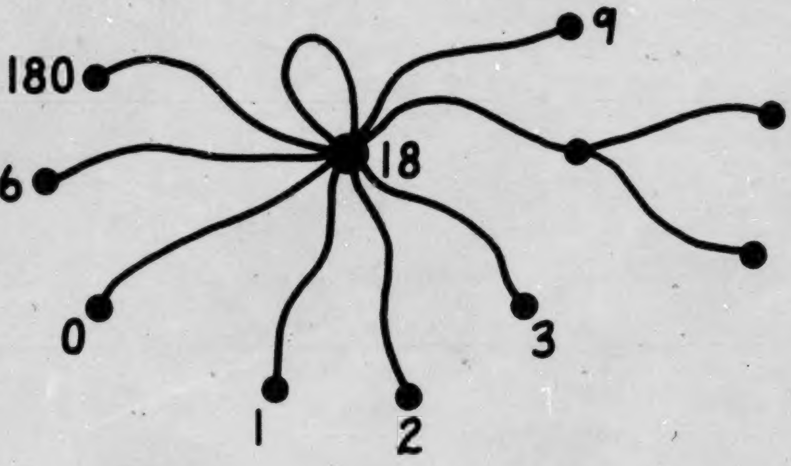
T: We drew a red cord. It's like an arrow which goes in both directions.

Ask students for some of the numbers which 18 may talk with and record them in the picture. Students should include numbers larger than, smaller than, and equal to 18. This picture shows some of the many possibilities.



L 16

Point to one of the numbers in the picture other than 18 and then draw two cords starting at the dot for that number.

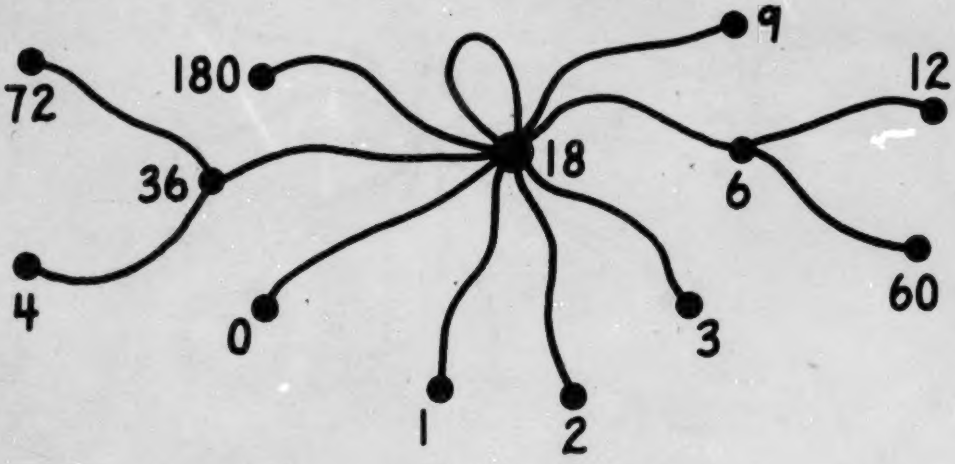


T: What are some numbers 6 may talk to which are not in our picture*?

S: 12.

S: 60.

Label the dots being considered with the numbers suggested. Repeat this activity using one of the other numbers in the picture. For example,



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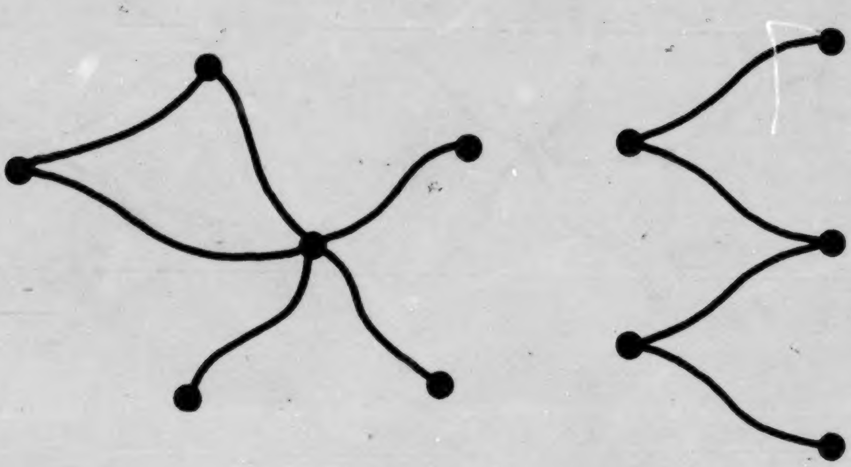
* It is possible that 6 may talk to other numbers already in the picture. If one of them is suggested, simply draw the cord and ask your question again.

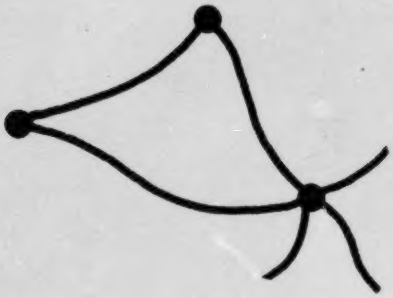
Exercise 2

Erase the board and then draw this picture.



T: This picture shows some of the cords which can be drawn between eleven numbers. What numbers could the dots be for?

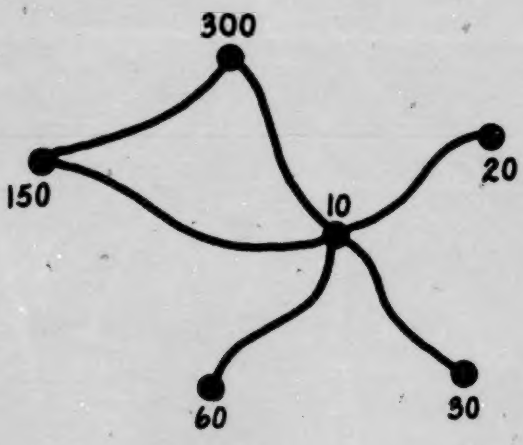
There are many possibilities. If your students have any difficulty at all, it most likely will be in labeling one of the dots in this group (see the next illustration) after the other two have been labeled.

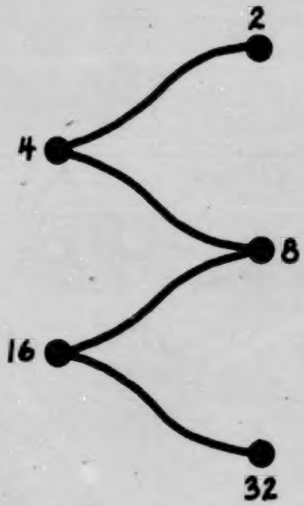


This illustration shows only part of the picture on the board.

L 16

The following illustration shows one way to label the dots.





Exercise 3

Draw two large dots on the board. Label one of them "12" and the other "18".

T: The numbers 12 and 18 are good friends, but under this new rule 12 and 18 may not talk to one another. But they have an idea! If they can find other numbers which they both may talk with, then they can relay messages through them. Can you help them find such numbers?

S: 6.

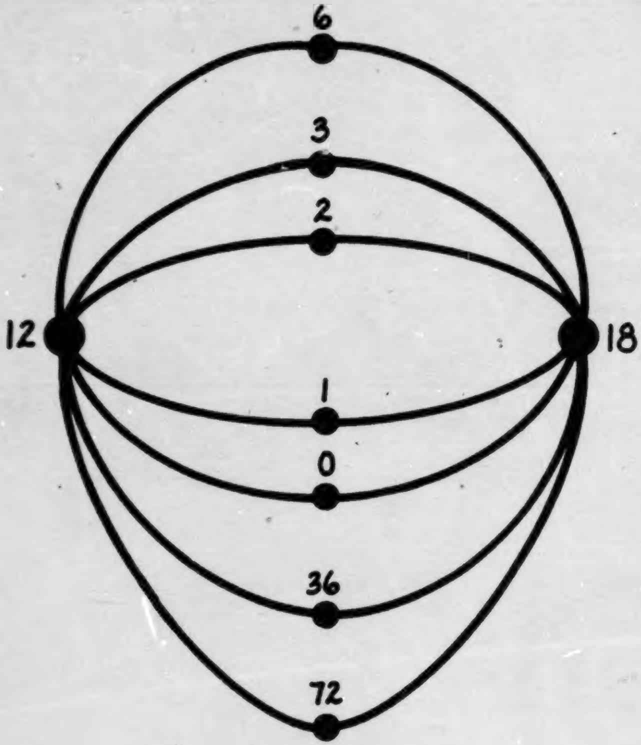
S: 3.

S: 0.

S: 1.

T: They could certainly relay messages through the principal (0) or the assistant principal (1), but they didn't think that would be such a good idea.

As students suggest numbers, record them in the picture.

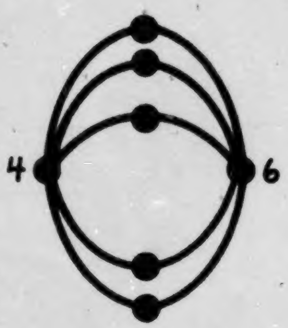


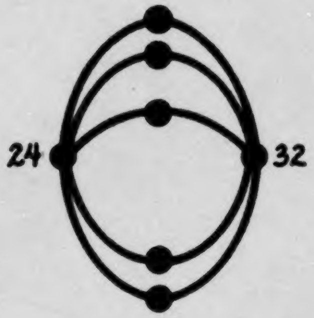
As an individual activity ask students:

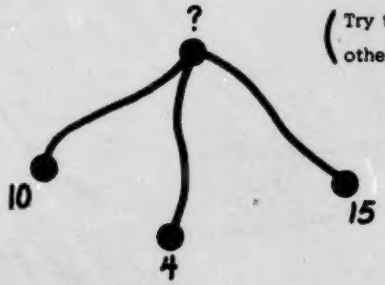
- to find some numbers 4 and 6 may both talk with;
- to find some numbers 24 and 32 may both talk with;
- to find some numbers 4, 10, and 15 may all talk with.

L 16

Indicate the preceding three problems by drawing these pictures on the board.







(Try to find a number other than 0 and 1.)

Students should copy the pictures, label the dots, and extend the pictures if they can show more possibilities.

NOTE: Whole numbers which 4 and 6 may talk with are: 0, 1, 2; and 12, 24, 36, . . ., 12 × n, ...

Whole numbers which 24 and 32 may talk with are: 0, 1, 2, 4, 8; and 96, 192, . . ., 96 × n, ...

Whole numbers which 4, 10, and 15 may talk with are: 0, 1; and 60, 120, 180, . . ., 60 × n . . . .

L17 Multiplication Modulo 10 #2

CAPSULE LESSON SUMMARY

Review the story of DANCING FRIENDS and the ten friends' operations, ⊕ and ⊗. Review also the 2 ⊗ picture with ten numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. Draw an arrow picture for 3 ⊗. Provide individual work on other similar arrow pictures.

MATERIALS

Teacher: Colored chalk

Student : Worksheets L17* and **; colored pencils

DESCRIPTION OF LESSON

Exercise 1

Begin the lesson by recalling briefly the story of DANCING FRIENDS.

T: How many number friends were invited to play in the dancing games, and which numbers were they?

S: Ten friends: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.

List the whole numbers from 0 to 9 on the board.

0 1 2 3 4 5 6 7 8 9

Review the operation ⊕ by doing these problems as a collective activity. Answers are in the boxes.

$5 \oplus 3 = \boxed{8}$ $5 \oplus 8 = \boxed{3}$ $\boxed{7} \oplus 4 = 1$

$7 \oplus 5 = \boxed{2}$ $6 \oplus \boxed{9} = 5$ $1 \oplus \boxed{9} = 0$

L 17

Continue by asking which numbers could go in the box and in the triangle in the following number story. A table of possible answers is given below it. (Also the $\square$ and $\triangle$ numbers can be reversed.)

$$\square \oplus \triangle = 6$$

$\square$	$\triangle$
0	6
1	5
2	4
3	3
7	9
8	8

Review the operation ⊗ by doing these problems as a collective activity. Answers are in the boxes.

$3 \otimes 4 = \boxed{2}$ $\qquad 2 \otimes \square = 6$ $\qquad$ (Answers: 3 and 8)

$5 \otimes 4 = \boxed{0}$ $\qquad 4 \otimes \square = 4$ $\qquad$ (Answers: 1 and 6)

$4 \otimes 7 = \boxed{8}$ $\qquad \boxed{5} \otimes 3 = 5$

Continue by asking which numbers could go in the box and in the triangle in the following number story. A table of possible answers is given below it. (Also, the $\square$ and $\triangle$ numbers can be reversed.)

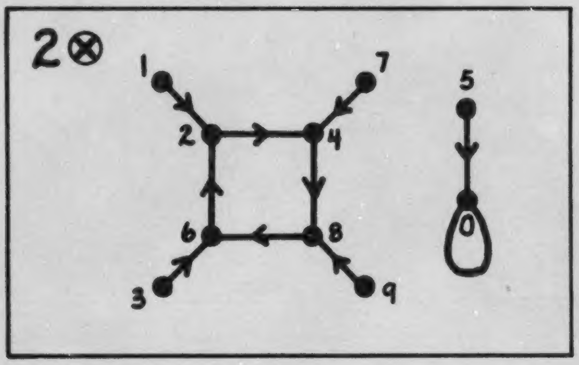
$$\square \otimes \triangle = 0$$

$\square$	$\triangle$
0	1, … 9
2	5
4	5
6	5
8	5

Exercise 2

T: Do you remember when we showed the ten friends in a 2⊗ picture?

With students participating, recreate the 2⊗ picture from Lesson L11.

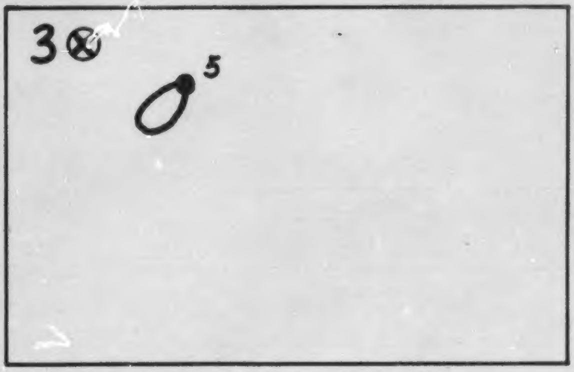


Keep this picture on the board for the remainder of the lesson.

T: Let's show the ten number friends in a 3⊗ picture. What number would you like to start with?

We cannot predict what number your class will choose to start the picture; we can give only a sample dialogue. The illustrations on the right show how the picture on the board progressively develops.

NOTE: Careful location of the numbers in the picture will yield an attractive final drawing (see last illustration in the sequence).



S: 5.

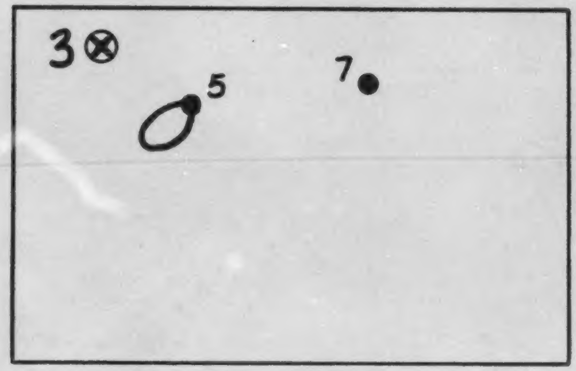
T: 3 ⊗ 5 = . . . ?

S: 5.

L 17

T: What number would you like to put next in the picture?

S: 7.



T: 3 ⊗ 7 = . . . ?

S: 1.

T: 3 ⊗ 1 = . . . ?

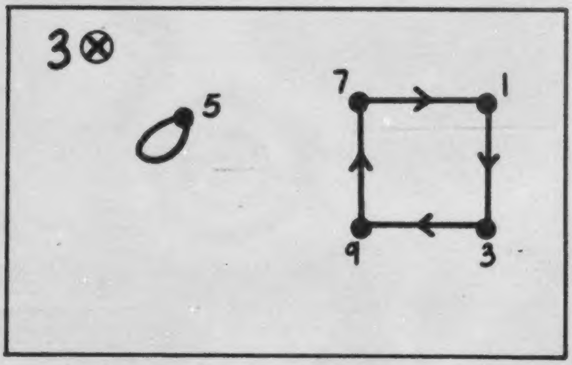
S: 3.

T: 3 ⊗ 3 = . . . ?

S: 9.

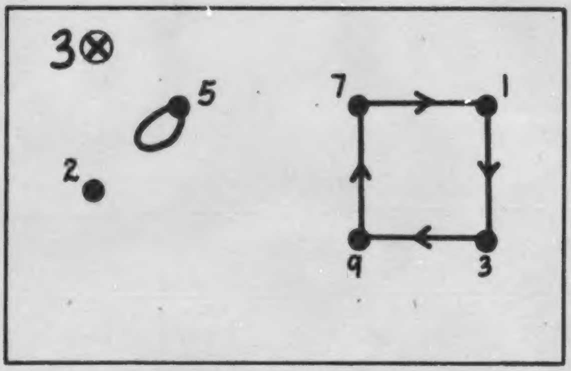
T: 3 ⊗ 9 = . . . ?

S: 7.



T: We have odd numbers in our picture; let's put an even number next.

S: 2.



T: 3 ⊗ 2 = . . . ?

S: 6.

T: 3 ⊗ 6 = . . . ?

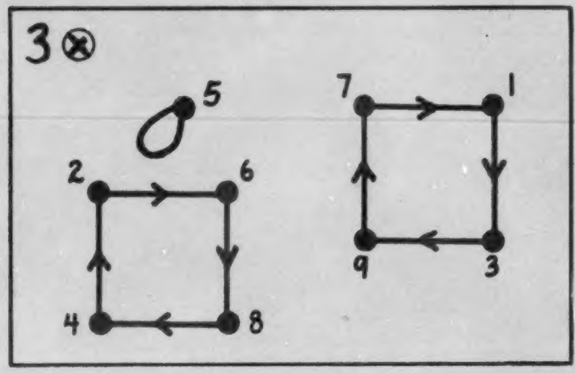
S: 8.

T: 3 ⊗ 8 = . . . ?

S: 4.

T: 3 ⊗ 4 = . . . ?

S: 2.

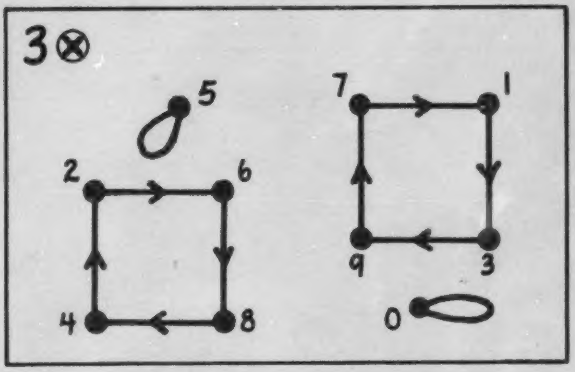

3⊗
5
7
1
2
6
9
3
4
8

T: Which of the ten number friends is missing from our picture?

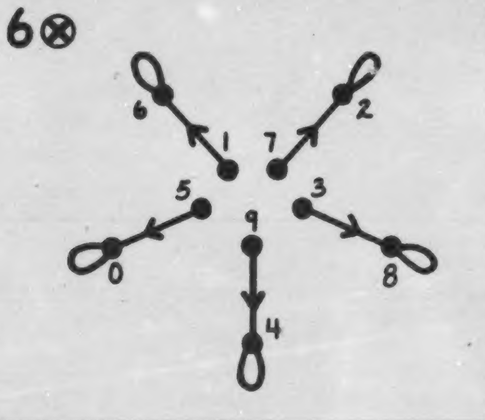
S: 0.

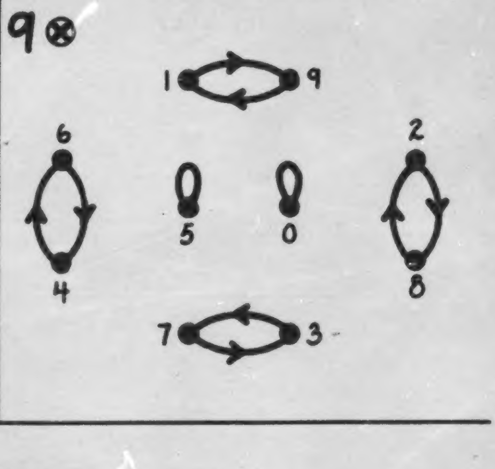
T: 3 ⊗ 0 = . . . ?

S: 0.


3⊗
5
7
1
2
6
9
3
4
8
0

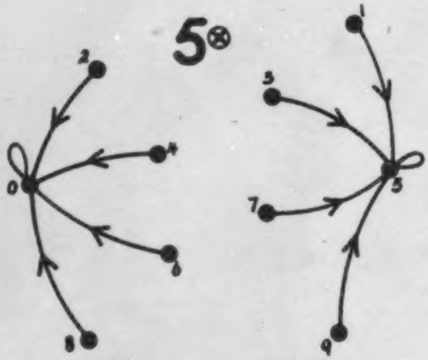
Distribute copies of Worksheets L17* and **. Ask students who finish quickly to show the ten number friends in a 6⊗ picture and in a 9⊗ picture.


6⊗
6
1
7
2
5
3
9
0
8
4


9⊗
1
9
6
2
5
0
4
8
7
3

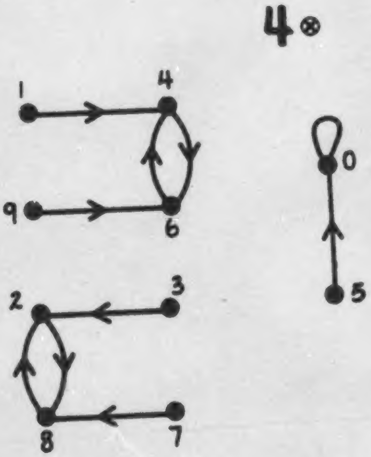
Name ____________________ **L 17** *

All the ten number friends are here. Draw blue arrow for their 5⊗ picture.


5⊗
2
4
0
6
8
1
3
5
7
9

Name ____________________ **L 17** **

Label the dots to show the ten number friends in this 4⊗ picture.


4⊗
1
4
9
6
0
5
2
3
8
7

APPENDIX: The String Game

The String Game is used in many versions throughout the CSMP Elementary School Program. It gives students an opportunity, in a game-like atmosphere, to become familiar with the language of strings, while at the same time it involves them in the kind of reasoning that will be developed and reinforced in various contexts throughout the program.

The String Game is first played with A-Blocks (shapes), but later on in the Intermediate Grades variations which make use of numbers appear. In this appendix we provide you with the necessary information and examples to enable you to play the game using A-Blocks, and we suggest some of the various possibilities open to you when you play the game with shapes. Although only a few lessons in UPG Part IV are scheduled for the String Game, we hope you will not feel restricted to playing the game only at these times. It has been found that it is most beneficial and enjoyable for students if you make a regular practice of playing the game about once every two weeks, or whenever you have an extra 10-15 minutes during the course of the day.

Equipment

PLAYING BOARDS

The equipment for this game may be most easily managed if you have a magnetic* chalkboard available. Many of the permanently mounted chalkboards in classrooms are magnetic, and you should test yours using one of the demonstration Minicomputer checkers. If your permanent chalkboards are not magnetic, try any portable chalkboard (grid board, etc.) that the school has available. If you do not have a magnetic chalkboard available, you can use your regular chalkboard.

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* Strictly speaking the chalkboard itself is not magnetic, but rather magnet-sensitive; i.e., magnets adhere to it. However, for simplicity we call such boards "magnetic".

TEAM BOARD

(attach a list of
string cards here)

TEAM A	TEAM B

a) Magnetic: If you do locate a magnetic chalkboard, you will also need a magnetic or metallic Team Board. With a large magnetic chalkboard, say at least three chalkboard squares, the Team Board could be drawn on the chalkboard. However, if you have a relatively small (portable) magnetic chalkboard, then you will need to obtain a sheet of metal (minimum size: 60 cm by 80 cm) or locate a convenient metallic surface in the classroom such as the side of a file cabinet on which to put the Team Board. In this case, draw the Team board on a large sheet of (easel pad) paper and tape this paper to your metallic surface.

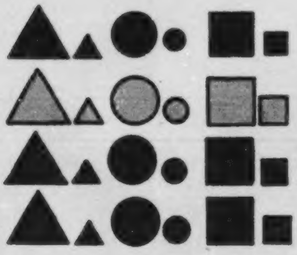
b) Non-magnetic: If you do not have a magnetic chalkboard available for the playing board, then your Team Board could be a large piece of poster board (minimum size: 60 cm by 80 cm). Using a felt-tip marker, divide the poster board into two regions and label them "Team A" and "Team B".

GAME PIECES AND STRING CARDS

One set of game pieces and string cards are needed for each version of the game. A list of the string cards (A-Blocks String Game Poster) is to be posted above the Team Board — it is a constant reminder during the game of the possible labels for the strings.

Game Pieces

String Game
with
A-Blocks



String Cards

RED	YELLOW	GREEN	BLUE
NOT RED	NOT YELLOW	NOT GREEN	NOT BLUE
○	△	□	BIG
NOT ○	NOT △	NOT □	LITTLE

Game pieces, string cards, and the poster list of the string cards can be found in the A-Blocks String Game Kit.

a) Magnetic: You can magnetize the game pieces (A-Blocks) by sticking a small piece of magnetic material to the back of each one. (Magnetic material is included in the A-Blocks String Game Kit, and it is available in many hobby and notions departments of stores.) Similarly, you can magnetize string cards by sticking a small piece of magnetic material to the front of each card, taking care not to obscure what is written thereon.

b) Non-magnetic: Game pieces can be attached to the Team Board by means of loops of masking tape stuck to the backs. A string card should have a loop of masking tape stuck to the front so that what is written on the card is not obscured. With this type of equipment, you should prepare yourself for running repairs by having sufficient masking tape available so that if a loop of tape loses its stickiness, it can be replaced on the spot. An alternative is to use a small wad of a plastic caulking compound (Rope Caulk or Mortite, for example) in place of the loop of masking tape. This material will maintain its stickiness over a much longer period of time.

Preparation for the Game

Draw two (or three, depending on which variation you are using) large, overlapping strings on the playing board using two (or three) different colors. Next to each of these strings attach one string card face-down. Draw the Team Board on the playing board or locate it conveniently nearby. Share the game pieces between the two sections of the Team Board labeled "TEAM A" and "TEAM B" by randomly placing one-half of the pieces in one section and the remainder in the other section. Divide the class into two teams using whatever method is acceptable to your class; e.g., boys vs. girls; rows 1, 2, 3 vs. rows 4, 5, 6; etc. Call one team "Team A" and the other "Team B". Before any student takes a turn, correctly place an equal number (at least one) of each team's game pieces in the string picture. This removes from the first players the necessity of operating on the basis of pure guesswork. You can influence how long the game will take by the number of pieces you place in the string picture before the game begins.

Object of the Game

Each team tries to place all of its game pieces correctly (according to the face-down string cards) in the string picture. The winning team is the one that first places all of its game pieces correctly and identifies the face-down cards correctly, after playing according to the rules.

Rules of the Game

1) The game is played in silence. This is to avoid one student telling another where to play a certain piece, or what the strings are for. Each student should have the opportunity to analyze the game alone. Infringement of this rule by anyone is penalized by the talker's team losing its next turn.

2) The teams alternate making plays, and the members take turns within each team. A player comes to the board and selects a piece from his or her team's collection to place in one of the regions of the string picture.

3) You are the judge. If the piece is correctly placed, say "yes". The piece remains in the string picture and the player immediately has a second turn (no player may have more than two consecutive turns). If the piece is incorrectly placed, say "no". The player returns the piece to the team's unplayed collection and play passes to the other team.

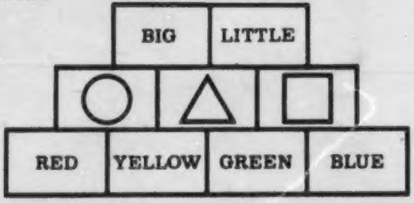
As an aid in judging, prepare a "crib-sheet" showing the correct position of each game piece or at least reminding you of what is on the face-down cards. If at any time you discover that you have made an error, say so immediately and rectify the mistake. Either move an incorrectly placed piece whose position you had approved to its correct region or replace in the string picture a correctly placed piece that had been rejected because you had disapproved its position.

4) When a team has correctly placed all of its pieces, the player who placed the last piece may thereupon attempt to identify each of the string cards. If these are all correct, the team has won. If a mistake is made (even if it is only in the case of one of the string cards), simply indicate that the identification is incorrect and let the game continue.

5) If a team has exhausted its stock of game pieces and the strings have not been identified, that team continues on its turn to attempt to identify the strings, while the other team works to place its game pieces.

Version A: The String Game with A-Blocks

The simplest version of the game uses twenty-four A-Blocks as game pieces and only nine string cards.

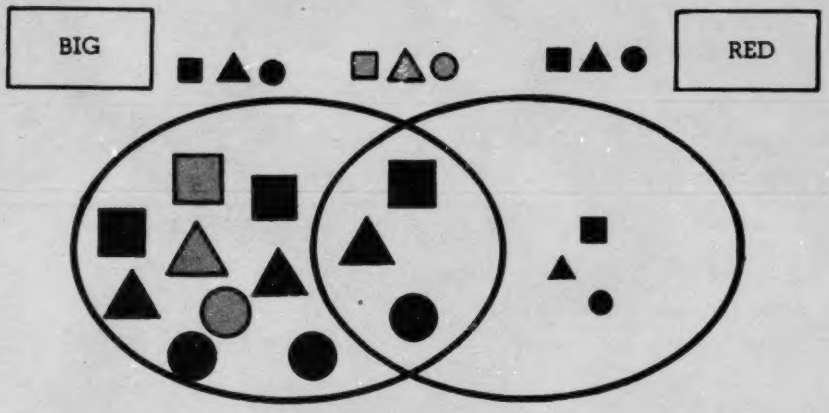


This list of string cards should be attached on or near the Team Board.

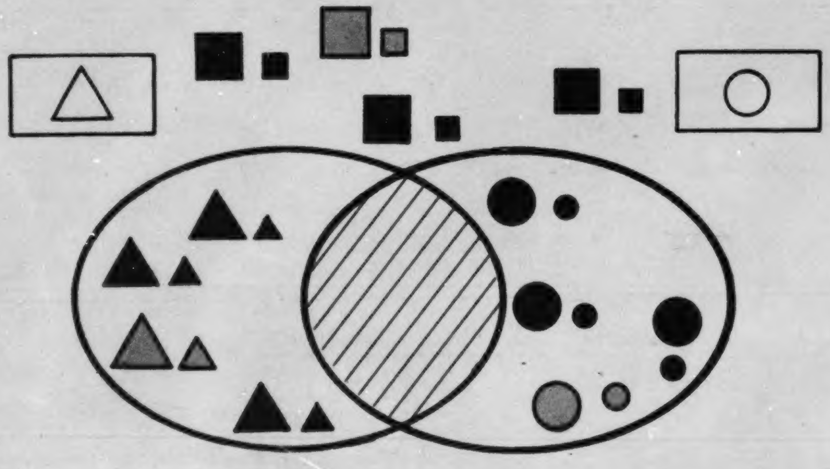
We provide here several example "crib-sheets" for variations of the game with two and with three strings.

Variation #1 : Two Strings

Example 1 : Two Empty Regions

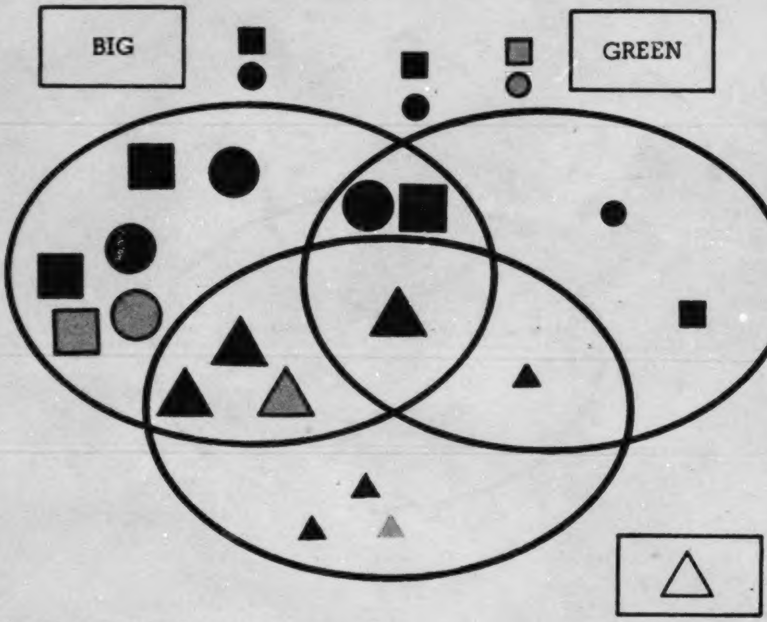


Example 2 : One Empty Region

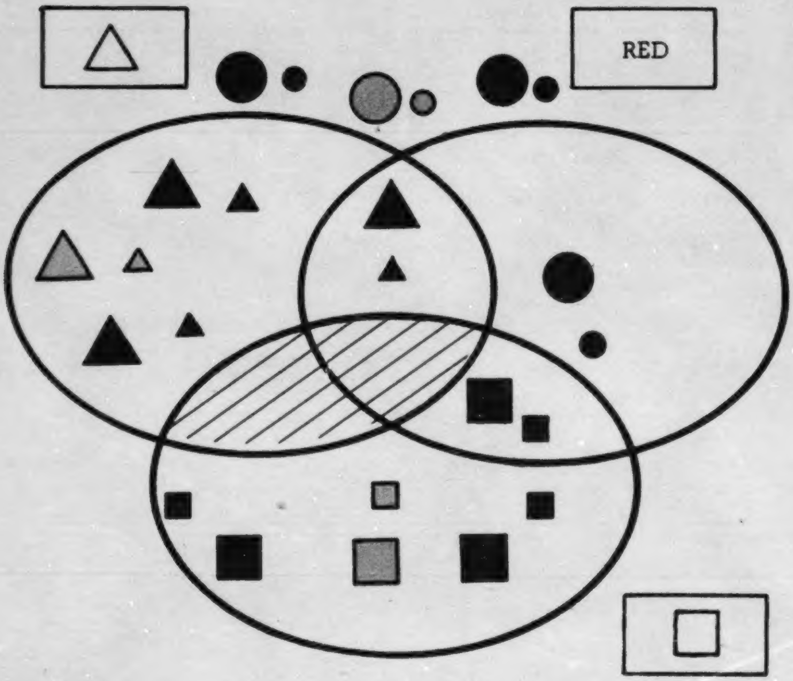


Variation #2: Three Strings

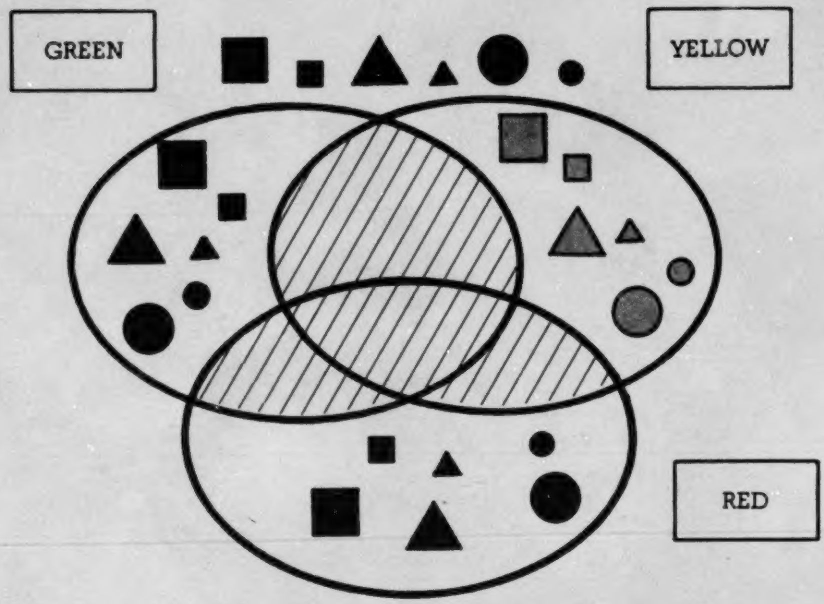
Example 1: No Empty Regions



Example 2: Two Empty Regions



Example 3: Four Empty Regions



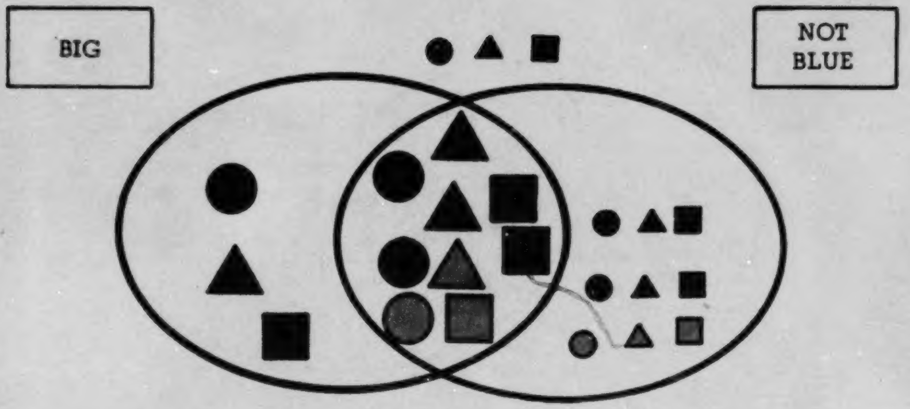
Version B: The String Game with A-Blocks, using "Not" Cards

A more complicated version of the game with A-Blocks uses all sixteen of the string cards. A list of all sixteen string cards should be posted or attached to the Team Board.

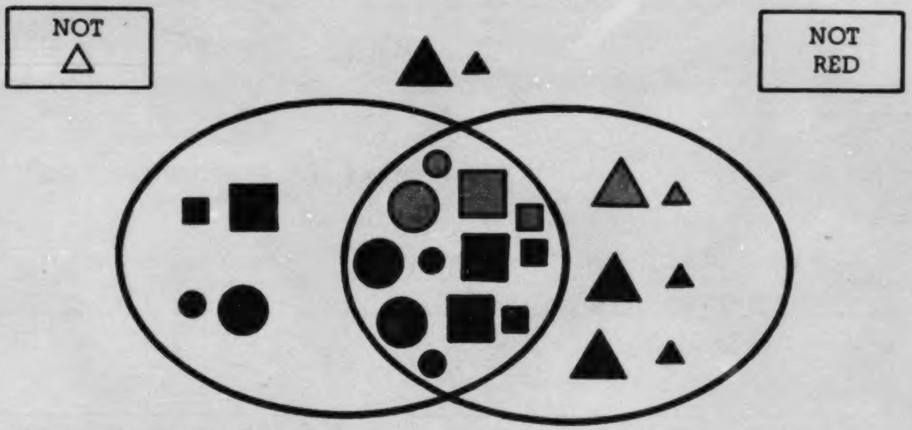
NOTE: Since "NOT BIG" and "LITTLE" describe exactly the same A-Blocks (and similarly, "NOT LITTLE" and "BIG"), such a "not" card is redundant. If a student questions why we do not include these possibilities, let the class discuss why they are unnecessary.

We provide here several example "crib-sheets" for this version of the String Game played with two strings.

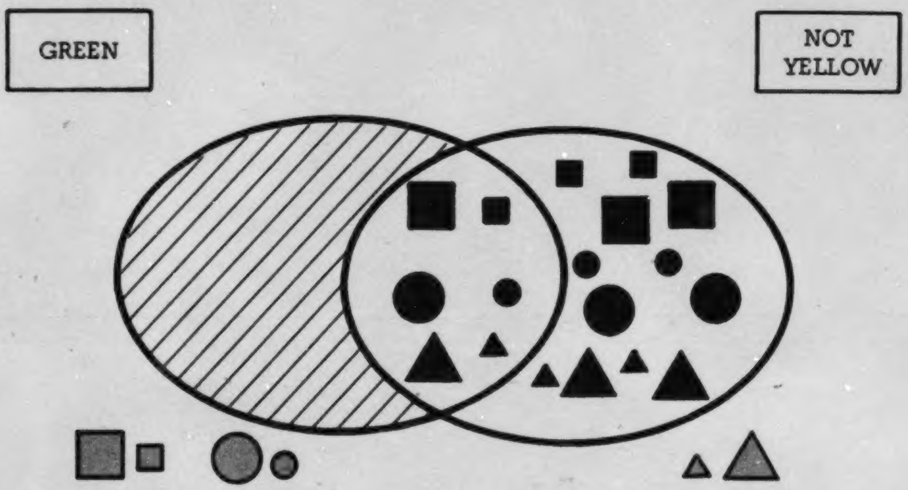
Example 1 : No Empty Regions


BIG
NOT BLUE

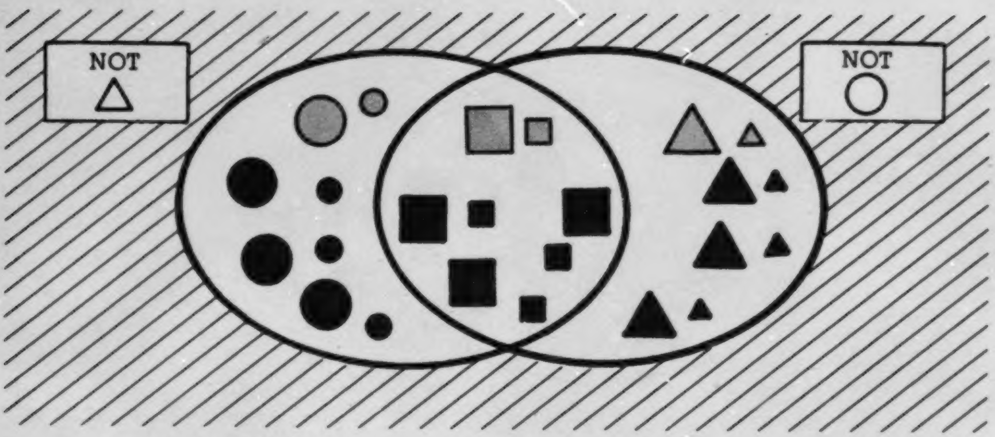
Example 2 : No Empty Regions



Example 3 : One Empty Region


GREEN
NOT YELLOW

Example 4: One Empty Region



Of course, by hatching the "outside" region of the diagram we mean to indicate that no game pieces can be placed there correctly. Strictly speaking, that region is not empty because, for example, the number 50 is in the outside region.

You should be warned that the "NOT" version of the game played with three strings is almost impossible to judge without a "crib-sheet" and is correspondingly difficult to play. Hence you would be well-advised to avoid it until such time as you think the two-string version is no longer challenging enough for the majority of your students.

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GEOMETRY AND MEASUREMENT

Comprehensive School Mathematics Program

CSMP Mathematics for the Upper Primary Grades

Part IV

Geometry and Measurement

CEMREL, Inc.

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GEOMETRY AND MEASUREMENT

INTRODUCTION

Geometry is useful and interesting, an integral part of mathematics; geometric thinking permeates mathematics and relates to most of the applications of mathematics. Geometric concepts provide a vehicle for exploring the rich connections between arithmetic concepts (such as numbers and calculations) and physical concepts (such as length, area, symmetry, and shape). Even as children experience arithmetic notions in their daily lives, they encounter geometric notions. Common opinion recognizes that each child should begin developing a number sense early. There is no sound argument to postpone the development of geometric sense. These are compelling reasons why the systematic development of geometric thinking should be a substantial component of an elementary school curriculum.

A benefit of early development of geometric thinking is that it provides alternative opportunities for "slow learners". Children develop number concepts at individual rates. Relative lateness of a child's numerical development often is signified by the child's trouble remembering number facts. A common response to this problem is assignment of arithmetic drill, which serves only to confirm the diagnosis of slowness. However, numerical development may not be an accurate indication of other development, in particular, of geometric development. Often the labels "slow" and "fast" are incorrect when alternative activities are presented.

We are careful to allow students a great deal of time to explore quantitative relationships among physical objects before studying arithmetic. In the same spirit, students should be allowed to explore geometrical relationships through free play and guided experiences. Before studying geometrical properties of lines and shapes, they should have free play with models and with drawing instruments in order to create new figures and patterns. Before considering parallelism and perpendicularity as relations on geometric lines, they should have free play in creating patterns with parallel and perpendicular lines.

Before learning length and area formulas, they should have informal experiences to establish and sharpen intuitive notions of area and length. These early experiences are aesthetically rewarding and just as stimulating and suggestive of underlying concepts as early experiences leading to arithmetic. Such experiences are a crucial first step in the development of a geometric sense.

As with all educational development occurring through informal experiences, the payoffs come unexpectedly and spasmodically. In the development of geometric thinking, the suggested activities vary widely, reach into so many other facets of the program, and continue for so long a span that it would be nonsense to set a priori milestones of accomplishments.

Thus, we must not rush the students through the free play activities that precede the more substantive lessons. These preliminary activities have implicit significance for geometric thinking and also provide aesthetic opportunities and explicit practice. We must restrain our inclination to view these activities as mere play. On the other hand, some activities appear to have important consequences or clearly to relate geometry to other parts of the program. Here the inclination might be to spend more time on these activities than is suggested. Again this should be resisted, remembering that similar activities will provide further experience later in slightly different contexts.

The key word is "experience" not "mastery".

But what sorts of experiences will lead students to think geometrically? The standard curriculum usually consists of measurement activities, recognition of some special geometric figures, and association of these figures with their technical names; in short, a good deal of vocabulary but little opportunity for geometric thinking.

A glance at the lessons of this strand indicates that the CSMP Upper Primary Grades curriculum includes measurement activities (leading to the use of metric units) as well as topics not usually in traditional programs at this level, for example, networks and taxi-geometry.

Measurement activities eventually lead to notions of distance. Traditionally one waits for a good deal of maturity before studying properties of figures, such as circles, that are defined in terms of distance. The reason is that Euclidean distance (distance "as the crow flies") requires sophisticated notions of "exact" measurement in terms of the real number system in order to be able to measure every line segment. But in the CSMP curriculum very young children (even kindergarteners) begin thinking about distance in the simplified setting of "taxi-geometry". Taxi-distance (as a taxicab travels) between points is measured in terms of whole numbers by an easy counting process. Experiences in taxi geometry lead to some creative situations, yet do not require special technical skills or vocabulary.

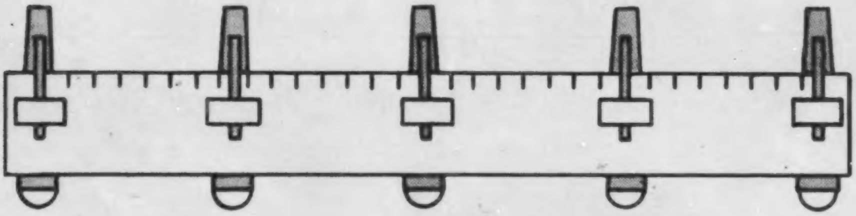
Putting the emphasis on experience rather than on mastery frees us to engage in imaginative activities with children that we ordinarily would not attempt. For example, even when considering such commonplace topics as area and length, we should feel free to devise activities that help students recognize patterns, develop cognitive strategies, and relate the topics to others in the program, while suppressing the urgency to get the "basic" formulas and to master "basic" skills. What is basic here, at this level, is that children are thinking rather than doing tasks on command. The skills and mastery will come eventually, in their own time and place in the curriculum.

NOTE ON GRIDS

A demonstration grid board is needed for the taxi-geometry lessons and for some of the area lessons. This should be a large square grid board at least 12 squares by 12 squares (and grid lines spaced 6 to 8 centimeters apart) on which you can draw with colored chalk or with colored pens. A portable, commercially made, permanent grid chalkboard is ideal. However, if such a grid board is not available to you, try one of these alternatives.

- Printed grids on large sheets of paper are provided with full classroom sets of materials for CSMP Mathematics for the Upper Primary Grades, Part IV. Tape one of these sheets to the board, use thick felt-tip pens for drawings, and discard the sheet after use; or mount one of the sheets on cardboard, cover it with clear contact paper to provide a protective surface, and use erasable wax crayons for drawings.

- For each lesson, carefully draw a grid on the chalkboard. One can make a tool for drawing such grids efficiently. Use a music staff drawer with chalk at the first, third, and fifth positions; or proceed as follows: clip five pen caps to a ruler at centimeter intervals and secure them with tape. Pen caps are usually of a diameter to hold a piece of chalk securely.



G1 Taxi-Circles

CAPSULE LESSON SUMMARY

Review the story of Nora's Neighborhood and the notion of taxi-distance between two points. From several clues about the location of the new house of Nora's cousin Linda, positively determine where Linda will live.

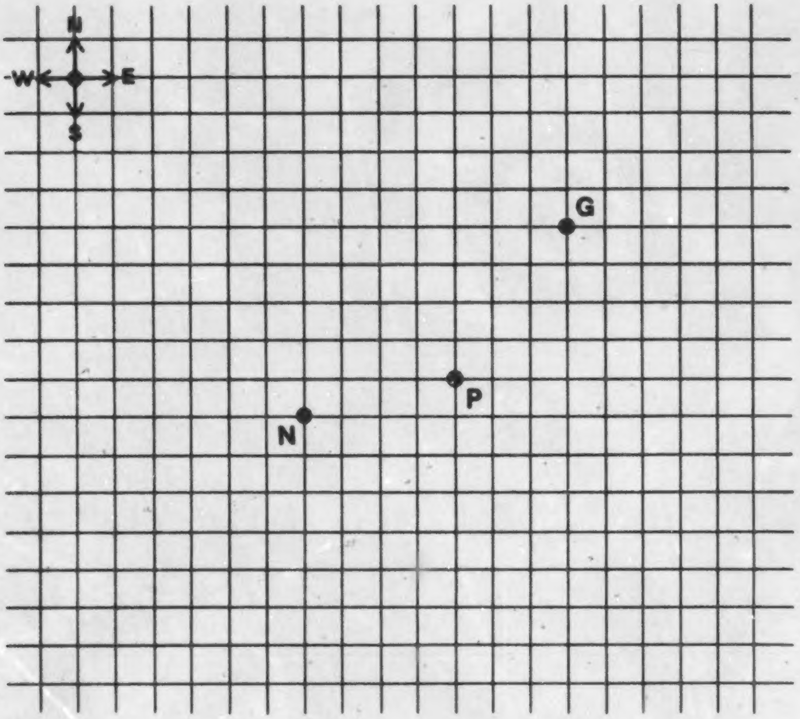
MATERIALS

Teacher: Grid board[*]; colored chalk

Student: Worksheet G1 ; colored pencils

DESCRIPTION OF LESSON

Display a grid board. Mark and label three points "N", "P", and "G" as in the illustration below. Indicate directions at a grid point in the upper left-hand corner of the board.



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[*] See the NOTE ON GRIDS in the introduction to this strand.

G1

Briefly review the story of Nora's neighborhood. Ask someone to trace on the grid a long walk from Nora's house (N) to her grandmother's house (G), and then ask someone to trace a shortest walk.

T: How long is a shortest walk from Nora's house to her grandmother's house?

S: Twelve blocks.

T: How do you know?

S: Nora has to walk five blocks north and seven blocks east.

Students may need help wording their responses in terms of directions.

T: We say that the taxi-distance from Nora's house to her grandmother's house is twelve blocks.

Record this information on the board. This will be the first line in a list.

taxi - distance from N to G = 12 blocks

T (pointing to "P"): This is where the park is on the map. What is the taxi-distance from Nora's house to the park?

Invite a student to count the blocks in a shortest path.

S: Five blocks.

Record this information on the board. This will be the second line in the list.

taxi - distance from N to G = 12 blocks
taxi - distance from N to P = 5 blocks

T: Nora has a cousin named "Linda". Linda is moving to Nora's neighborhood. Where will Linda live? Nora's mother gave her a clue.

First clue

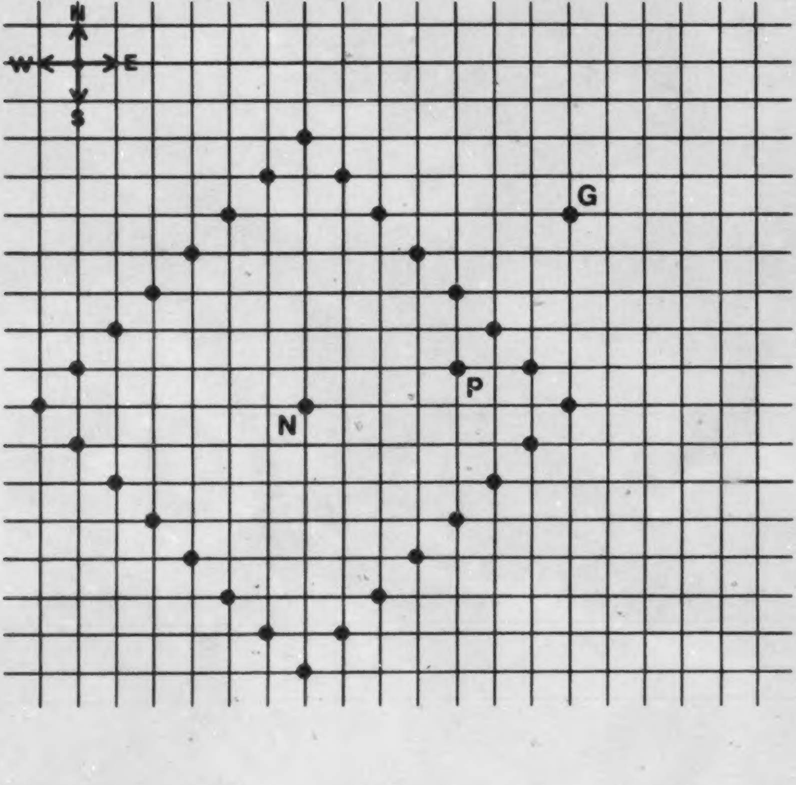
T: Her mother said, "Linda will live seven blocks from our house." Can you find some places where Linda's new house might be?

Let some students, one at a time, point to possible places on the grid. Draw a dot for each correct suggestion. After four or five places have been found, hold up a copy of Worksheet G1 as you say:

T: I would like you to find all the places where Linda's new house might be. On this paper draw red dots to record each possibility.

Distribute copies of Worksheet G1 and colored pencils.

Let the class work independently for five to ten minutes on the worksheet. Invite a student who finishes quickly to draw the complete taxi-circle on the grid board or as much of it as the size of your grid board allows.



G1

T: What do you think of this picture?

S: It makes a diamond.

If your grid board is not large enough for the complete taxi-circle, ask:

T: Are there any dots missing?

S: Yes, the grid is not big enough.

Ask the student to point to where the missing dots would be.

T (pointing to the grid): If you do not have these dots on your paper, draw them and then count all the places seven blocks from Nora's house.

When many students have marked the twenty-eight points on their worksheets, ask a student to count the red dots on the grid board. You will need to help the student count if there are corners which cannot be marked because of the limitations of your grid board.

Add this information to the list on the board.

taxi-distance from N to G = 12 blocks
taxi-distance from N to P = 5 blocks
taxi-distance from N to L = 7 blocks

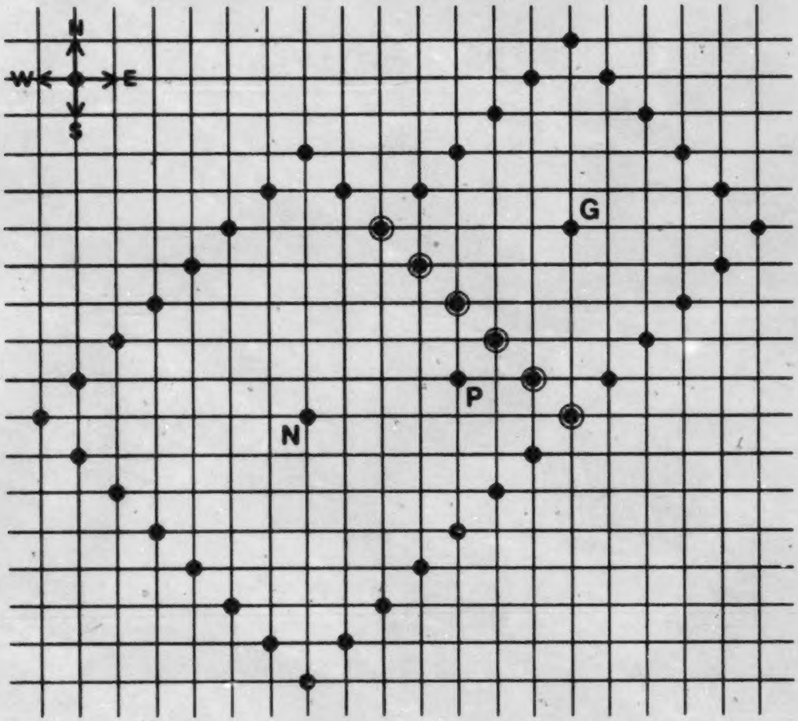
28 red dots

T: Nora was not very happy. Linda will be moving to her neighborhood, but there are twenty eight places where her new house might be. Which place is it? Nora's grandmother gave her another clue.

Second clue

T: Nora's grandmother said, "Linda will live five blocks from my house." On your worksheet, draw blue dots for all the places which are five blocks from Nora's grandmother's house. If a place is already marked on the map with a red dot, draw a blue circle around it.

Let the class work independently for about five minutes, and then invite someone to draw the complete taxi-circle on the grid board or as much of it as the size of your grid board allows.



Ask someone to count the blue dots on the grid. You will need to help the student count if there are corners which cannot be marked because of the limitations of your grid board.

S: 1, 2, 3, . . ., 20.

Add this information to the list on the board.

taxi-distance from N to G = 12 blocks

taxi-distance from N to P = 5 blocks

taxi-distance from N to L = 7 blocks

28 red dots

taxi-distance from G to L = 5 blocks

20 blue dots

T: Now do you know where Linda will live? (No) Where might Linda's new house be?

Let a student point to the new set of possibilities (red and blue dots).

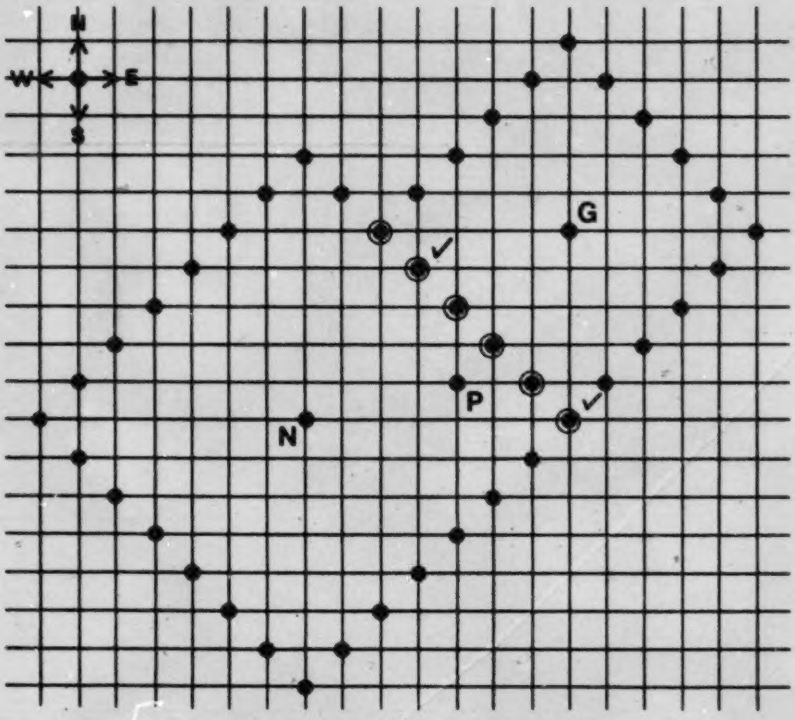
T: Nora decided to visit Linda and to ask her which of the six places is the correct one.

Third clue

T: Linda enjoyed the suspense so she gave Nora another clue: "When I move to your neighborhood, I'll be one block closer to the park than you are." (Point to the red-blue dots.) Which of these six places could be the one for Linda's new house?

When a place is suggested by a student, check the taxi-distance from that place to the park and from Nora's house to the park. Since the taxi-distance from N to P is five blocks, the places you are looking for are a taxi-distance of four (5-1) blocks from the park. The students should conclude that there are only two of the six dots a block closer to the park than Nora's house.

Put a check mark (√) by each of them.



S: But we still don't know which of the two places is for Linda's new house.

Fourth clue

T: This is the last clue. Nora found out that there is only one shortest path to walk from her house to Linda's new house. Which dot is for Linda's future house?

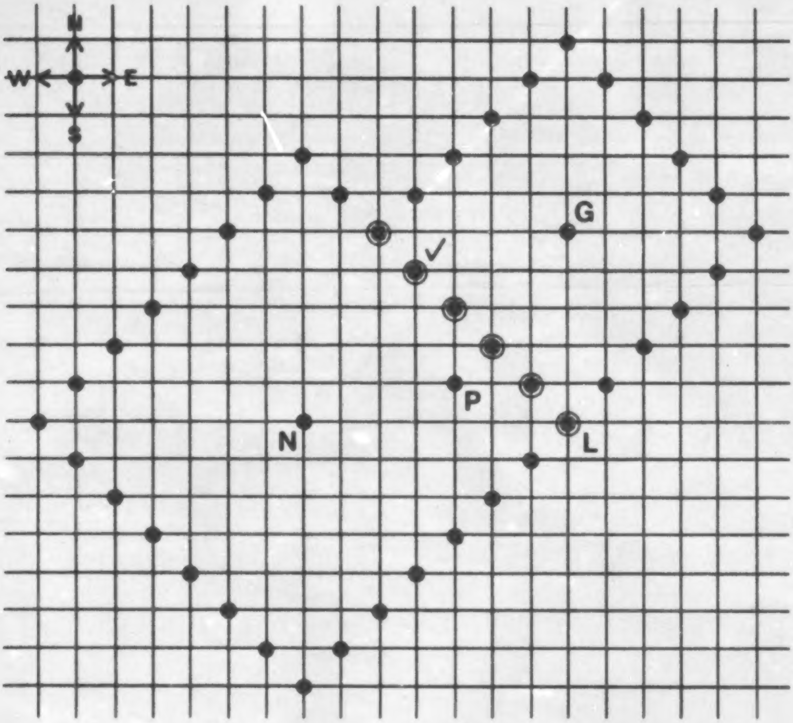
Let a student point to the dot.

T: How do you know that dot is for Linda's new home?

S: Because the shortest path is straight across (due east).

G 1

Erase the check mark and label the appropriate dot "L".



T (pointing to the dot which still has a check by it): Who can show me some shortest ways to walk here from N?

Let three or four students each trace a shortest path.

T: How many shortest paths from N to here do you think there are? (Thirty-five)

Accept several responses without comment. Do not expect to receive the correct answer.

S: At least ten or twelve.

S: Less than fifty.

G2 Fractional Parts of Shapes and Area

CAPSULE LESSON SUMMARY

Divide a rectangle of area 480 cm^2 into halves, fourths, eighths, and sixteenths and find the areas of those fractional parts. Determine many fractional names for $\frac{1}{2}$. Locate 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, and $\frac{1}{16}$ on the number line and draw $\frac{1}{2}\times$ arrows between points on the number line. Divide a circle of area 600 cm^2 into thirds, sixths, and twelfths and find the areas of those fractional parts. Determine many fractional names for $\frac{1}{3}$.

MATERIALS

Teacher: Colored chalk
Student: Worksheet G2* and **; colored pencils

DESCRIPTION OF LESSON

Exercise 1

Draw five rectangles each 16 cm by 30 cm on the board.

T: What kinds of shapes did I draw on the board?

S: Rectangles.

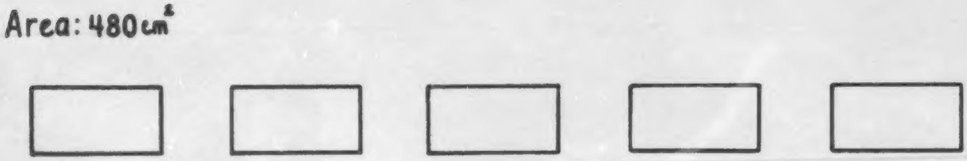
T: Are they about the same size?

S: Yes.

T: Yes, they are supposed to be the same size. Each of them has area 480 square centimeters.

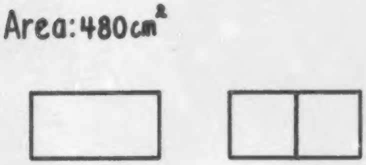
G 2

Record the area above the first rectangle from the left.



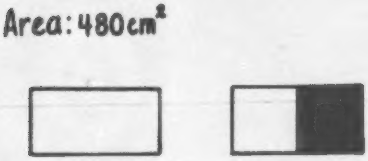
T (pointing to the second rectangle from the left): Who can show how to divide this shape into two pieces of the same size?

Give a ruler to a volunteer and ask the student to use it to indicate where the shape could be divided. Draw the line segment yourself while the student holds the ruler in position. One possibility is shown below.



T: Who would like to color one-half of this shape red?

Show the volunteer how to use the side of the chalk to color.



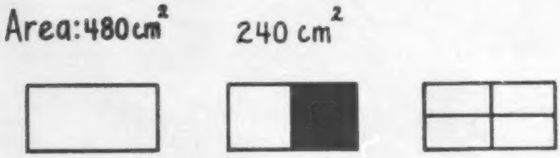
T: What is the area of the shape in red?

S: 240 square centimeters.

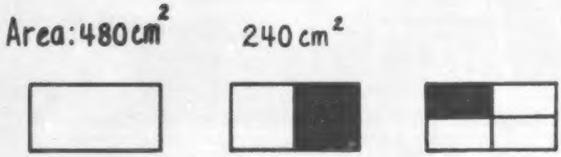
Record "240 cm^2" in red above the second rectangle.

T (pointing to the third rectangle from the left): Who can show how to divide this shape into four pieces of the same size?

Let a student hold a ruler in positions which indicate a way to divide the shape while you draw the appropriate line segments. If the student does not divide the shape in half first, strongly urge that this be done because it is easier to see one-half of a shape than to see one-fourth of the shape. Point out to the class that the shape is first being divided in half and then each piece is being divided in half. One possibility is shown below.



T: Who would like to color one-fourth of this shape blue?



T: What is the area of the shape in blue?

S: 120 square centimeters.

T: How do you know?

S: I took $\frac{1}{2}$ of 240.

Record "120 cm^2" in blue above the third rectangle.

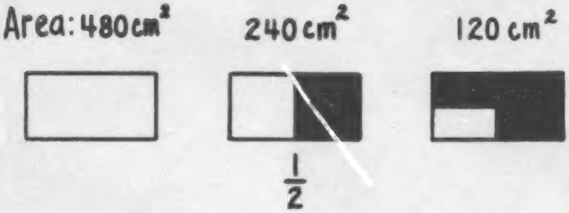
G 2

T: We colored one-half of this shape red (write "$\frac{1}{2}$" below the second rectangle). Who would like to color one-half of this shape red (point to the third rectangle)?

Ask the student who volunteers:

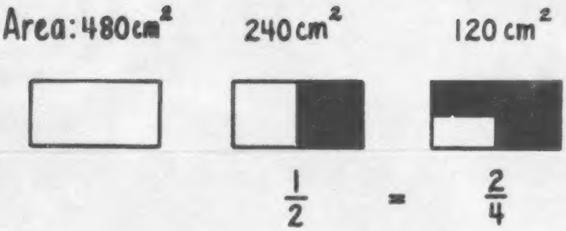
T: How many pieces are you going to color?

S: Two.


Area: 480cm²
240cm²
120 cm²
1/2

T: What is another name for $\frac{1}{2}$ suggested by this picture (point to the third rectangle)?

S: "$\frac{2}{4}$".


Area: 480cm²
240cm²
120 cm²
1/2 = 2/4

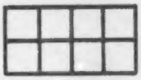
T (pointing to the fourth rectangle from the left): Who can show how to divide this shape into eight pieces of the same size?

If a student starts by estimating about how large one-eighth is, ask if someone has a more "clever" way of dividing it. Draw the appropriate line segments when someone suggests:

first divide the shape in half . . .

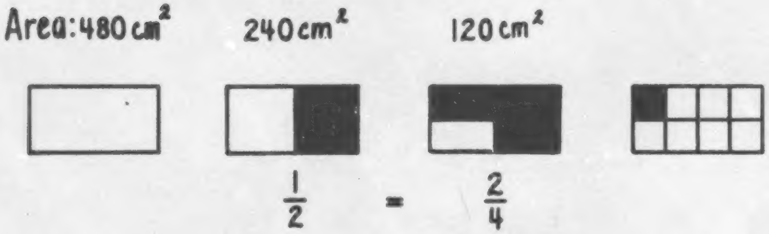
then divide each of those two pieces in half . . .

and then divide each of those four pieces in half.



NOTE: The preceding illustration only shows one of many ways of doing this halving process to divide the shape into eighths.

T: Who would like to color one-eighth of this shape green?



T: What is the area of this shape in green?

S: 60 square centimeters.

T: How do you know?

S: I took $\frac{1}{2}$ of 120.

Record "60 cm^2" in green above the fourth rectangle.

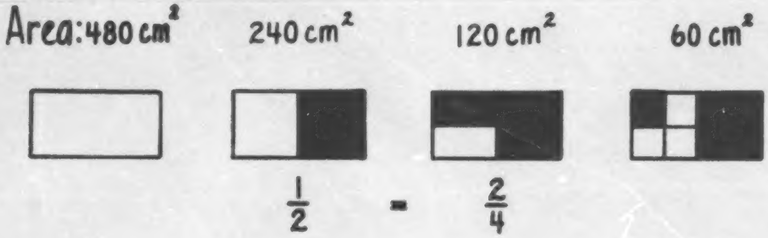
G 2

T: Who would like to color one-half of this shape red (point to the fourth rectangle)?

Ask the student who volunteers:

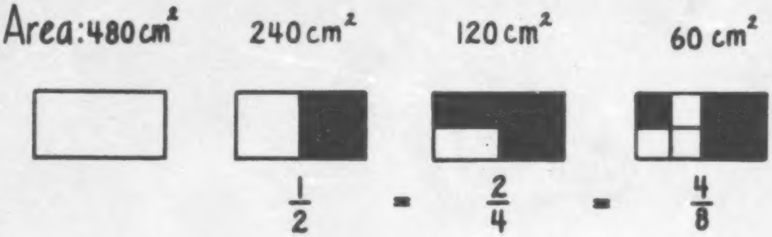
T: How many pieces are you going to color?

S: Four.


Area: 480 cm²
240 cm²
120 cm²
60 cm²
$\frac{1}{2} = \frac{2}{4}$

T: What is another name for $\frac{1}{2}$ suggested by this picture (point to the fourth rectangle)?

S: "$\frac{4}{8}$".

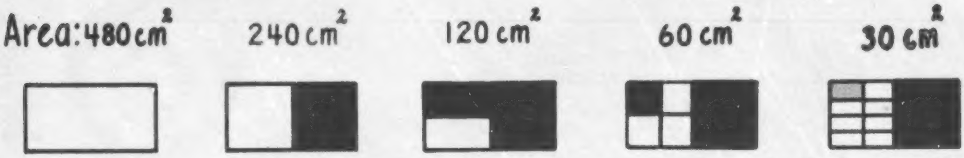

Area: 480 cm²
240 cm²
120 cm²
60 cm²
$\frac{1}{2} = \frac{2}{4} = \frac{4}{8}$

T (pointing to the fifth rectangle from the left): How could we divide this shape into sixteen pieces of the same size?

S: We could divide it into eighths as we did with that one (points to the fourth rectangle) and then divide each of the eight pieces in half.

When someone suggests this idea, ask that student to hold the ruler in the positions needed to divide the shape while you draw the appropriate line segments.

Proceed as before and conclude with the class that one-sixteenth of the shape has area 30 cm² and that another name for $\frac{1}{2}$ is "$\frac{8}{16}$".



$$\frac{1}{2} = \frac{2}{4} = \frac{4}{8} = \frac{8}{16}$$

T: We found some names for $\frac{1}{2}$. Who can give another name for $\frac{1}{2}$?

S: "$\frac{16}{32}$".

S: "$\frac{32}{64}$".

S: "$\frac{64}{128}$".

On the board write the names for $\frac{1}{2}$ which are suggested. Students may or may not continue the pattern as below.

$$\frac{1}{2} = \frac{2}{4} = \frac{4}{8} = \frac{8}{16} = \frac{16}{32} = \frac{32}{64} = \frac{64}{128}$$

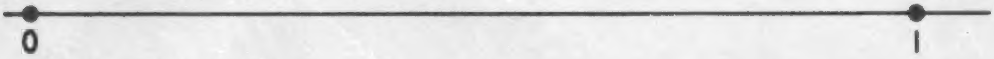
Keep the pictures and the names for $\frac{1}{2}$ on the board for references in Exercise 2.

Exercise 2

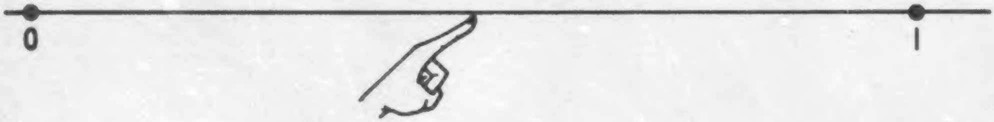
NOTE: This exercise shows the relationship between the geometry of the previous exercise and the location of various fractions on the number line. If your students do not see this relationship easily, do not belabor the topic but move to the next exercise.

G 2

On the board draw a line segment about 160 to 170 cm long. Draw a dot near the left end of the segment and label it "0"; draw a dot near the right end of the segment and label it "1".



T (tracing the segment between 0 and 1): If we divide this segment in half, . . .

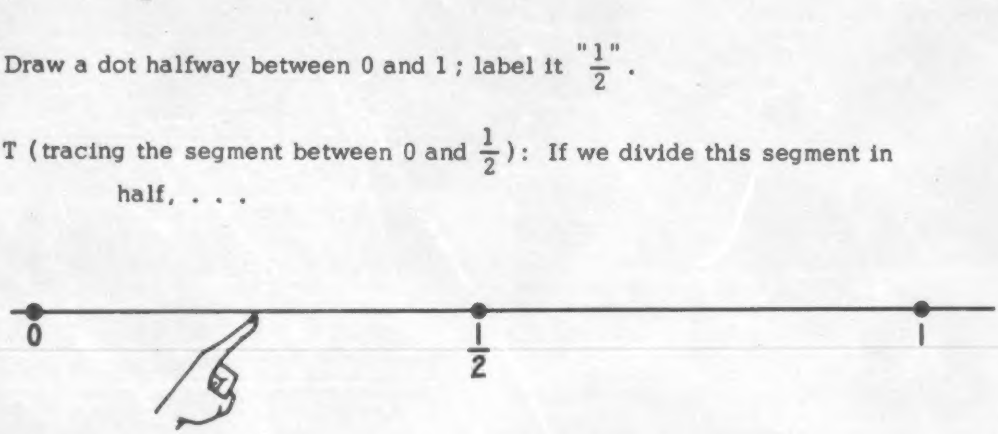


what could we label the midpoint?

S: "$\frac{1}{2}$".

Draw a dot halfway between 0 and 1; label it "$\frac{1}{2}$".

T (tracing the segment between 0 and $\frac{1}{2}$): If we divide this segment in half, . . .



what could we label the midpoint?

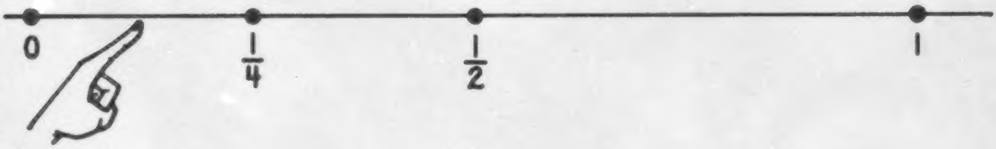
S: "$\frac{1}{4}$".

Draw a dot halfway between 0 and $\frac{1}{2}$; label it "$\frac{1}{4}$". If no one suggests "$\frac{1}{4}$", refer to the third rectangle on the board from the preceding exercise as you say:

T: We divided this shape in half and then we divided each piece in half. Here we divided this line segment in half and now we are dividing this piece (trace from 0 to $\frac{1}{2}$) in half.

Then repeat your original question. If students still are having difficulty, simply proceed to Exercise 3.

T (tracing the segment between 0 and $\frac{1}{4}$): If we divide this segment in half, . . .

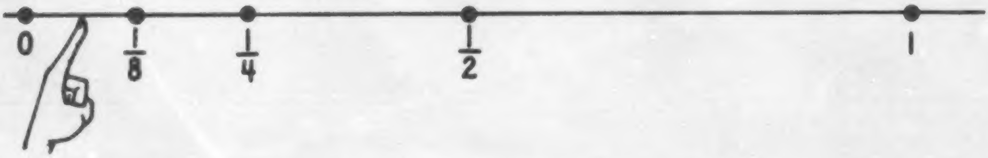


what could we label the midpoint?

S: "$\frac{1}{8}$".

Draw a dot halfway between 0 and $\frac{1}{4}$; label it "$\frac{1}{8}$".

T (tracing the segment between 0 and $\frac{1}{8}$): If we divide this segment in half, . . .

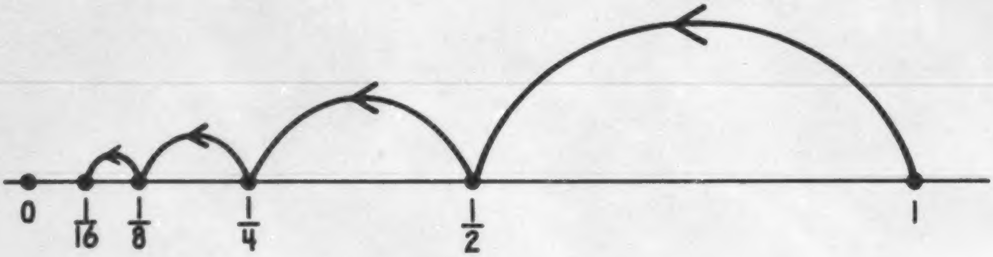


what could we label the midpoint?

G 2

S: "$\frac{1}{16}$".

Draw a dot halfway between 0 and $\frac{1}{8}$; label it "$\frac{1}{16}$". Then draw these red arrows.

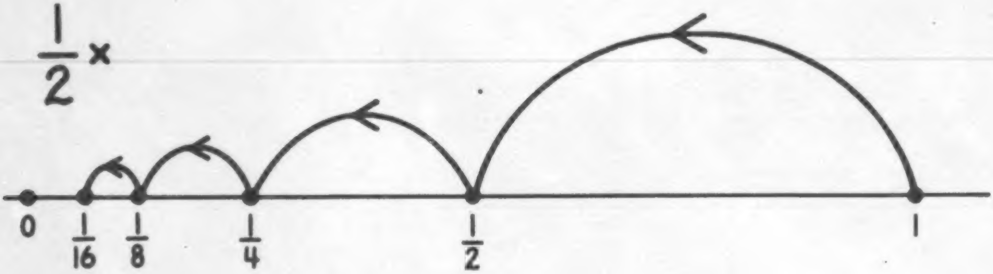

0
$\frac{1}{16}$
$\frac{1}{8}$
$\frac{1}{4}$
$\frac{1}{2}$
1

T: What could the red arrows be for?

S: Is larger than.

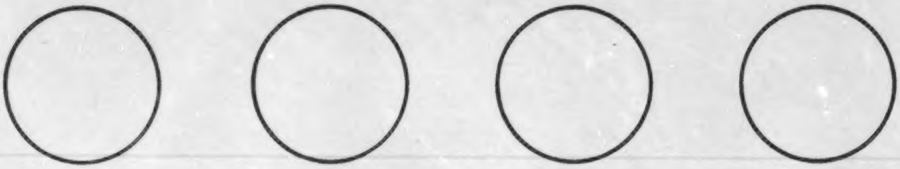
S: $\frac{1}{2}$x.

If someone suggests 2x, trace the red arrows in the opposite direction (left to right) and say that if the red arrows were in that direction they could be for 2x. When someone suggests $\frac{1}{2}$x, write the key on the board in red.

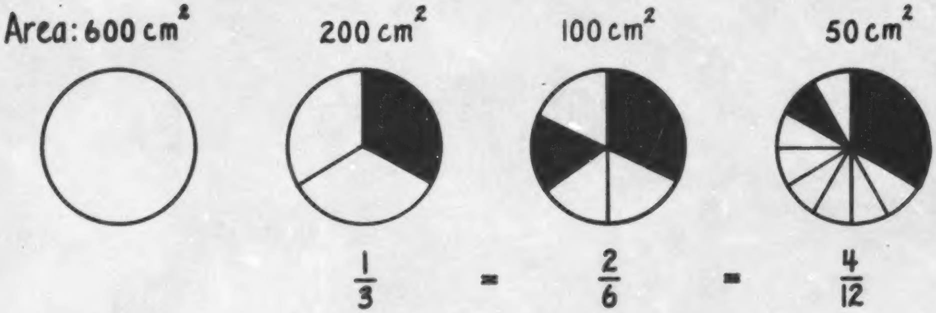

$\frac{1}{2}$x
0
$\frac{1}{16}$
$\frac{1}{8}$
$\frac{1}{4}$
$\frac{1}{2}$
1

Exercise 3

On the board draw four circles each about 28 cm in diameter.



In the same manner as in Exercise 1, divide the second disc into thirds, the third disc into sixths, and the fourth disc into twelfths. Use the pictures to suggest other names for $\frac{1}{3}$.



Ask the students to extend the list of names for $\frac{1}{3}$; for example,

$$\frac{1}{3} = \frac{2}{6} = \frac{4}{12} = \frac{8}{24} = \ldots = \frac{3}{9} = \frac{6}{18} = \frac{12}{36} = \ldots$$

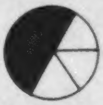
Worksheets G2* and ** are available for individual work. If a student has difficulty finding one-third of a shape, show that student the method of coloring one out of every three pieces.

Name ______________________ **G 2** *

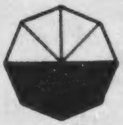
Color exactly one-half of each shape.

Use the picture to write another name for $\frac{1}{2}$.

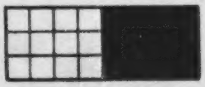
EXAMPLE:



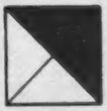
$\frac{1}{2} = \frac{3}{6}$



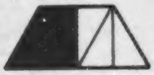
$\frac{1}{2} = \frac{4}{8}$



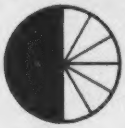
$\frac{1}{2} = \frac{12}{24}$



$\frac{1}{2} = \frac{2}{4}$



$\frac{1}{2} = \frac{3}{6}$



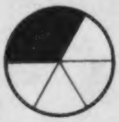
$\frac{1}{2} = \frac{6}{12}$

Name ______________________ **G 2** **

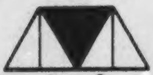
Color exactly one-third of each shape.

Use the picture to write another name for $\frac{1}{3}$.

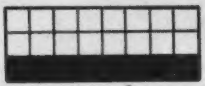
EXAMPLE:



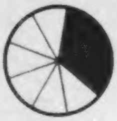
$\frac{1}{3} = \frac{2}{6}$



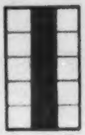
$\frac{1}{3} = \frac{2}{6}$



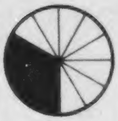
$\frac{1}{3} = \frac{8}{24}$



$\frac{1}{3} = \frac{3}{9}$



$\frac{1}{3} = \frac{5}{15}$



$\frac{1}{3} = \frac{4}{12}$

G3 Concentric Taxi-Circles

CAPSULE LESSON SUMMARY

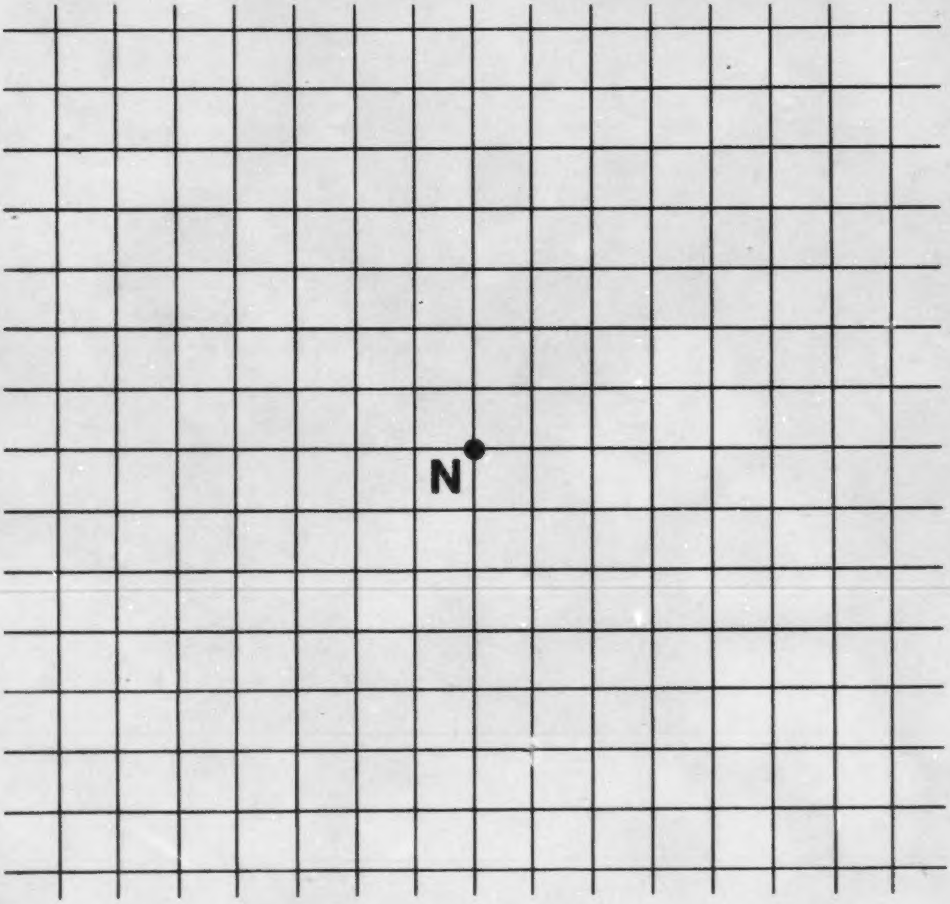
Review the idea of a taxi-circle from Lesson G1. Find the number of places at a specific taxi-distance from a given point. Repeat the activity for several taxi-distances and observe a rule for determining this number from the taxi-distance. Relate the notion of a taxi-circle to that of a usual (Euclidean) circle.

MATERIALS

Teacher: Grid board; colored chalk
Student: Worksheet G3; colored pencils

DESCRIPTION OF LESSON

Display a grid board. Draw a dot at a cross-point close to the center of the grid. Label it "N".



G3

T: Do you remember how Nora used clues to find the place where her cousin Linda's future home would be?

Let students comment on that lesson (Gl).

T: Nora's mother had given her the clue that the taxi-distance from Nora's house to Linda's house will be seven blocks.

Write this information on the board.

The taxi-distance from N to L is 7 blocks.

T: Who can show us some places which are seven blocks from Nora's house?

Let several students trace seven-block shortest paths from Nora's house. Mark each ending point with a red dot.

T: How many red dots like these can we draw?

Take estimates from a few students. Someone might remember the exact number (28) from the previous lesson on taxi-circles.

T: What kind of pattern are the dots forming?

S: A diamond.

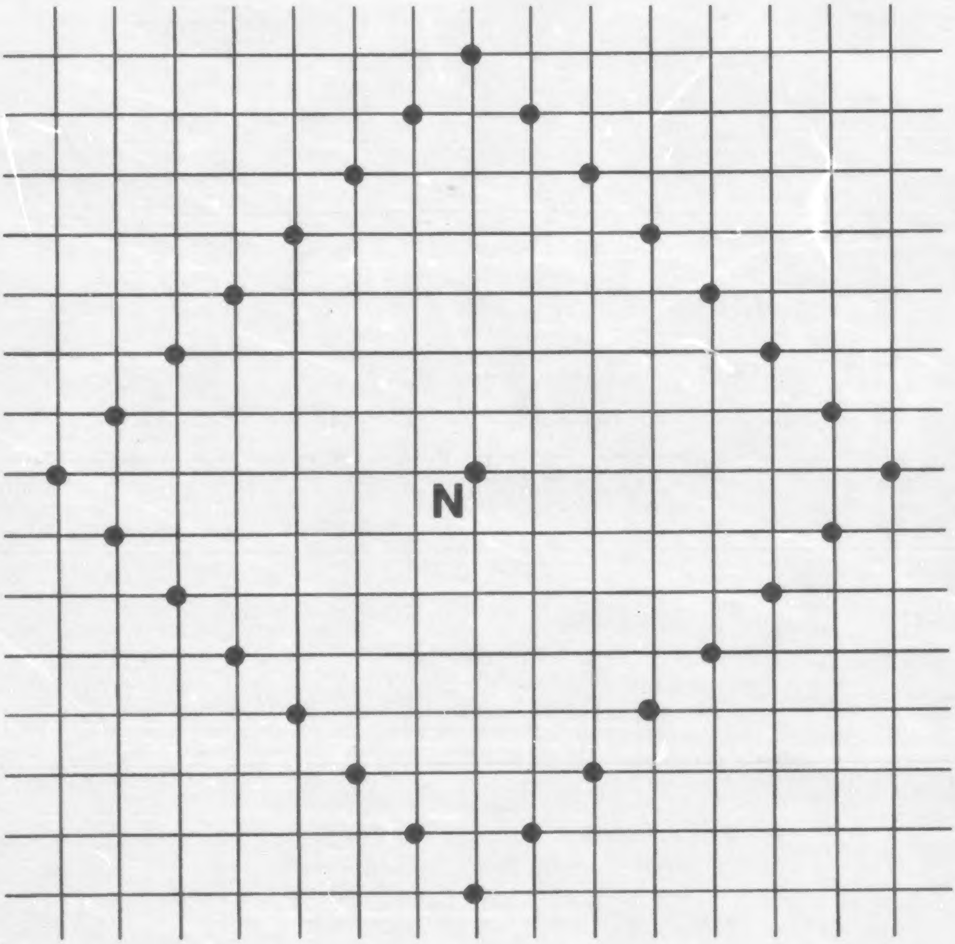
T: Who can show us some other places which are seven blocks from Nora's house?

Let students who volunteer each indicate three or four places. Before the taxi-circle is complete, begin asking students:

T: Can you show me a place without having to count?

This question should prompt students to use the diamond shape pattern to find the remaining points in the taxi-circle. If someone, as a result of this suggestion, points to a place which is not seven blocks from Nora's house, check the taxi-distance for that point.

Continue until the taxi-circle is complete or as much of it is drawn as the size of your grid board allows.



T: Count the red dots while I draw a chart on the board.

taxi-distance from N	How many places?
7 blocks	

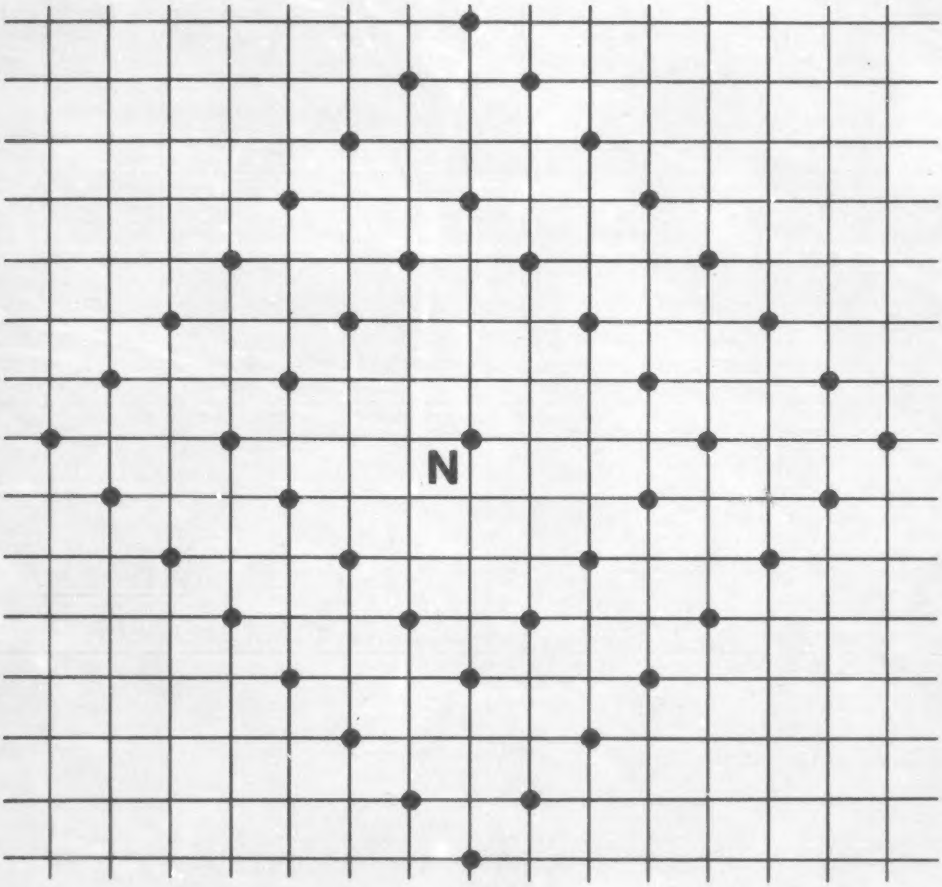
Invite a student to count aloud the red dots at the board. You will need to help the student count if there are places which cannot be marked because of the limitations of your board.

S: 1, 2, 3, . . ., 28.

Record "28" in the chart to the right of "7 blocks"

taxi-distance from N	How many places?
7 blocks	28

Repeat this activity to find the places which are a taxi-distance of 4 from Nora's house. See the blue dots in the next illustration.



Your chart should look like this.

taxi-distance from N	How many places?
4 blocks	16
7 blocks	28

G 3

T: How many places are a taxi-distance of zero blocks from Nora's house?

S: Just one — Nora's house.

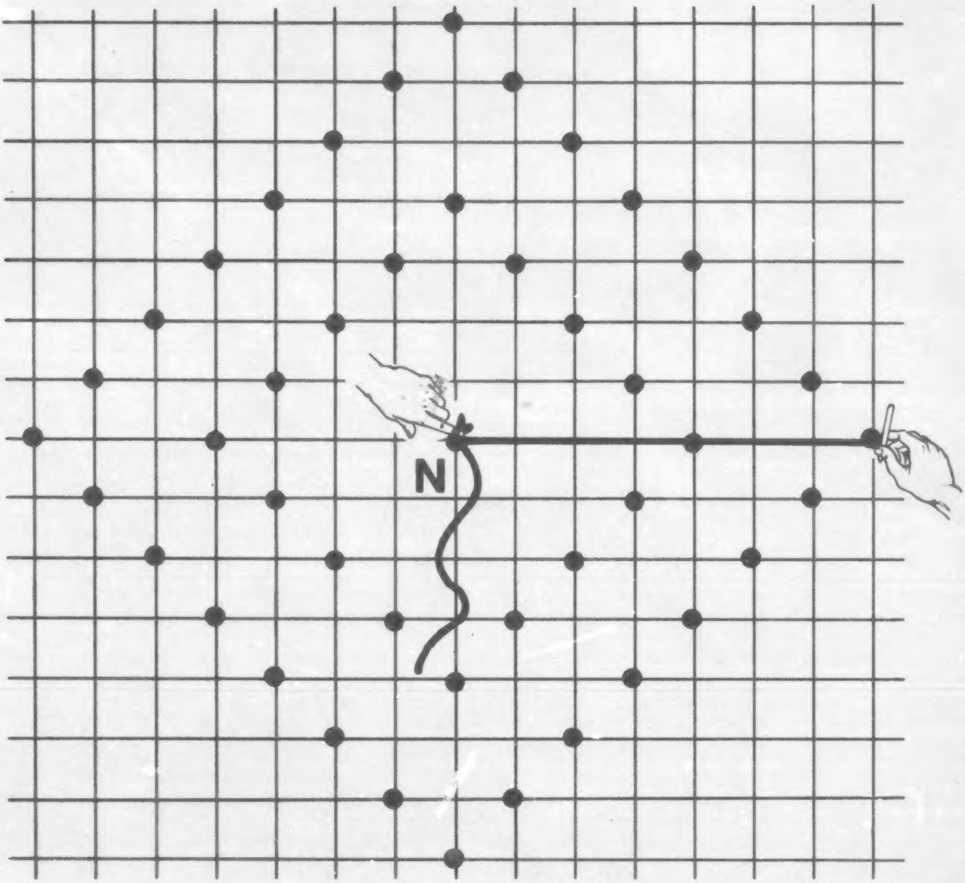
Record this information in the chart.

taxi-distance from N	How many places?
0 blocks	1
4 blocks	16
7 blocks	28

Trace each taxi-circle in turn as you say:

T: These places (red dots) are a taxi-distance of 7 from Nora's house. These places (blue dots) are a taxi-distance of 4 from Nora's house. These diamonds are called taxi-circles. In a few minutes we'll see why they are called this.

Take out a piece of string with chalk attached to one end. Hold the piece of chalk at the red dot seven blocks east of "N" and pull the string taut to "N".

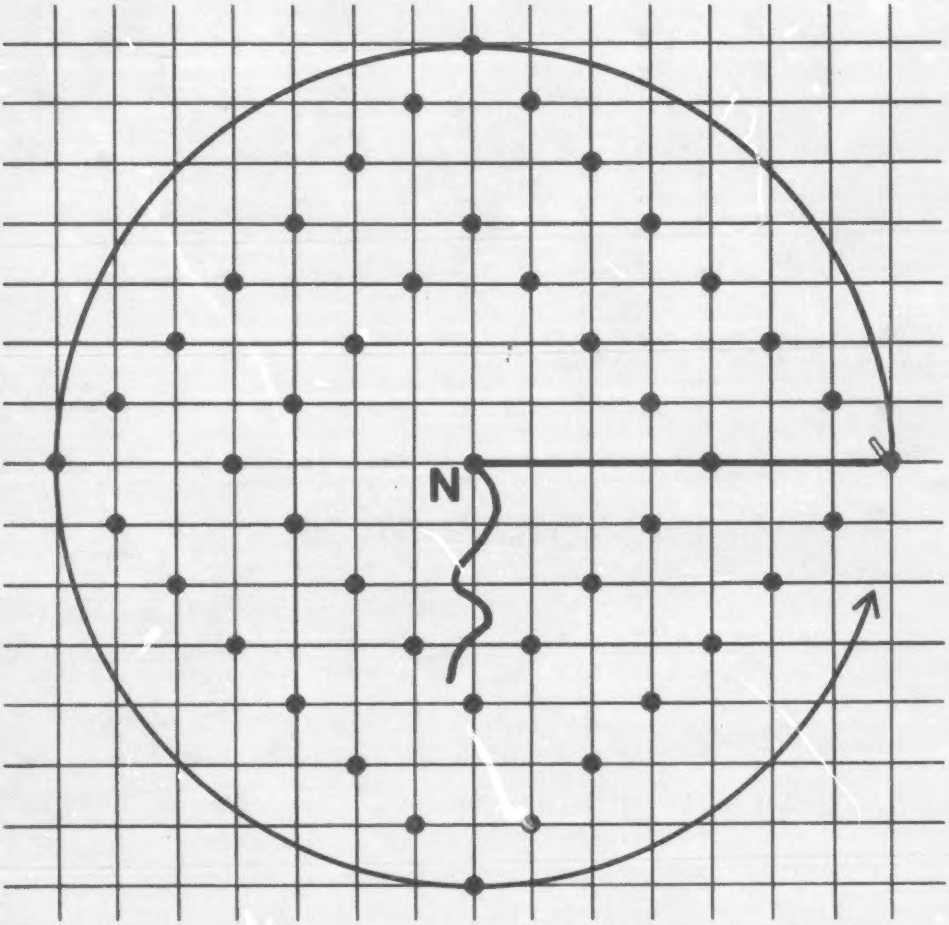


T: I have set this piece of string to the length of a seven-block path. Now, suppose Nora were not restricted to staying on the sidewalks, perhaps she's in the country where there are big fields with no streets, and she wants to find all the places at the same distance as this seven block path from her house. Where are those places? How can we find them?

Let the students discuss the problem and experiment with finding possible points at the same distance from N as the red dot seven blocks directly east of N. Use the piece of string to check that a suggested point is the correct distance from N. If necessary, lead the class to observe that other points can be found by holding one end of the piece of string at N and moving the chalk end (keeping the string taut). In fact, the class should observe that all possible points can be found this way.

S: All the places are on a circle with the center at N.

If no one suggests this, suggest it yourself. Carefully swing the chalk to complete the circle.

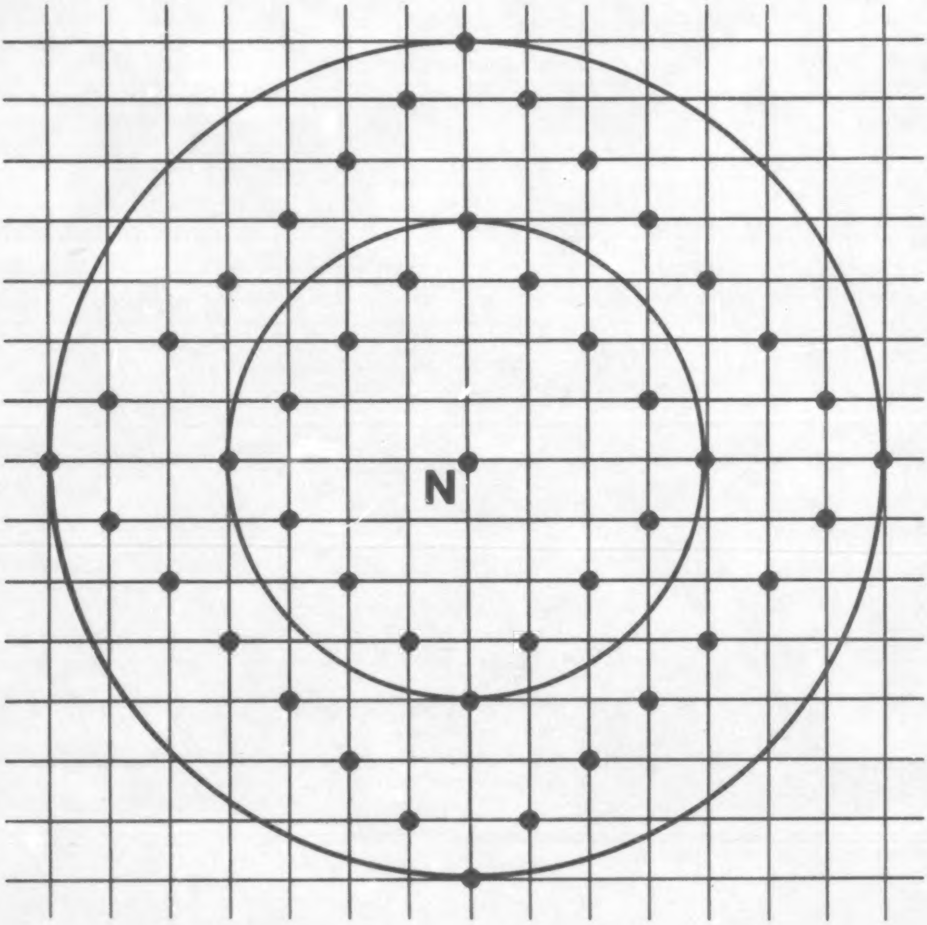


T: Who can show us how to draw a regular circle for a four block path centered at N?

Invite a student to tell you how to hold the string and the chalk. Then draw the circle.

T (drawing the circle): How many blue dots am I going to hit?

S: Four.



T: You get regular circles if you are in the country and do not have to follow streets. Taxi-circles are the kind of circles you get if you have to follow the streets like a taxi-cab.

Distribute copies of Worksheet G3 and colored pencils. Go over the directions collectively. The small circles in the chart are for the students to indicate a color code for the taxi-circles.

G 3

If some students finish the worksheet quickly, ask them to extend the table for other numbers. Do not give away the fact that the number of places is always 4 × the taxi-distance from N. When most of the students have completed the chart, go over the answers collectively filling in the chart on the board. Extend the chart to include other entries that your students have found. Perhaps your chart will look like this one.

taxi-distance from N	How many places?
0 blocks	1
1 blocks	4
2 blocks	8
3 blocks	12
4 blocks	16
5 blocks	20
6 blocks	24
7 blocks	28
8 blocks	32
9 blocks	36
10 blocks	40
11 blocks	44

T: How many places are a taxi-distance of 20 from N?

S: 80.

T (pointing to the right column of the chart): What kind of numbers are these?

S: Multiples of 4.

T: All these numbers except for 1 are multiples of 4.

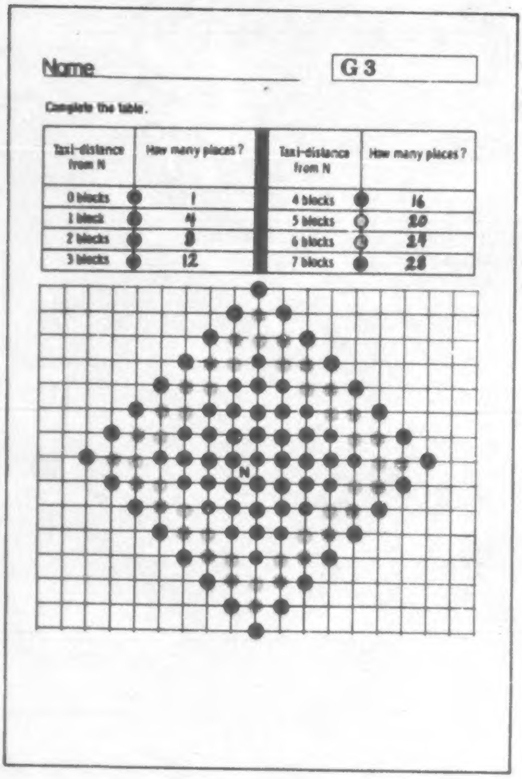
Briefly indicate the 4 × rule between the entries in the first and in the second columns (except for the "0-1" entry). Point to the appropriate places in the chart as you indicate the 4 × rule.

T: 1 × 4 = 4. →

T: 2 × 4 = 8. →

T: 3 × 4 = 12. →

taxi-distance from N	How many places?
0 blocks	1
1 blocks	4
2 blocks	8
3 blocks	12



Name

G 3

Complete the table.

Taxi-distance from N	How many places ?	Taxi-distance from N	How many places ?
0 blocks ●	1	4 blocks ●	16
1 block ●	4	5 blocks ○	20
2 blocks ●	8	6 blocks ○	24
3 blocks ●	12	7 blocks ●	28

G4 Trees

CAPSULE LESSON SUMMARY

Examine a special kind of graph, a tree, and find that in a tree there is one more dot than there are edges. Find also that between any two dots there is exactly one path following the edges. Assign numbers (weights) to the edges of a tree and play a game which generates a cycle of arrows going through all the dots.

MATERIALS

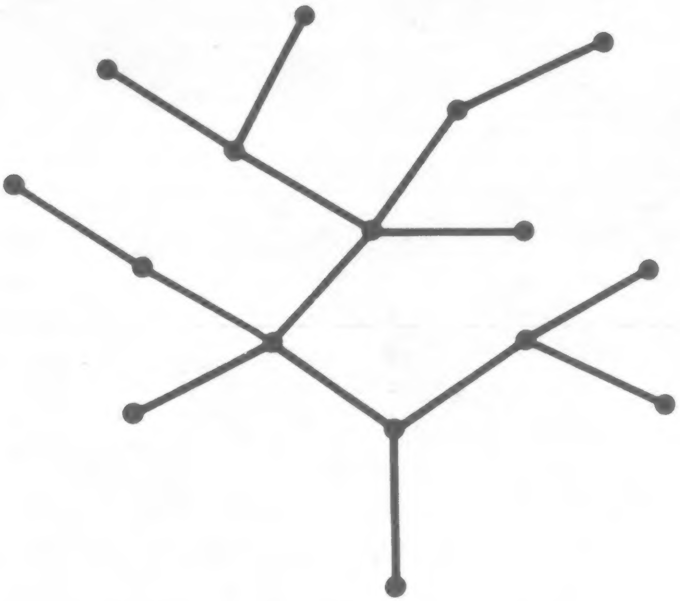
Teacher : Colored chalk
Student : Worksheets G4* and **; colored pencils

DESCRIPTION OF LESSON

Exercise 1

T: I am going to draw something on the board; sometimes it is called a "tree". While I'm drawing it, see if you can figure out why it is called that.

Draw this tree on the board.



S: It looks like it has branches.

T: How many dots are there? (Sixteen)

Collectively let the students count the dots aloud as you point to each dot.

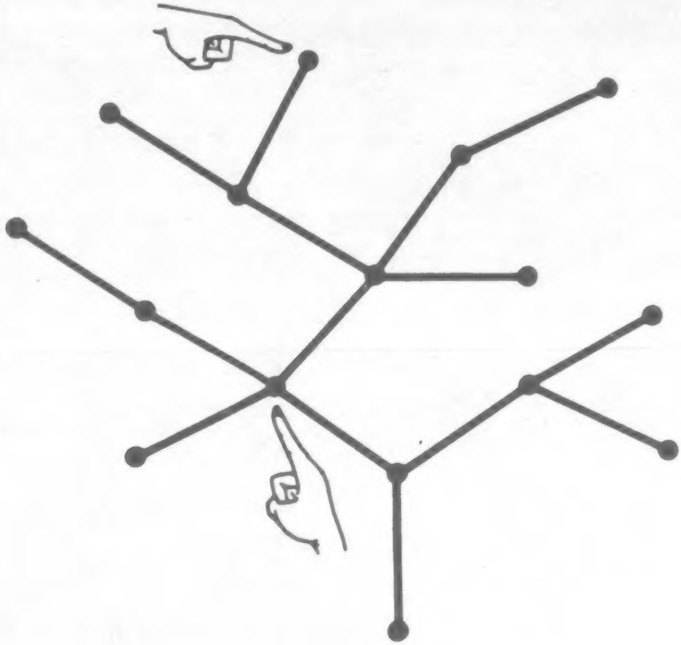
Trace a line segment connecting two of the dots as you say:

T: This is called an "edge". How many edges are there in our tree? (Fifteen)

Collectively let the students count the edges aloud as you point to each edge. Record the number of edges and the number of dots on the board.

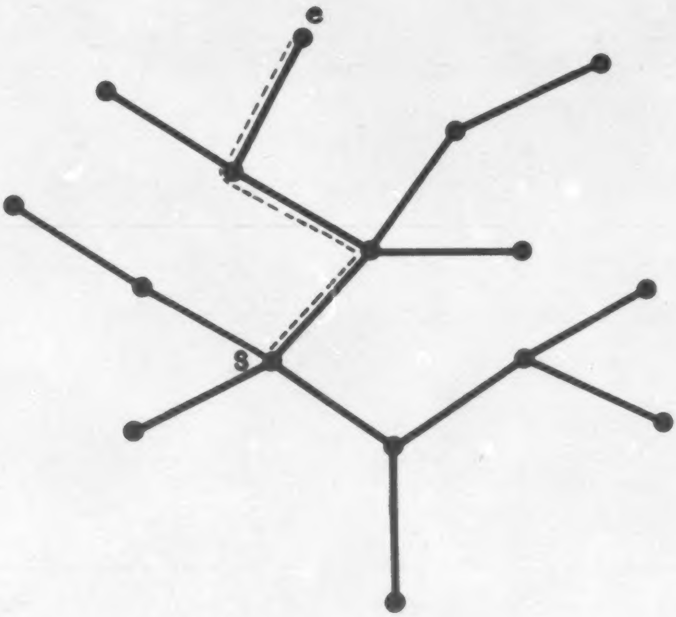
16 dots
15 edges

Choose any two dots in the tree for the discussion which follows; for example,



T: Suppose I start here (tap one of the dots). How many ways are there to go to this dot (tap the other dot) following the edges? Once we follow an edge we do not want to back up on that edge and then to go somewhere else.

Label the starting dot "s" and the ending dot "e" and invite someone to trace a path from s to e.



T: Are there any other paths between these two dots (s and e)?

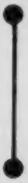
S: No.

Repeat the activity for two other pairs of dots in the tree; one pair of your choosing and another pair of a student's choosing. The class should discover that there is only one path between the dots in each pair considered. Generalize this observation as you say:

T: That is what is special about a tree — there is only one way to get from one dot to another following the edges.

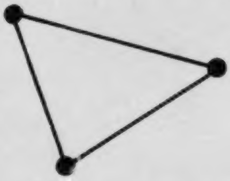
G 4

Draw two dots on the board and a line segment connecting them. The size of your dots and the length of your segment will set a size perspective for the tree to be built.

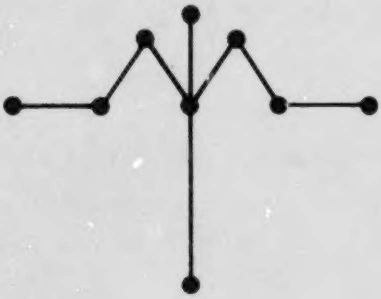


T: Let's make another tree. I started the tree. Who would like to draw another dot and another edge?

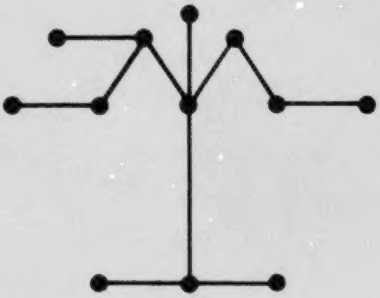
Continue until thirteen or fourteen edges have been added to the picture. After each edge is added, ask the students if the picture is still a tree. There should be no cyclic paths, i.e. paths which start and end at the same dot; for example,



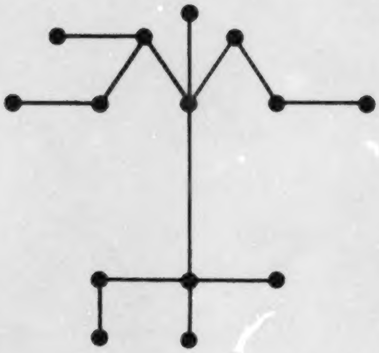
Periodically ask how many dots and how many edges are in the tree at that particular stage. The illustration below and on the next page shows the tree one of our development class built and also indicates at what stages the number of edges and the number of dots were counted during the activity.



9 dots
8 edges



12 dots
11 edges



14 dots
13 edges

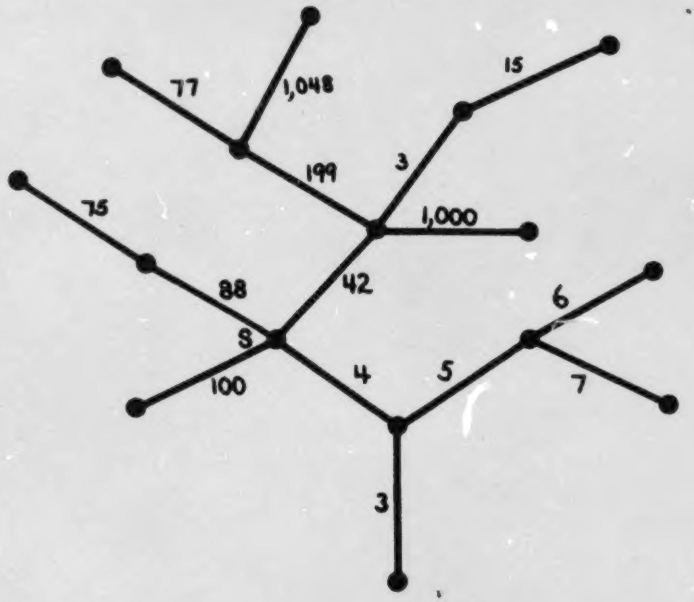
If no one mentions that the number of edges is always 1 less than the number of dots (a property of a tree), point this out yourself.

T: We have two different trees on the board. Now I would like you to help assign some numbers to the edges of this tree (the first one considered). The only restriction is that the edges which meet at the same dot must be assigned different numbers.

Point to the edges one at a time and let students choose numbers for them. Encourage the class to check that the restriction is obeyed each time they a assign a number to an edge.

G 4

This picture shows the assignment of numbers by one of our development classes.



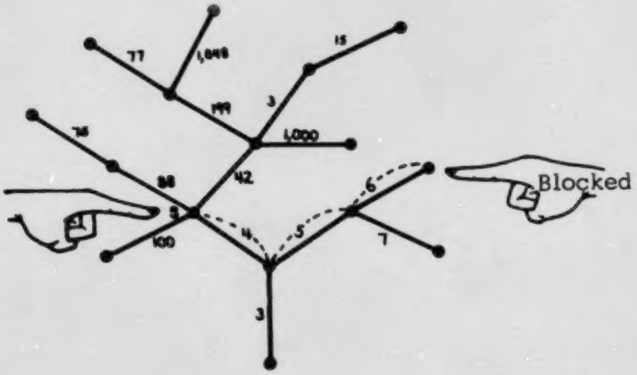
Ask someone to choose a starting dot. Label that dot "s".

T: We are going to play a red arrow game with this tree. This is how you play:

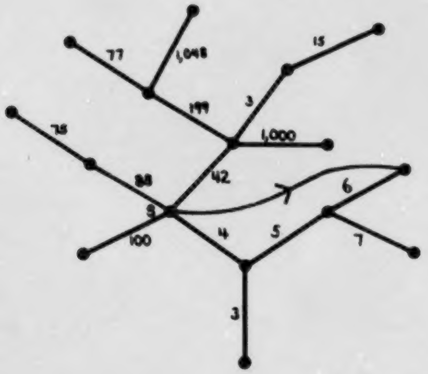
- begin at the starting dot and follow the edge with the smallest number;
- when you get to the next dot, try to follow an edge with a larger number but still as small a number as you can.

You may keep going until you are blocked, that is, you cannot follow an edge with a larger number. Let's start together and see what happens.

The assignment of numbers to edges in the preceding illustration will be used here for the explanation of the game. Put your left forefinger on the starting dot and hold it there. Invite a student to begin tracing edges following the rules of the game; we follow the edge labeled "4" since 4 is the smallest number assigned to an edge at s. When we get to the next dot, we look just at the edges with larger numbers and choose the edge (in this case there is only one) with the smallest number. We follow this edge, labeled "5", to the next dot. Then, again we look just at the edges with larger numbers (in this case there are two edges with numbers larger than 5) and choose the edge with the smallest number (in this case the edge labeled "6") to the next dot. Again we look at the edges with larger numbers — now there are none so we are blocked.

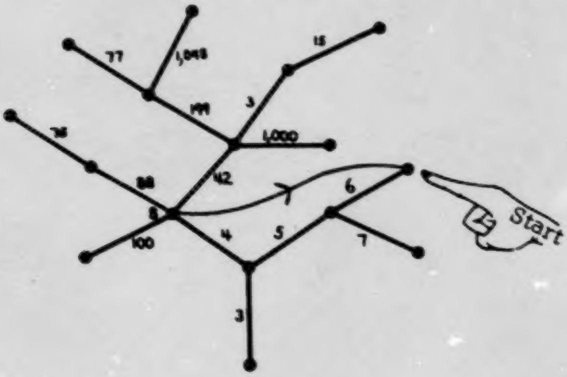


T: Now we are blocked because there are no edges here with numbers larger than 6. So, we draw a red arrow from where we started (point to "s") to where we ended (point to the dot where you are blocked).

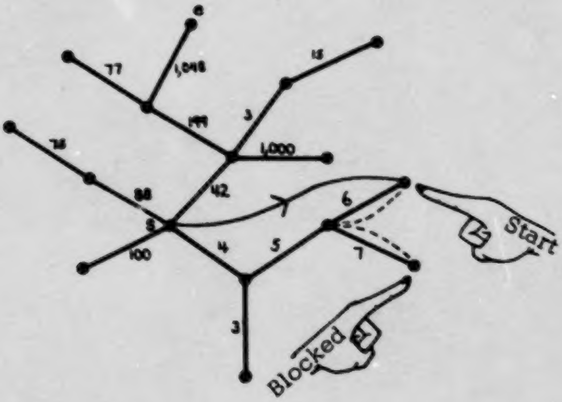


G 4

T: Now we start the game over again, but this time we start at the dot where the red arrow ends.

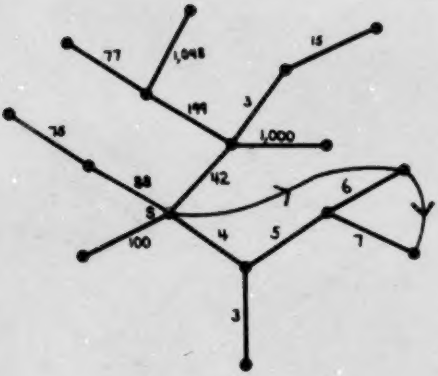

77
1,048
15
3
199
1,000
76
88
42
8
6
100
4
5
7
3
Start

Hold your finger at the new starting dot and invite a student to trace edges following the rules of the game until we are again blocked.

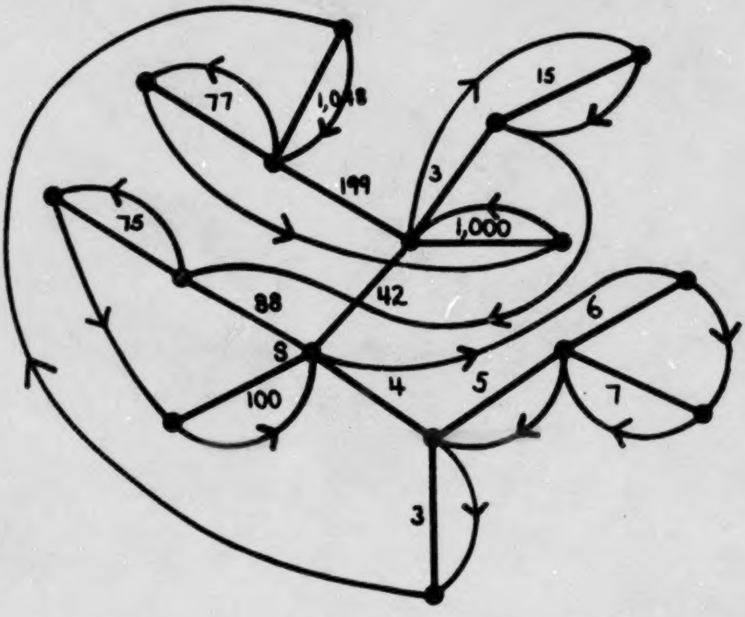

77
1,048
15
3
199
1,000
76
88
42
8
6
100
4
5
7
3
Start
Blocked

T: We are blocked again, so we draw another red arrow from where we started to where we are blocked this time. Then we begin again starting where this new red arrow ends.

582

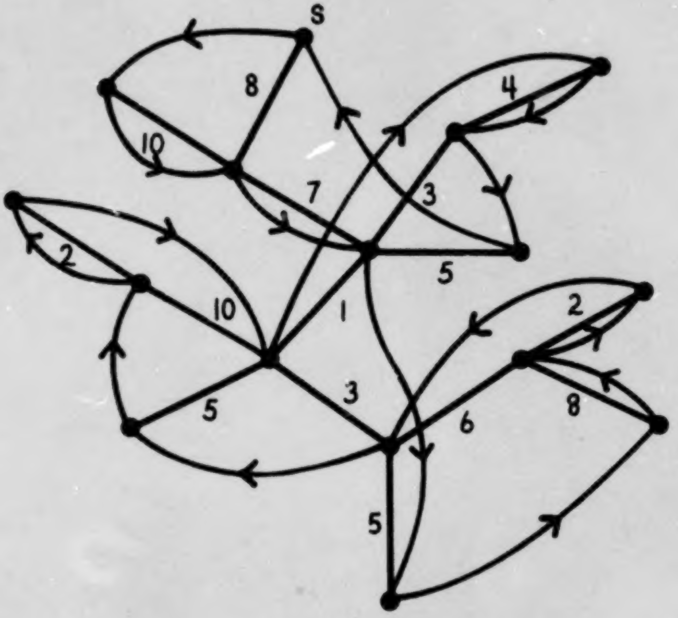

77
1,048
15
199
3
75
1,000
88
42
6
s
100
4
5
7
3

Continue until a red arrow returns to the starting dot "s".


15
77
1,048
3
199
75
1,000
88
42
6
S
100
4
5
7
3

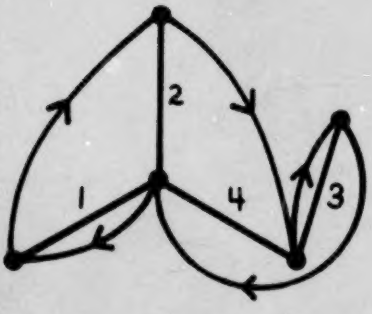
Invite someone to trace the cycle of red arrows starting at s and returning to s.

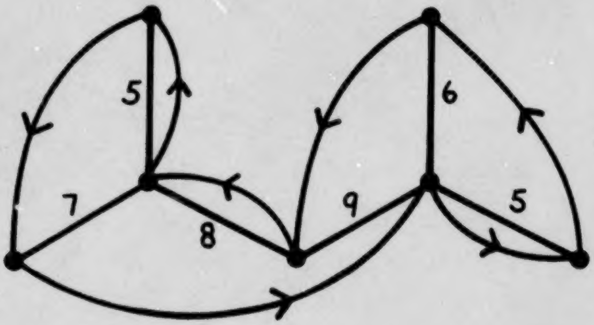
NOTE: In the above (preceding) picture it does not matter which dot you choose as your starting dot "s"; you will obtain the same cycle of red arrows going through the dots in the same order. What does make a difference, though, is the assignment of numbers to edges. For example, the next illustration pictures the same tree with a different assignment of numbers to edges; a different red cycle results.



Exercise 2

If your students understand the rules of the red arrow game fairly well, play the game again only this time with the tree your students drew on the board. If your students need more practice with the game rules, draw two very simple trees on the board and play the game with those instead; for example,

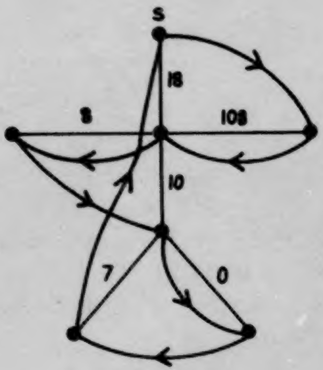




Worksheets G4* and ** are available for individual work.

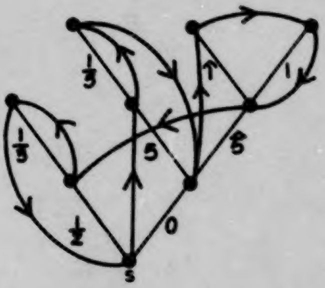
Name ____________________ **G 4 ***

Play the red arrow game with this tree. Start at S.


S
18
8
108
10
7
0

Name ____________________ **G 4 ****

Play the red arrow game with this tree. Start at S.


1/3
1/3
7
1
8/5
5
1/2
0
S

G5 Length in Centimeters and Decimeters

CAPSULE LESSON SUMMARY

Examine a tape measure and review how to use it to measure in centimeters. Introduce the decimeter unit of length and the equality "10 cm = 1 dm". Convert some decimeter lengths to centimeters and vice-versa. Find shortest paths between points in an irregularly shaped yard and measure the length of these paths in centimeters and decimeters.

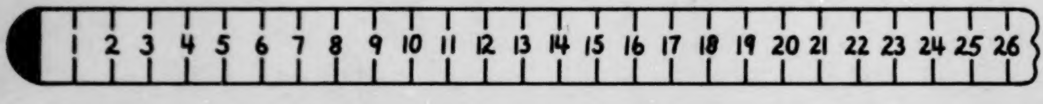
MATERIALS

Teacher: Metric tape measure; colored chalk
Student: Metric tape measure; Worksheets G5*, **, ***, ****

DESCRIPTION OF LESSON

Exercise 1

Use a straightedge to draw this enlargement of part of a metric tape measure.



Distribute metric tape measures to the students.

T: Look at the side of the tape which is numbered. What is the distance from one black mark to the next?

S: One centimeter.

Write "centimeter" and the abbreviation for it on the board.

centimeter (cm)

T: Turn the tape over to the other side. What is the distance from one orange mark to the next?

Some students might suggest the length of an orange rod, remembering some of the measurement lessons in CSMP Mathematics for the Upper Primary Grades, Part I and Part II.

T: You might not know the unit name for this length.

Say "decimeter" as you write it and its abbreviation on the board.

centimeter (cm)
decimeter (dm)

T: Look at each side of the tape. How many centimeters is one decimeter?

S: Ten.

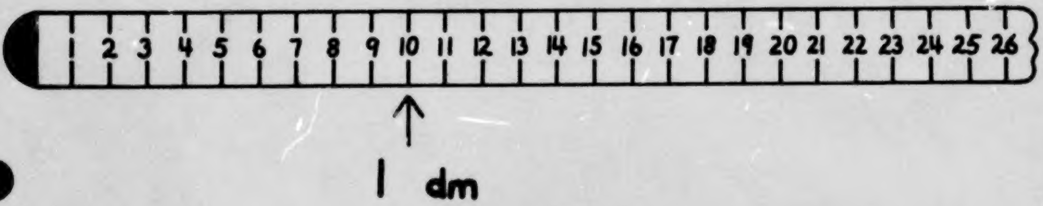
T: Ten centimeters is the same length as one decimeter just as ten dimes is the same amount of money as one dollar.

Begin a list of equalities on the board.

10 cm = 1 dm

T (pointing to the mark for 10 cm): Since 10 cm = 1 dm, this mark is also for 1 dm.

Draw an orange arrow pointing to the mark labeled "10".

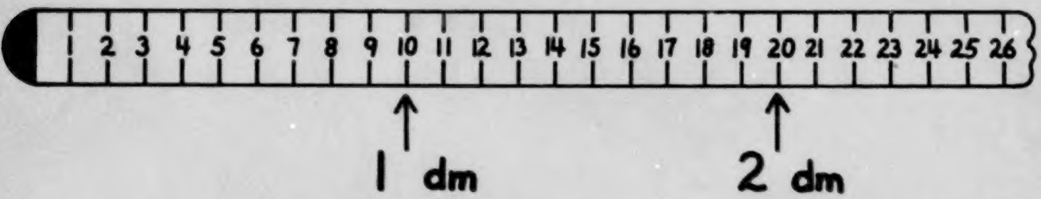


G5

T (pointing to the mark for 20 centimeters): This mark is for 20 centimeters; it is also for how many decimeters?

S: 2.

Draw an orange arrow pointing to the mark labeled "20".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2
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4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
1 dm
2 dm

Write "40 cm = ☐ dm" under "10 cm = 1 dm" as you say:

T: 40 centimeters is how many decimeters?

S: 4.

Complete the equality on the board.

$$10 \text{ cm} = 1 \text{ dm}$$
$$40 \text{ cm} = \boxed{4} \text{ dm}$$

Repeat this activity to complete these equalities. Answers are in the boxes.

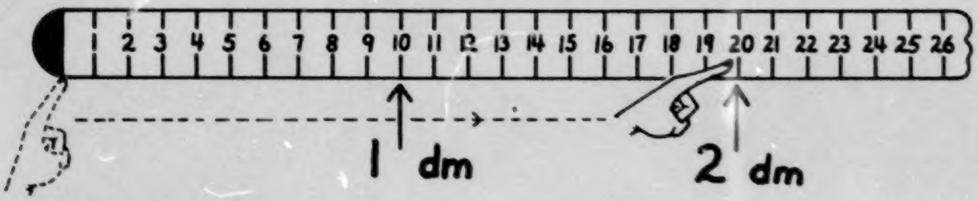
$$70 \text{ cm} = \boxed{7} \text{ dm}$$
$$\boxed{130} \text{ cm} = 13 \text{ dm}$$

Write "25 cm = ☐ dm" as you say:

T: 25 centimeters is how many decimeters?

Let students make suggestions. Perhaps someone will note that 25 centimeters is between 2 decimeters and 3 decimeters.

Put your right index-finger on the mark for 0 in the picture on the board. Move your finger to the right. As you pass the first orange arrow say "1 decimeter"; continue until you get to the second orange arrow, stop, and say "2 decimeters."



T: 25 centimeters is more than 2 decimeters. How many more centimeters is 25 centimeters than 2 decimeters?

S: 5.

Move your finger 5 centimeters to the right, stopping at the mark for 25 centimeters.

T: We could write "25 cm = 2 dm 5 cm". What is another way we could write this?

S: "$2\frac{1}{2}$ dm".

S: "2.5 dm".

If no one suggests "2.5 dm", ask how to write 2 dollars and 5 dimes and then return to your original question.

Complete the equality on the board. You should have this list.

10 cm = 1 dm

40 cm = $\boxed{4}$ dm

70 cm = $\boxed{7}$ dm

$\boxed{130}$ cm = 13 dm

25 cm = $\boxed{2.5}$ dm

Do a similar activity, completing the following equalities. Students should refer to their own tape measures since the picture on the board is not sufficient to discuss these problems. Answers are in the boxes.

53 cm = $\boxed{5.3}$ dm

78 cm = $\boxed{7.8}$ dm

6 cm = $\boxed{0.6}$ dm *

Erase the list of equalities and the orange arrows.

Exercise 2

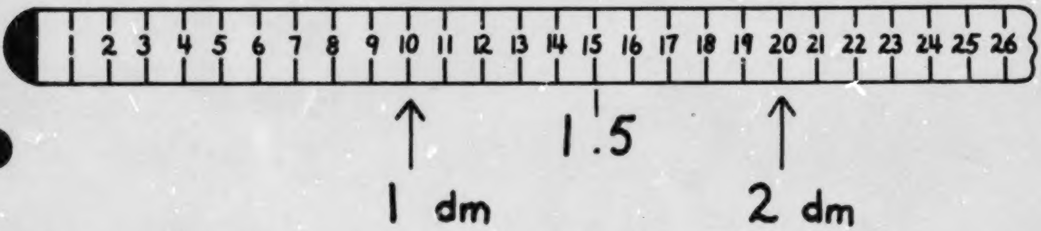
T (pointing to the picture on the board): Who can point to the mark for 1.5 decimeters?

If necessary, help the student who volunteers by pointing to the arrow indicating 1 decimeter and by counting 5 centimeters past it.

---

* Be sure to include the "0" before the decimal point.

Label the appropriate mark "1.5" as in the next illustration.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6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
1.5
1 dm
2 dm

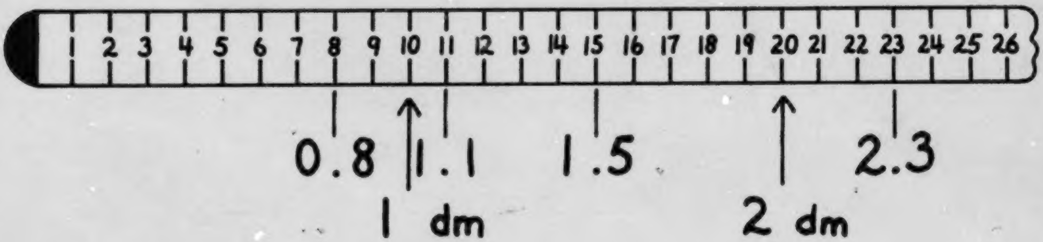
T: 1.5 decimeters is how many centimeters?

S: 15.

Record this equality on the board.

1.5 dm = 15 cm

Continue this activity asking students to locate in turn the marks for 2.3 dm, 1.1 dm, and 0.8 dm.


1
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10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
0.8
1.1
1.5
2.3
1 dm
2 dm

You should have these equalities listed on the board.

1.5 dm = 15 cm

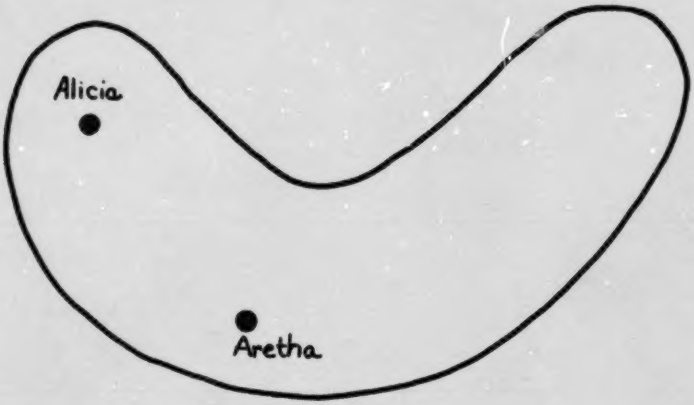
2.3 dm = 23 cm

1.1 dm = 11 cm

0.8 dm = 8 cm

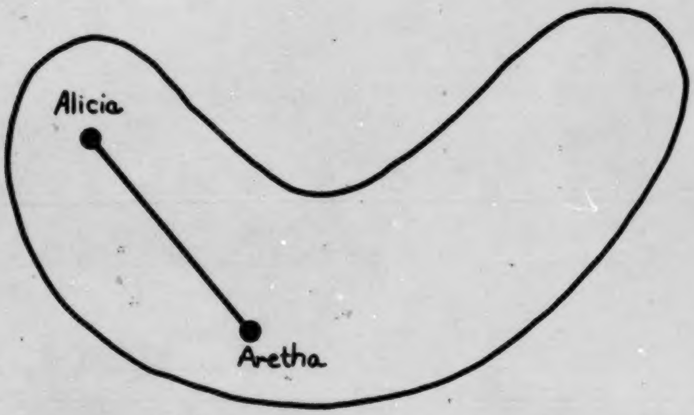
Exercise 3

Erase the board and draw this picture on the board.



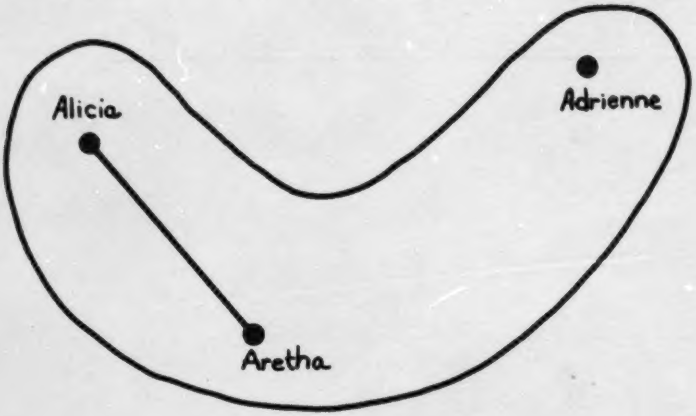
T: This is a picture of someone's yard. The dots show you where Alicia and Aretha are standing in the yard. Aretha wants to walk to where Alicia is standing and she wants to take as short a path as possible. Who can trace the shortest path from Aretha to Alicia?

The student who volunteers should trace the line segment between the dots. Draw the segment yourself.



Help a student measure the segment in centimeters. Emphasize that you place the mark for 0 cm (or the end of the tape, whichever the case may be) at one end of the segment and read the measure at the mark that lands at the other end of the segment. Then ask how long the segment is in decimeters.

Draw another dot in the picture and label it "Adrienne".

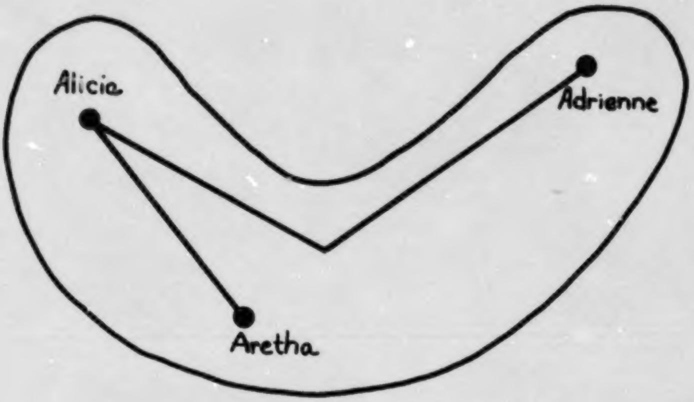


Demonstrate Adrienne's manner of walking along a zigzag path as you describe it.

T: Adrienne also wants to walk to where Alicia is standing without going out of the yard. She cannot take a completely straight path so she walks along a zigzag path. She walks in a straight line, turns, then walks in a straight line again, turns, and so on until she reaches Alicia. How might Adrienne walk to Alicia following a short path and always staying in the yard?

After a student traces a zigzag path which stays inside the yard, draw the zigzag in the picture. This is one possibility (see the next illustration); it is not the shortest such path.

G 5



T (tracing the zigzag): How can we find out how long this path is?

S: Measure the two pieces and then add their lengths.

Help students measure the pieces of the zigzag in centimeters. Ask the class what the lengths are in decimeters and record them on the board. Perhaps you will have numbers similar to these.

$$\begin{array}{r} 5.7 \text{ dm} \\ 7.4 \text{ dm} \end{array}$$

Invite a student to do the addition on the board. You may need to help if there is confusion because of the decimal points.

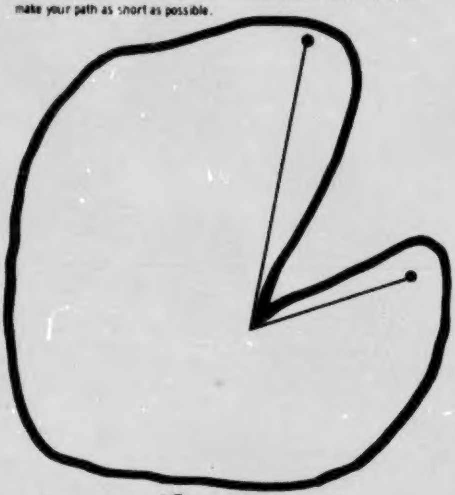
$$\begin{array}{r} 5.7 \text{ dm} \\ +7.4 \text{ dm} \\ \hline 13.1 \text{ dm} \end{array}$$

Repeat this activity if there is obviously a shorter path than the zigzag drawn and if there is student interest.

Worksheets G5*, **, ***, and **** are available for individual work.

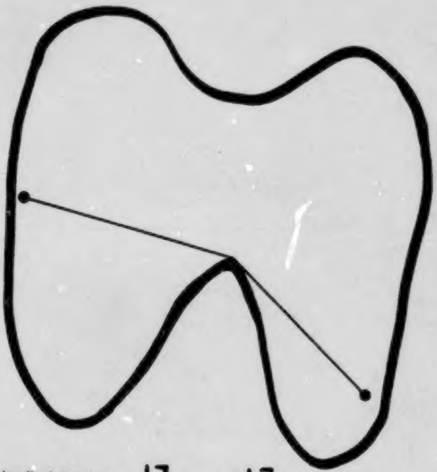
Name ______________ **G 5 ***

Connect the dots with a zigzag path, but do not go out of the yard. Try to make your path as short as possible.



Length of zigzag path = 19 cm = 1.9 dm.

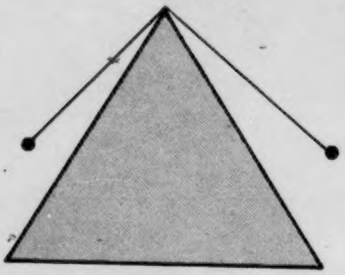
Connect the dots with a zigzag path, but do not go out of the yard. Try to make your path as short as possible.



Length of zigzag path = 17 cm = 1.7 dm.

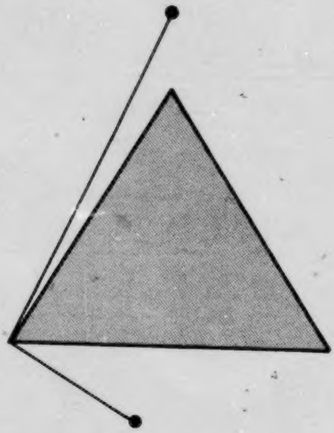
Name ______________ **G 5 ****

Connect the dots with a zigzag path, but do not go through the building. Try to draw a path shorter than 1.8 dm.



Length of zigzag path = 1.7 dm (less than 1.8 dm)

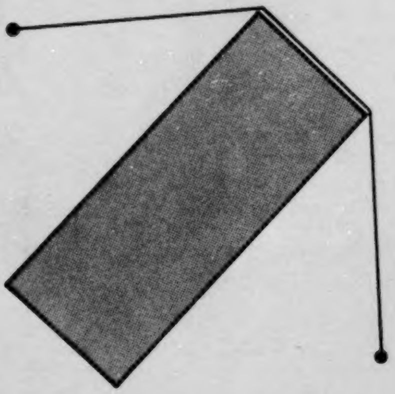
Connect the dots with a zigzag path, but do not go through the building. Try to draw a path shorter than 2.3 dm.



Length of zigzag path = 2.1 dm (less than 2.3 dm)

Name ______________________ **G 5 *****

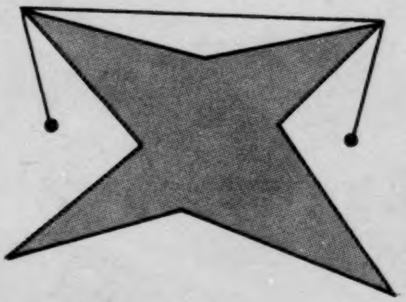
Connect the dots with a zigzag path, but do not go through the building. Try to draw a path shorter than 2.7 dm.



Length of zigzag path = 2.6 dm (less than 2.7 dm)

Name ______________________ **G 5 ******

Connect the dots with a zigzag path, but do not go through the building. Try to draw a path shorter than 2.6 dm.



Length of zigzag path = 2.5 dm (less than 2.6 dm)

G6 Taxi-Distance

CAPSULE LESSON SUMMARY

Find the taxi-distance from a given point to a given line by finding the taxi-distances from various points on the line to the given point and taking the shortest distance. Observe some patterns in the sets of numbers generated by looking at the distances from given points to the points along given lines. Notice that all the points which have the shortest distance (for a given point and a given line) are along the diagonal of a square with one corner being the given point.

MATERIALS

Teacher: Grid board*; colored chalk
Student: Worksheets G6*, **, and ***; colored pencils

DESCRIPTION OF LESSON

Exercise 1

On your grid board, draw the picture shown in the next illustration. Be careful to give the blue line the correct slope. If you draw one dot, you can get to another dot on the blue line segment by:

- moving to the right two spaces and then moving up one space; or by
- moving to the left two spaces and then moving down one space.

The dot labeled "N" is positioned five spaces up and one space to the left from the dot labeled "e". If your grid board is larger than the one pictured on the next page, extend the blue line segment as far as possible in both directions.

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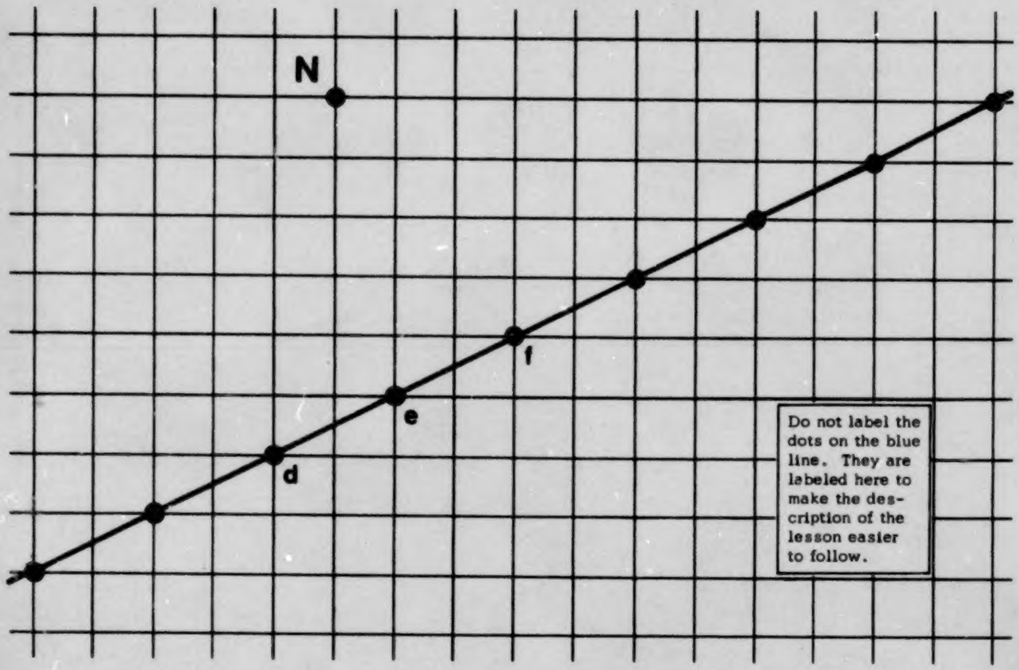
* See the NOTE ON GRIDS in the introduction to this strand.

G 6



Do not label the dots on the blue line. They are labeled here to make the description of the lesson easier to follow.

Briefly review the story of Nora's neighborhood and how to find the taxi-distance between two points. Write this information on the board.

taxi-distance = length of a shortest path*

T: On this map the blue line segment is for part of the path of an underground railway, called a "subway." The dots along that line segment are the subway stations — places where people can get on or off the train. Nora wants to ride the subway and to find which subway station is closest to her house (point to "N").

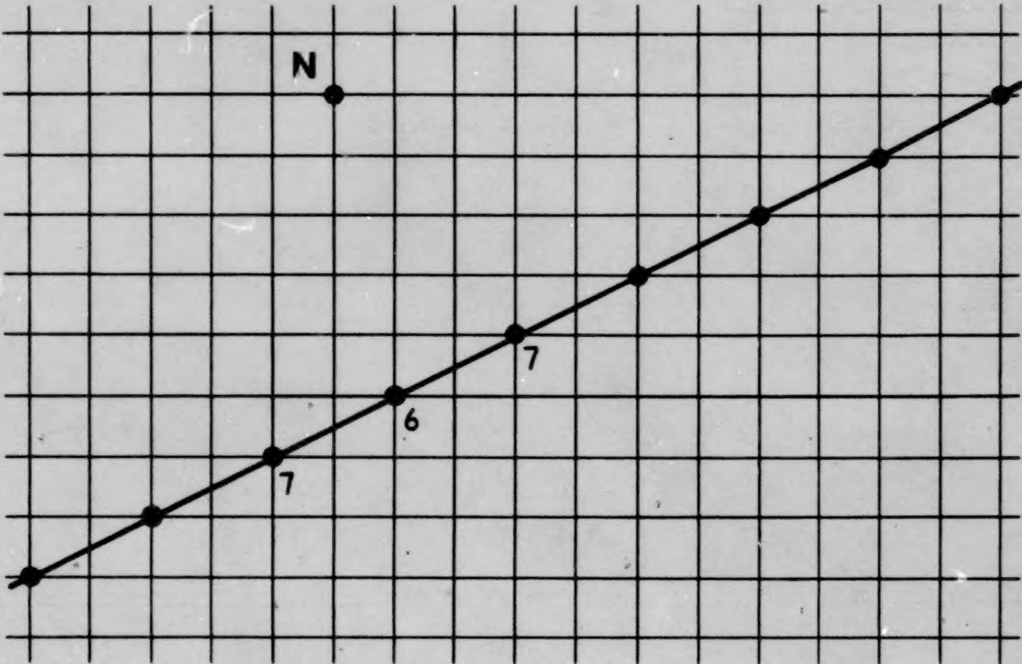
Let some students point on the map to which subway stations they think are closest to Nora's house. Most likely d, e, and f will be considered.

---

* This is understood to be an abbreviated way of saying that the taxi-distance from one point to another is the length of a shortest path between those two points.

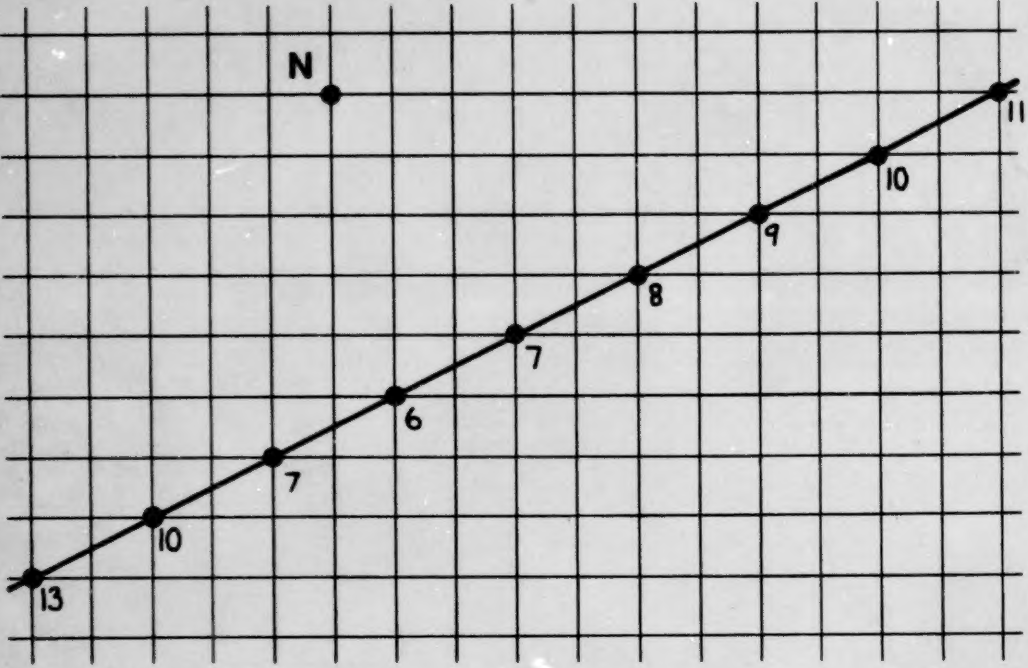
When a student suggests a possible closest station, ask the student to find the taxi-distance from Nora's house to that station. If correct, record the taxi-distance in the picture near the dot for the station suggested.

The next illustration shows the taxi-distance from N to d, to e, and to f, although your students may have considered dots other than these.



Give each student a copy of Worksheet G6*. Ask the students to find the taxi-distance to each of the stations on the map and to record those distances next to the dots.

Let the class work independently for a few minutes. Then invite students to finish labeling the dots on the grid board. Note that the picture on the board may have a shorter blue segment than the one on the worksheet.



T: Which station is closest to Nora's house?

S: The one six blocks from Nora's house.

On the board erase the dot for that station and redraw it in red. Students should draw a red ring around that dot on their papers.

T: Does anyone see a pattern to these numbers?

S: There is a "7" on each side of "6".

S: If you move that way (pointing to the right) from 6, the numbers start at 7 and go up by 1 each time.
If you move that way (pointing to the left) from 6, the numbers start at 7 and go up by 3 each time.

Ask students for the taxi-distances from Nora's house to stations not shown on the map on the grid board.

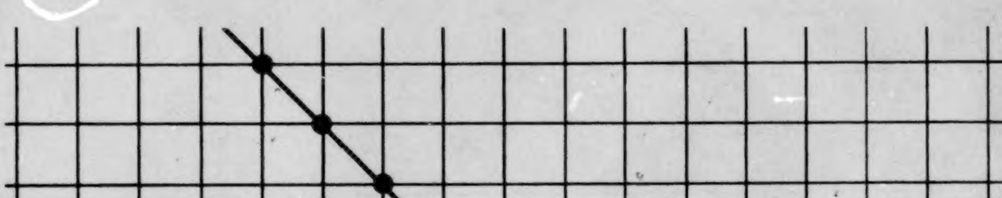
In our picture moving further to the right, you meet stations whose taxi-distances from Nora's are 12, 13, 14, 15, . . . ; moving further to the left, you meet stations whose taxi-distance from Nora's are 16, 19, 22, 25 . . . .

Exercise 2

Erase the grid board and then draw the picture shown in the next illustration. Be careful to give the line the correct slope. If you draw one dot, you can get to another on the blue line segment by;

- moving one space to the right and one space down; or by
- moving one space to the left and one space up.

The dot labeled "A" is positioned five spaces to the right from d and five space up from e. If your grid board is larger than the one pictured below, extend the blue line segment as far as possible in both directions.



Do not label the dots on the blue line segment. They are labeled here to make the lesson description easier to follow.

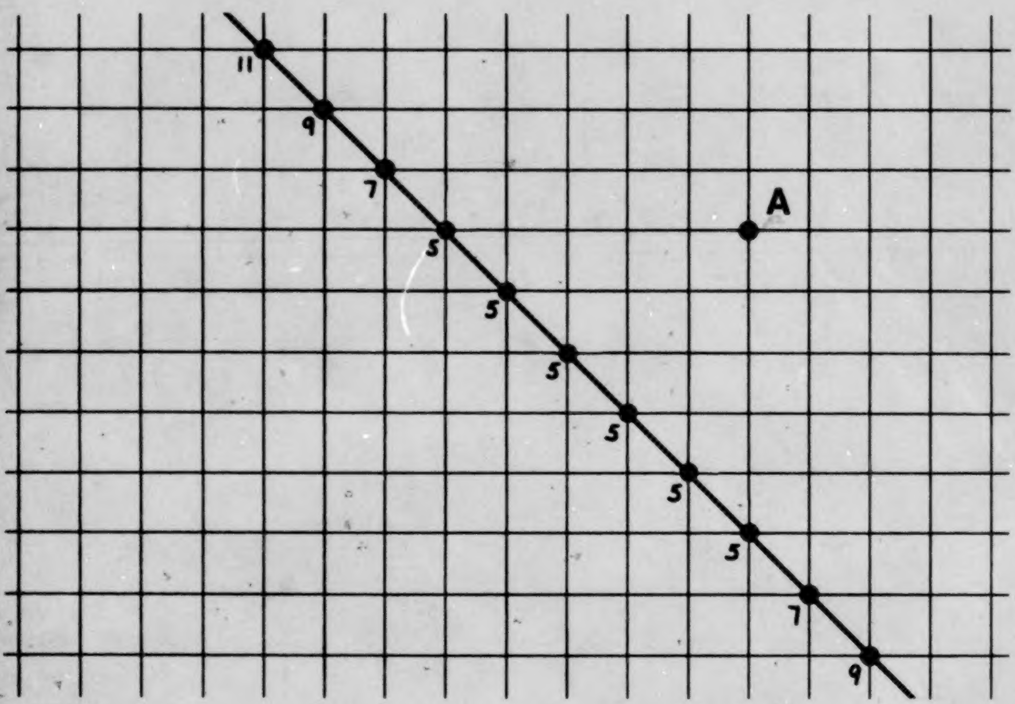
G 6

T: In another part of town there is a sports arena. After people go to the arena to see a basketball game or some other event, many of them ride the subway home and want to board the subway at a station as close as possible to the arena.

Turn your worksheet over to find a map like the one on the board. Is there a station close to the arena?

As students suggest possibilities, let them record on the board the taxi-distance from the arena to those stations. Students very likely will express surprise that there are several closest stations.

Continue until all six closest stations have been found. Redraw the corresponding dots in red. Collectively label the rest of the dots with the taxi-distances from the arena to the represented stations.

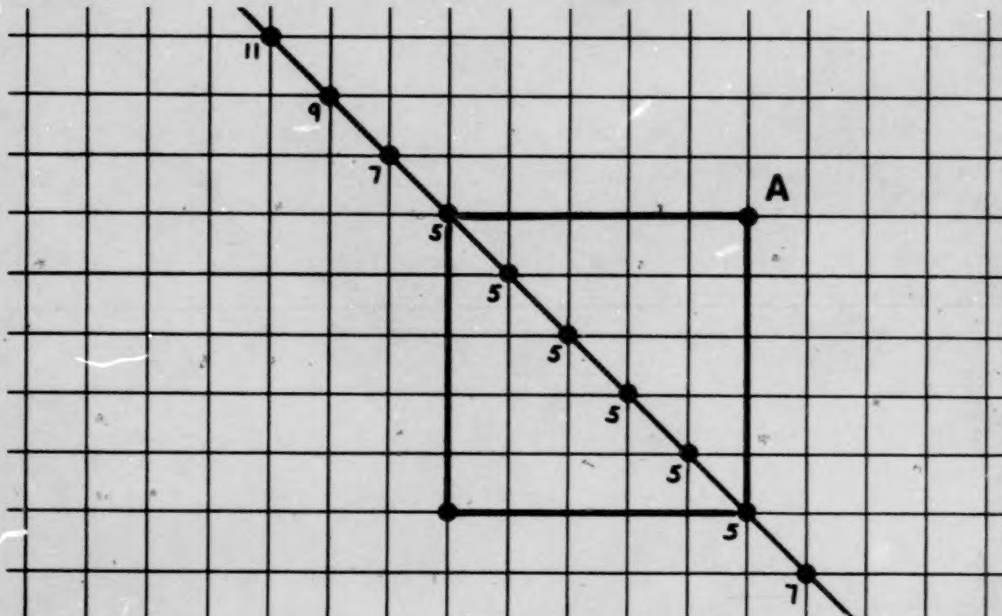


T: What kind of number patterns do you see in this picture?

S: There are six 5's and then you count by twos moving in either direction.

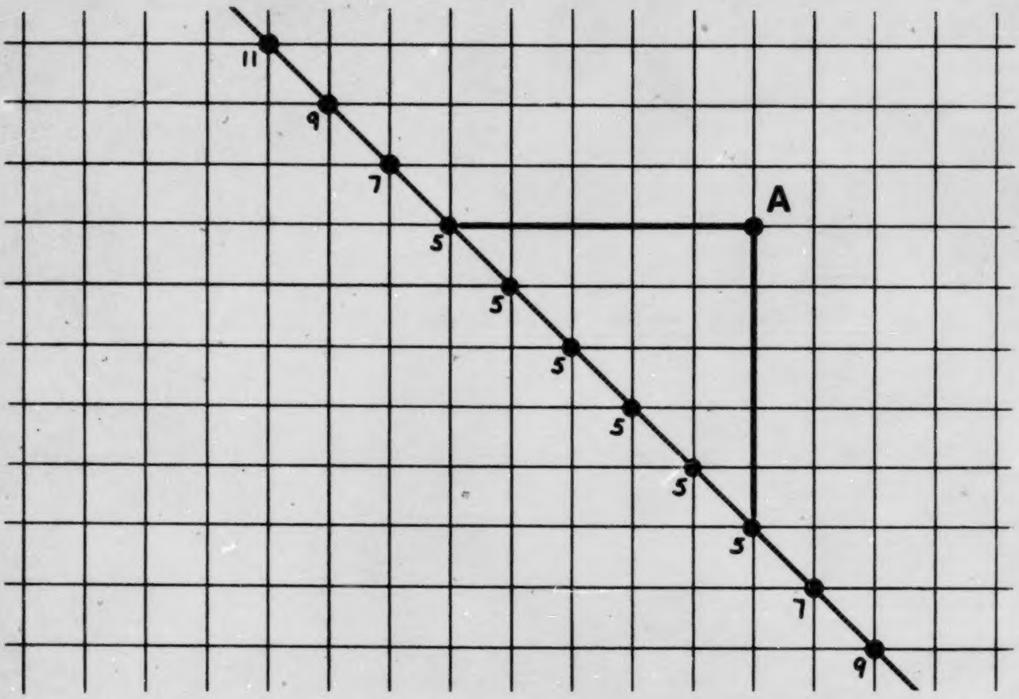
T: We found six different stations which are closest to the arena. Is there another place besides the arena which is a taxi-distance of 5 from these same stations (point to each of the six red dots)?

Allow time for thought. When a possible place is suggested, check the taxi-distance from there to each of the six stations marked in red on the map. Continue until the correct place has been found and then draw a dot there and the square shown below.



If no suggestion is forthcoming, draw part of the square . . .

G 6



. . . and repeat the question. If necessary, indicate that the left side of the blue line segment should be considered.

Worksheets G6** and *** are available for individual work. Conclude the lesson by asking for the number patterns in each picture (see the answer keys for the worksheets).

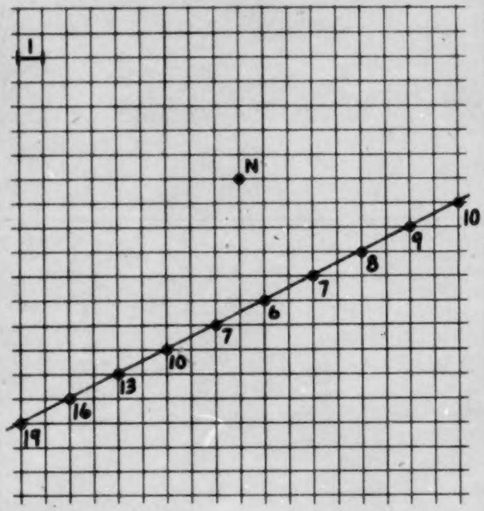
G6**: There are five 4's and then the numbers increase by twos in either direction.

G6***: There are two 7's and then the numbers increase by fours moving upward and increase by twos moving downward.

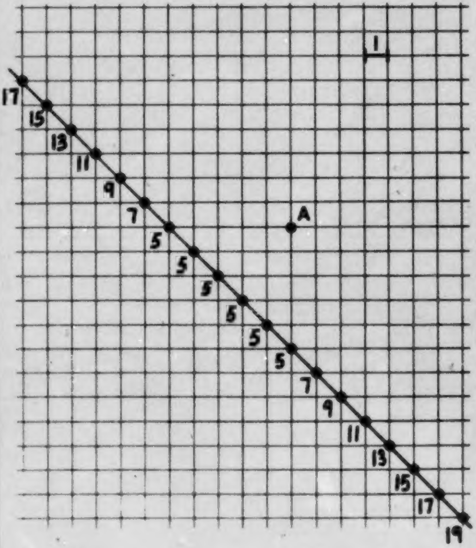
Name ______

G 6 *

Find the taxi-distance from N to each station.


1
N
10
9
8
7
6
7
10
13
16
19

Find the taxi-distance from A to each station.

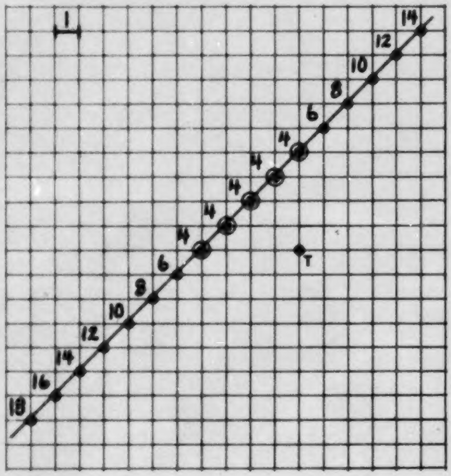

1
A
17
15
13
11
9
7
5
5
5
5
5
5
7
9
11
13
15
17
19

Name ______

G 6 **

Circle in red the stations which are closest to T.

Find the taxi-distance from T to each station.

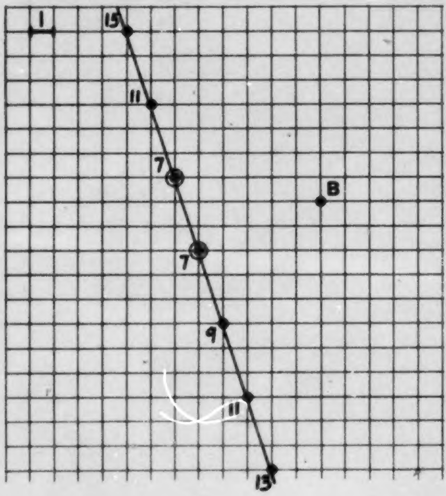

1
14
12
10
8
6
4
4
4
4
4
6
8
10
12
14
16
18
T

Name ______

G 6 ***

Circle in red the stations which are closest to B.

Find the taxi-distance from B to each station.


1
15
11
7
B
7
9
11
13

G 7

G7 Touring Houses #1

CAPSULE LESSON SUMMARY

Find paths through a house using each doorway at most once. Specify starting and ending points for some walks and try to use as many doors as possible. Find some walks which go through all the doors and introduce such paths as tours.

MATERIALS

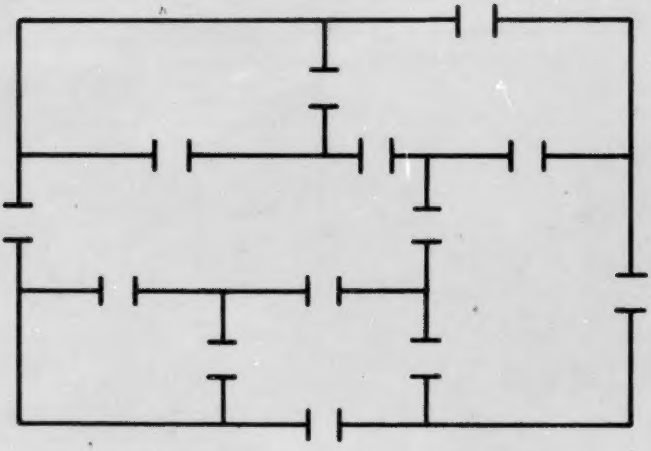
Teacher: Colored chalk; (Optional) Geometry transparency #1 for UPG IV; overhead projector

Student: Worksheet G7; pencil

NOTE: A transparency is provided for this lesson. If you have access to an overhead projector, you could use the transparency to project the floor plan discussed in the lesson. Paths could be drawn and erased easily without spoiling the floor plan.

DESCRIPTION OF LESSON

Draw this picture on the board (or project it using the transparency).



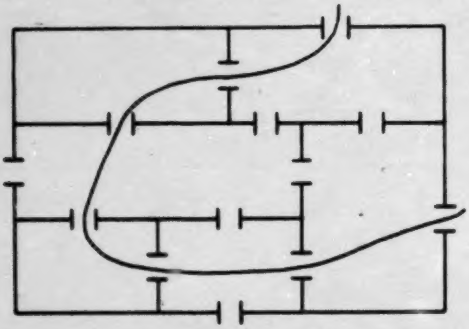
: This is a floor plan for a house. A floor plan shows you where the rooms and the doors are in the house.

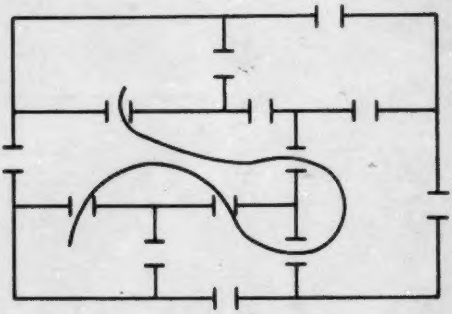
et students comment on the situation. Perhaps some of your students will ave seen floor plans in advertisements or in newspapers.

T: How many rooms does this house have? (Six)
How many doors does this house have? (Thirteen)

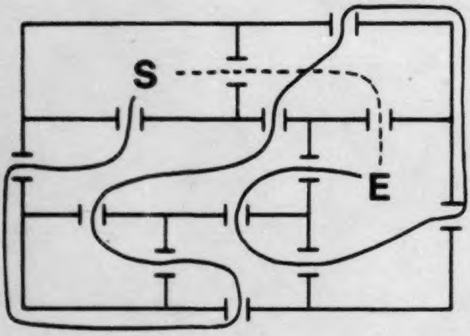
Let's pretend that we are going to take a walk through this house, but that we can go through a doorway no more than once. Who can show a way we could walk through this house?

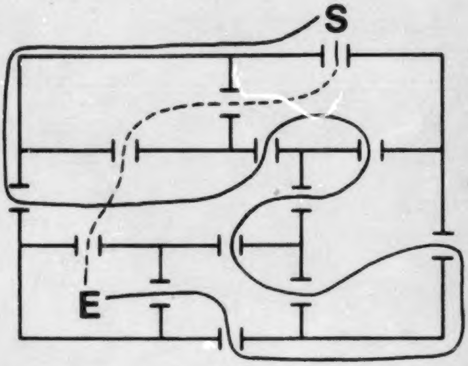
Let several students trace (or draw) paths. There are many possibilities; two examples are given below.





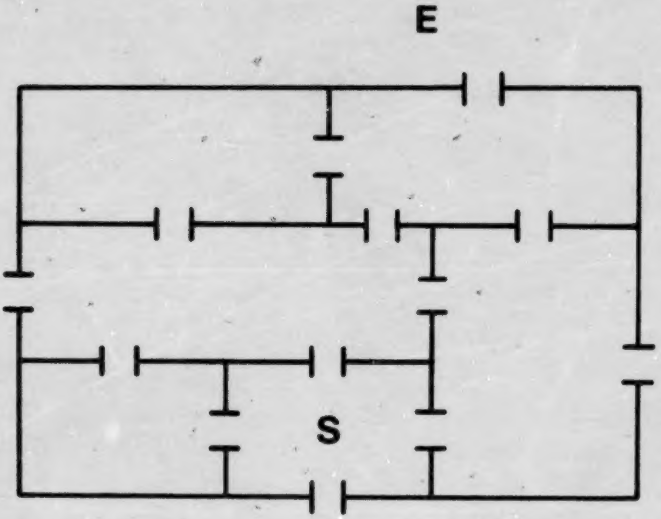
Ask how to walk through the house starting at S, in a given room or outside the house, and ending at E, in a given room or outside the house. Students should trace (or draw) such paths. Two choices for S and E are indicated in the next illustration; two of many paths are shown for each choice.





G7

Erase the path. Label the outside "E" and one of the rooms (see the next illustration) "S".

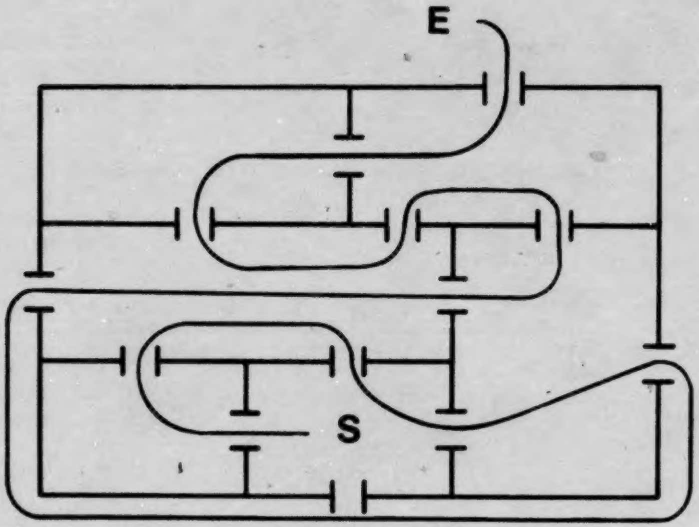


Distribute copies of Worksheet G7 and instruct the students to consider the side with one copy of this floor plan.

T: All of you have a copy of this floor plan. Try to find a path from S to E which goes through as many doorways as possible.

Let the students experiment. After about five minutes, invite a student to trace a path on the board (or draw on the transparency if you are using an overhead projector). As the student traces a path, mark each doorway used with an "X". Note how many doorways the path goes through and then ask if anyone can find a path which goes through more doorways. Continue until several students have traced (drawn) paths. For each path point out to the class that some of the doorways of the starting room, S, are missed.

NOTE: It is possible to have a path which goes through all of the doorways except one. An example of such a path is shown in the next illustration.

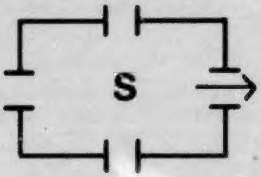


T: Why is it that none of the paths go through all of S's doorways?

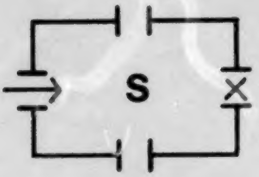
Do not expect well-formed answers; a student might respond, "You get trappe if you do!"

The next illustration shows only room S in the floor plan.

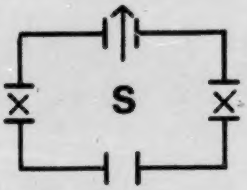
T: If we start in S and go out . . .



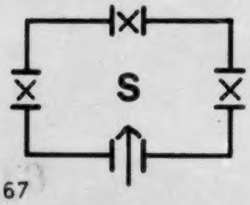
and then come back in . . .



and go back out, . . .



could we come back in?



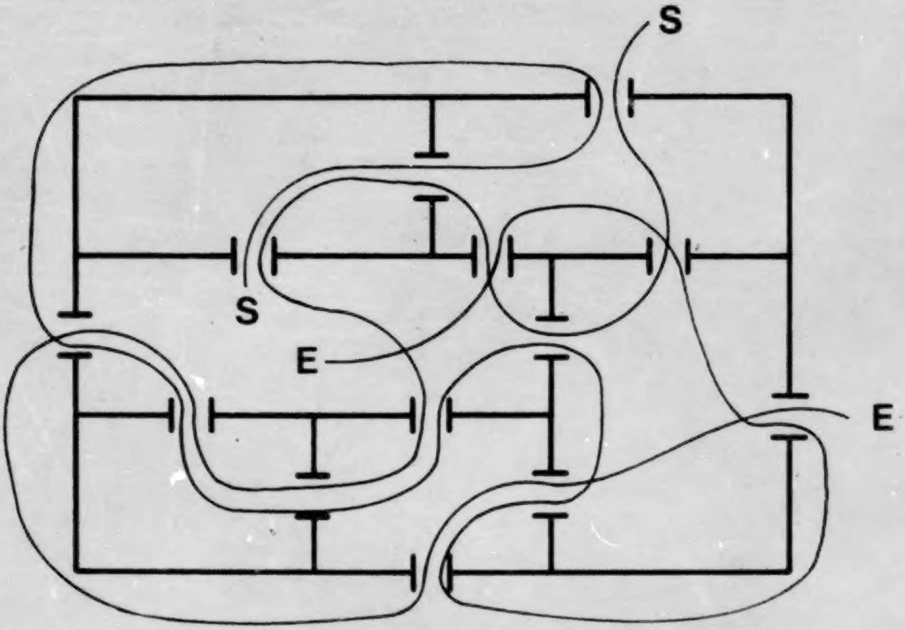
S: No, because we couldn't get back out and we are supposed to end at E.

Erase "S" and "E" from the floor plan. Ask the students to turn over their worksheets. Two copies of the floor plan are on this side of the worksheet.

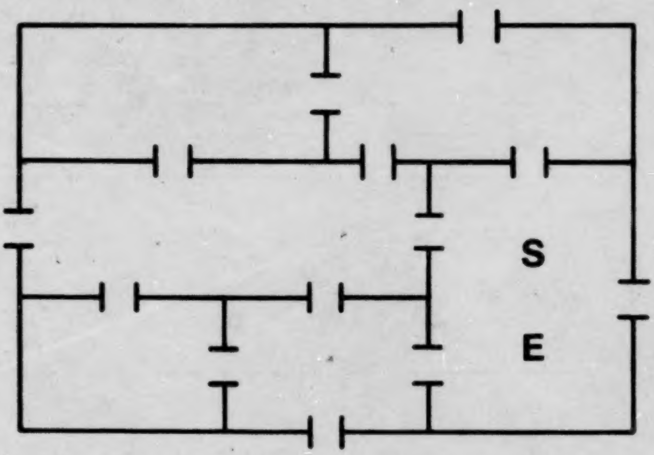
T: Try to plan a walk which goes through all the doorways once each. You may choose where you start and where you end. Use one of the two pictures on your worksheet and use pencil so that you can erase easily.

NOTE: Any path which goes through all the doorways starts and ends outside or starts and ends in the same room.

Let students experiment. After several minutes invite two students who have found paths that go through all the doorways and whose paths start and end in different places to draw their paths on the board (or on the transparency). These are two of the many possibilities.

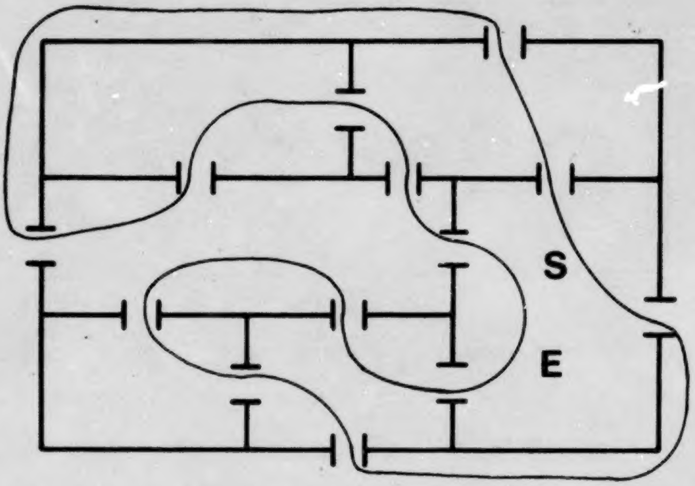


Erase the paths. Choose a room in which neither of the two paths (just erased) started and label that room "S" and "E" on the floor plan. In the dialogue it is assumed that the room chosen is the one on the lower right.



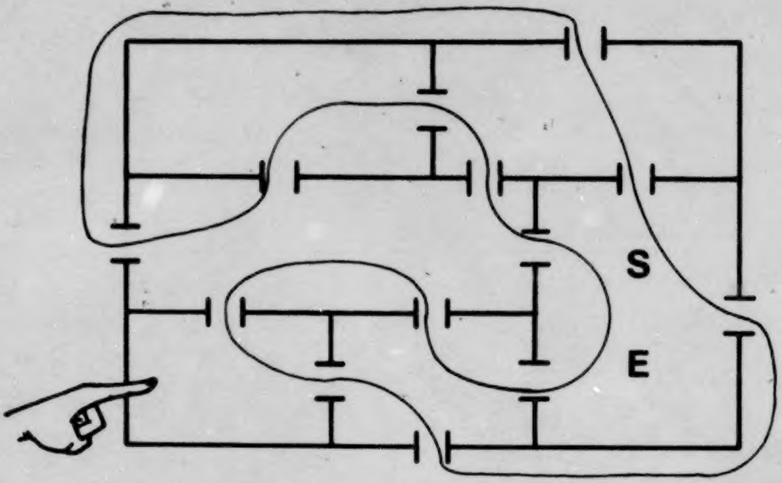
T: Do you think we could start and end in this room and go through all of the doorways? Try to draw one. Use the other picture on your worksheet.

Let students experiment before asking for a volunteer to trace such a path on the board. Suppose a girl named "Mindy" volunteers. Draw the path in red as the student traces it. Connect the beginning and ending points of the path to give a continuous curve. One possibility is shown below.



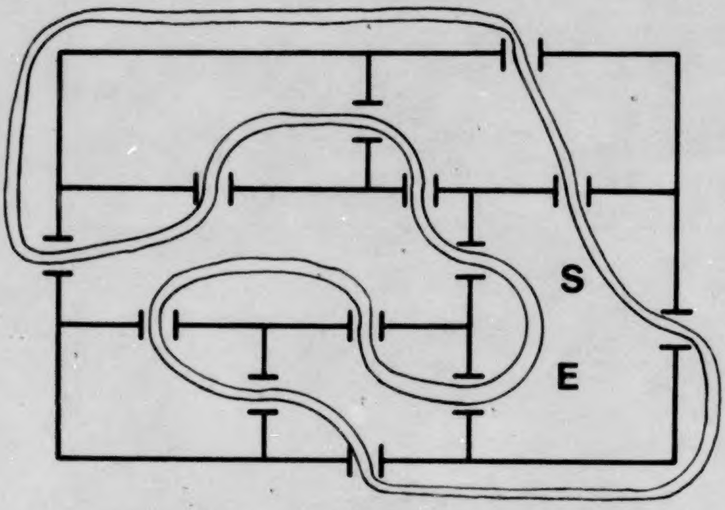
G 7

Point to a room other than S(E) as you ask:



T: Does someone see a clever way to use Mindy's path to show a path which starts here and ends here?

Ask students to trace paths until someone indicates that Mindy's path can be used directly. Redraw the walk in blue for emphasis.



Point to other rooms and repeat the question. Students should conclude that the path on the board can be used to show that it is possible to start and end in any of the rooms (or outside) and to go through each doorway once.

T: A path that goes through each doorway exactly once is called a tour.

G8 Length of Spirals

CAPSULE LESSON SUMMARY

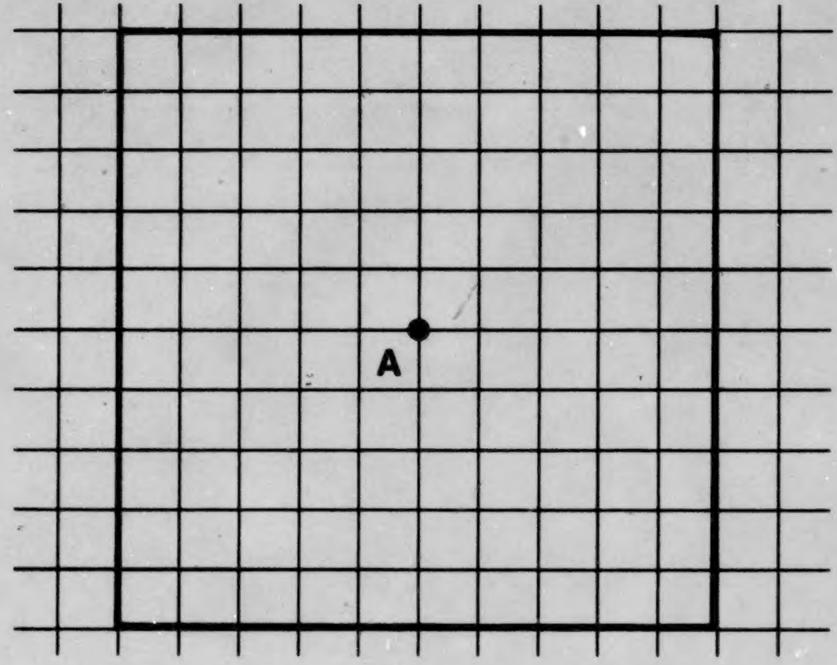
Find the perimeter of a square with sides of length 10 cm. Draw a specified spiral starting at the center of the square and ending at one corner of the square. Find the length of this spiral using an organized counting procedure. Compare the length of the spiral to that of a meter stick. Repeat the activity with a square having 20 cm sides.

MATERIALS

Teacher: Grid board[#]; colored chalk; meter stick
Student: Worksheets G8* and **; colored pencils

DESCRIPTION OF LESSON

Distribute copies of Worksheet G8* and red pencils. Display a grid board and draw on it a square, 10 units on each side. Use the side of a piece of chalk to draw the square so that it will be easy to see. Label the center point of the interior of the square "A". Your grid board should look similar to the picture on Worksheet G8*.



[#] See the NOTE ON GRIDS in the introduction to this strand.

G8

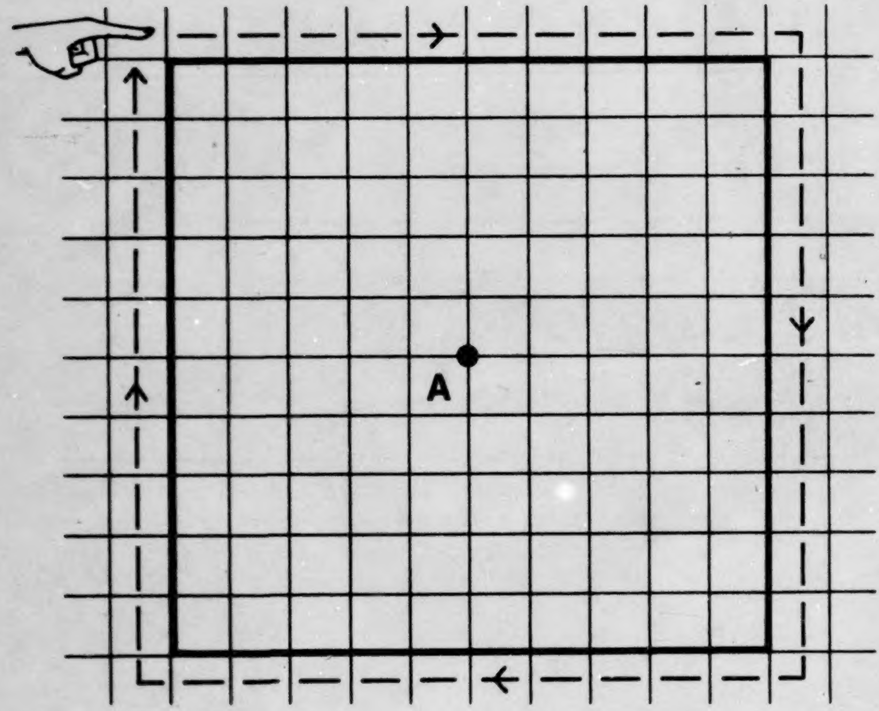
T (tracing the large square on the grid board): What shape is this?

S: A square.

T: Each little square on your worksheets has sides of length 1 cm. What is the length of a side of the large square?

S: 10 cm.

T: If a bug starts here (point to one of the corners of the large square) and walks completely around the square (trace around the square),



. . . how far will the bug have walked?

S: 40 cm.

614

T: How do you know?

S: I counted: 10, 20, 30, 40.

T: 40 cm is the perimeter of this square.

Direct the students to draw spirals on their worksheets as you do so on the boa
Draw the spiral with red chalk.

T: Take your red pencil. Start at A;
go up 1 centimeter; then go 1 centimeter to the right;
go down 2 centimeters; then go 2 centimeters to the left;
go up 3 centimeters; then go 3 centimeters to the right.

What are we drawing?

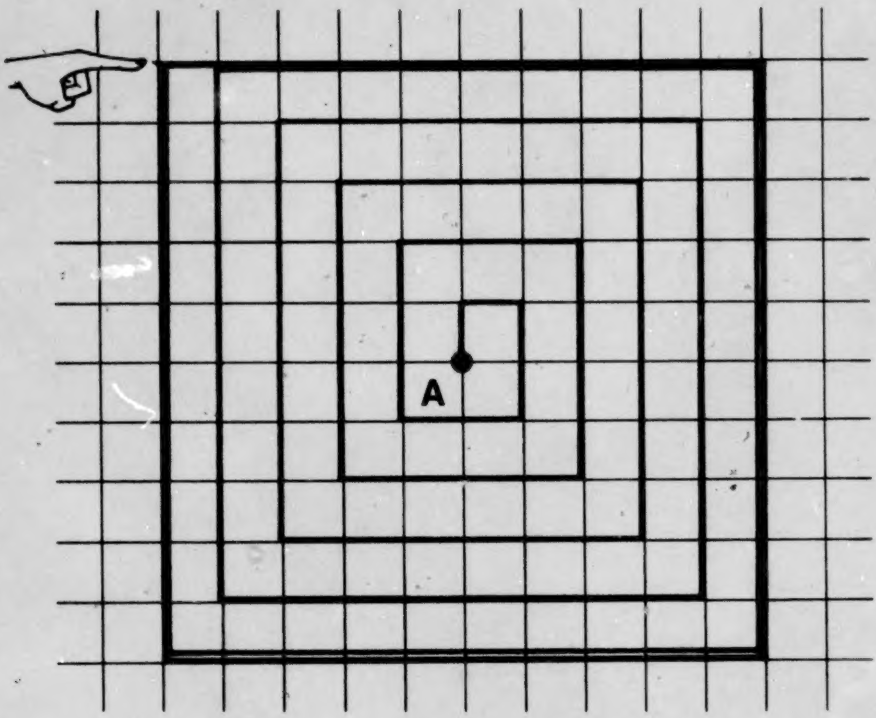
S: A whirl.

S: A spiral.

T: Continue drawing the spiral. You can draw along the sides of the large square but do not go outside of it.

On the board complete the spiral within the boundaries of the large square.

T: Does everyone have a spiral which ends here?



T: How long do you think the red spiral is?

Let students guess its length.

S: 100 cm.

S: 200 cm.

S: 90 cm.

S: 400 cm.

T: Let's see if we can figure out exactly how long the spiral is. How long is a shortest segment in this spiral?

S: 1 cm.

S: There are two segments that long.

Record "2 × 1" and "2" on the board.

2×1

2

Trace the first (vertical) 2 cm segment in the spiral on the board.

T: How many 2 cm segments are there in the spiral?

S: Two.

2×1 2×2

2 + 4

Continue until you have this addition problem on the board.

2×1 2×2 2×3 2×4 2×5 2×6 2×7 2×8 2×9 2×10

2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 + 20 + 10

Note that there are three 10 cm segments because the boundary of the large square cuts off the last segment of the spiral at 10 cm.

T: How shall we do this addition problem?

G 8

Let the students suggest methods; for example,

- simply add the numbers one after the other;
- notice a pattern of sums of 22;

$$\underbrace{2 + \overbrace{4 + \ldots + 18} + 20} + 10 = (5+22) + 10$$

$$= 120$$

- notice a pattern of sums of 20.

$$\underbrace{2 + \underbrace{4 + \ldots + 10 + \ldots + 16} + 18} + 20 + 10 = 6 \times 20$$

$$= 120$$

Using one of the preceding methods, conclude with your class that the spiral is 120 cm long.

T: The spiral you drew is 120 cm long. (Hold up a meter stick.) Is the length of the spiral more or less than 1 meter?

S: More.

T: How much more?

S: 20 cm.

Record this information on the board.

10 cm square — 120 cm spiral

1 m 20 cm

Ask your students to turn their worksheets over to the ** side.

T (holding up a copy of G8**): Each side of this large square measures 20 cm. What is its perimeter (trace the large square on the worksheet)?

S: 80 cm.

T: How long do you think a spiral which starts at A and fits in this square would be?

These are some of the responses from one of our development classes.

S: 400 cm.

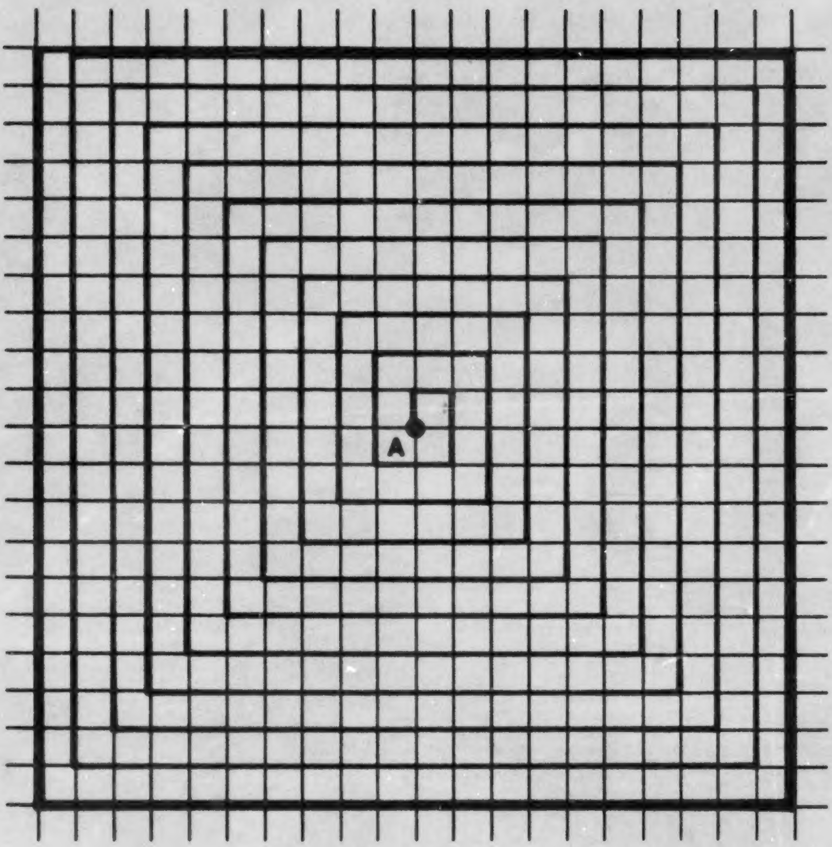
S: The other spiral was six twenties; this one will be about eleven twenties.

S: About 420 cm.

S: A square like this one (pointing to the * worksheet) fits four times into this square (pointing to the ** worksheet). So maybe the spiral will be about 480 cm.

T: Draw another spiral, starting at A. Once again, do not go beyond the borders of the large black square.

This is a reduction of what your students should draw on their worksheets. (See the illustration on the next page.)



T: How long is the spiral you drew?

Let students work on the problem individually for a few minutes.

T: There are two 1 cm pieces, two 2 cm pieces, two 3 cm pieces; . . .

2 + 4 + 6 +

T: How far do we have to go?

S: To 2 × 20.

S: And then there is still one more 20 cm piece.

If no one responds, ask how long each side of the large square is, or write out the full addition problem as students read it off. Here the second line continues the calculation from right to left.

$$2+4+6+8+10+12+14+16+18+20+22+$$
$$20+40+38+36+34+32+30+28+26+24$$

T: How shall we do this addition problem?

Let your students suggest methods; for example,

- notice a pattern of sums of 40;

$$2+4+\ldots+18+20+22+\ldots+36+38+40+20$$

$$11\times 40=440$$

- notice a pattern of sums of 42;

$$2+4+6+\ldots+36+38+40+20$$

$$(10\times 42)\qquad +20\qquad =440$$

- use the previous problem and a pattern of sums of 62.

$$2+4+\ldots+18+20+22+24+\ldots+38+40+20$$

$$(120-10)\qquad+\qquad(5\times 62)\qquad+20$$
$$110\qquad+\qquad310\qquad+20=44$$

If your students only suggest adding the numbers one after the other, show them the pattern of sums of 40.

Conclude with your class that the length of the spiral is 440 cm.

T: How many meters is 440 cm?

S: 4 meters with 40 cm left over.

T: Do you think the ceiling in our classroom is that high?

Perhaps you and your students would like to check this with a meter stick, but unless your classroom has a very high ceiling the likely answer is "no."

Record the information about the second spiral on the board.

10 cm square — 120 cm spiral
1 m 20 cm

20 cm square — 440 cm spiral
4 m 40 cm

T: 440 cm is almost 4 × 120 centimeters. Why do you think that's true?

S: Because four 10 cm squares fit inside one 20 cm square.

If no one suggests this, simply leave the question open.

G9 Touring Houses #2

CAPSULE LESSON SUMMARY

Review the idea of touring a house and find tours of two different houses. Observe that some tours start and end at the same place and some do not. Find ways to decide where a tour could start (or end). Modify a house by building or blocking doors so that there is a tour which starts and ends at the same place.

MATERIALS

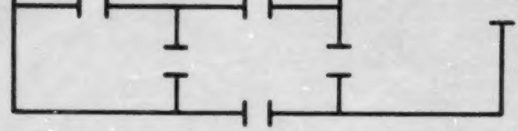
Teacher: Colored chalk; (Optional) Geometry transparency #2 for UPG-IV; overhead projector

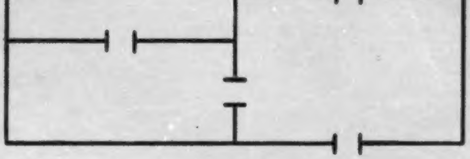
Student : Worksheets G9*, **, ***, and ****; pencil

NOTE: A transparency is provided for this lesson. If you have access to an overhead projector, you could use the transparency to project the floor plan discussed in the lesson. Paths could be drawn and erased easily without spoiling the floor plan.

DESCRIPTION OF LESSON

Draw these two pictures on the board (or draw the one on the left and project the one on the right using Geometry transparency #2 for UPG-IV.)





T: What do you think these are?

S: Floor plans. We looked at that one (floor plan on the left) before and found walks through the house.

G 9

S: The paths didn't go through a doorway more than once.

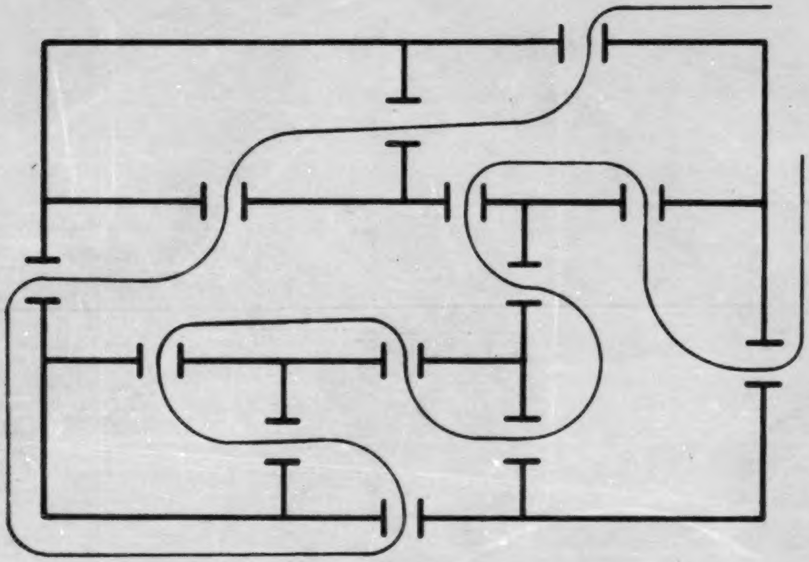
S: We tried to go through as many doorways as possible.

T: Do you remember what we call a walk that goes through each doorway exactly once?

S: A tour.

T: Who can trace a tour of this house (point to the floorplan on the left)?

As a student directs you, draw a path on the board. If the path misses a doorway, erase it and ask for another. One possibility is shown below.



T: Where did this walk start and end?

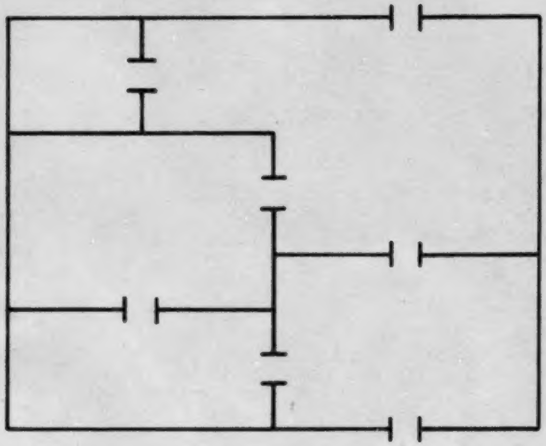
S: Outside (answer will vary with the path).

Connect the beginning and ending points of the path to give a continuous curve.

T: Could we find a tour which starts and ends somewhere else?

Your students should quickly see that the tour indicated on the board can be used to show that it is possible to start a tour in any room or outside, but that tour ends in the same place it starts.

Consider the floor plan on the right.



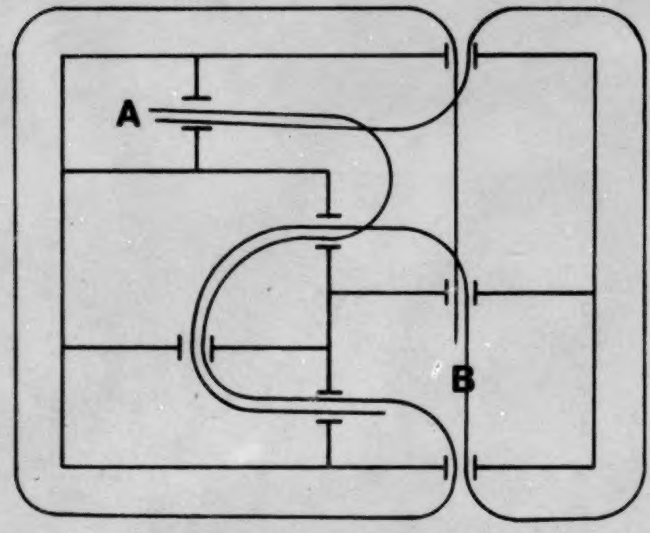
T: Who can trace a tour of this house?

S: Where should the tour start?

T: Anywhere you choose.

As a student directs you, draw a path on the board (transparency). If the path misses a doorway, erase it and ask for a walk which is a tour. Continue until two distinct tours have been found. Two of the many possibilities are shown in the next illustration.

T (labeling two rooms): These tours start either in room A or in room B and end in the other room.

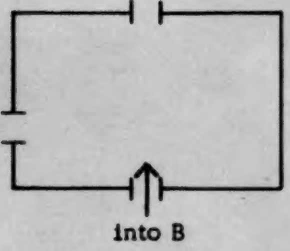


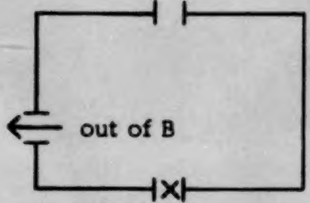
T:	Do you think we could find a tour starting in a different room?

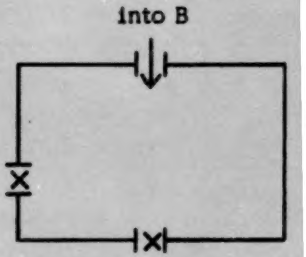
If necessary, let students experiment until someone discovers that it is impossible.

S:	You must start in A or in B to get a tour!
S:	A has only one door, so you must either start there or end there.
T:	If we start in A, why must we end the tour in B?
S:	There are three doors in B. You come into B and go out of B through different doorways; then all you can do is come back in through the third door and stay there.

If necessary, help a student who is on the right track to convince the class. The next illustration shows only room B.







Discuss also why a tour which ends in A must start in B. (If you did not start in B, then you would end in B and you cannot end in both A and B.)

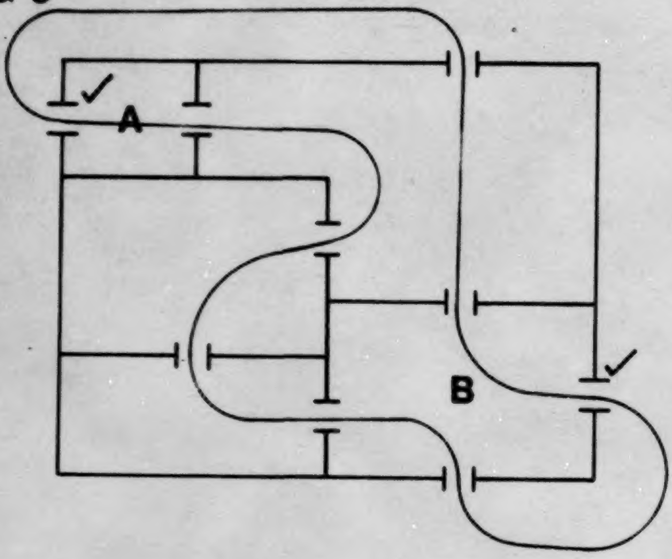
T: With the other floor plan we found a tour which started and ended in the same place and we used it to show that a tour could start and end anywhere in the picture. Do you think that we could build some doorways in this house so a tour could start anywhere?

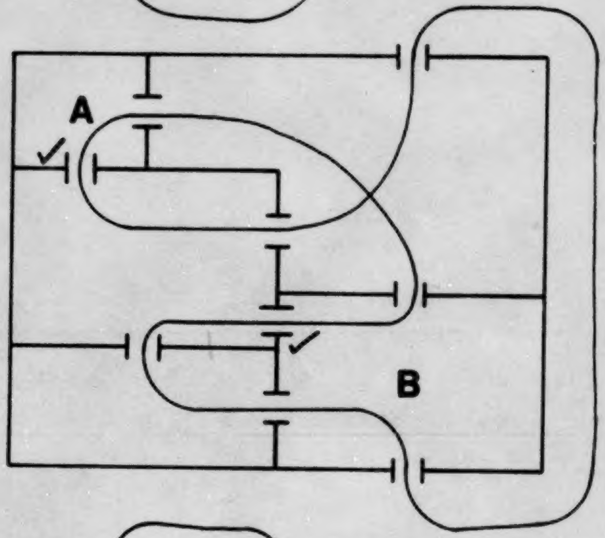
Erase the two tours your students suggested. Let students experiment by building doors and tracing possible tours until they are successful. Perhaps your students will focus on rooms A and B since, with the floor plan as it is, tours must start in one of those rooms and end in the other. There are many ways to add doors to accomplish the task at hand, but building exactly two doors is sufficient. (See the next illustration.)

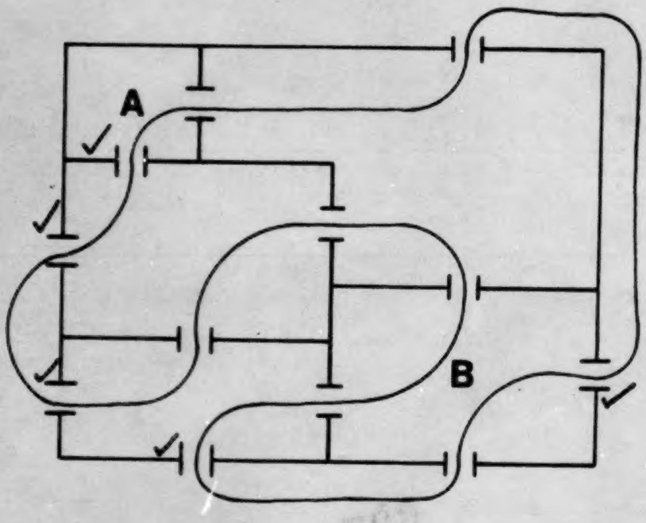
After your students have built extra doors and traced a tour starting and ending in the same place, draw that tour and use it to argue that now a tour can start and end anywhere in the picture so long as it starts and ends in the same place.

The next illustration shows several ways to add doors so that tours can start and end in the same place and anywhere in the picture. Check marks (√) distinguis the extra doors from the doors in the original floor plan.

G 9



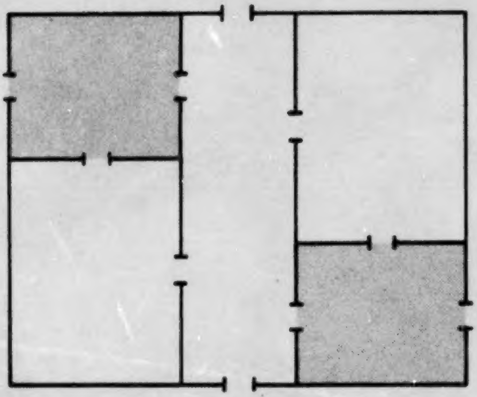




Distribute copies of Worksheets G9* and **. Let the students work on these problems individually before going over them collectively. Students who finish quickly should be given Worksheets G9*** and ****.

The floor plans on Worksheets G9* and ** are identical but the tasks vary. All four problems ask for walks using as many doors as possible, but in three of them the starting place and the ending place change. Walks using as many doors as possible may or may not be tours depending on the starting and ending places.

- G9* (front) — It is possible to find a path that goes through all of the doorways, i.e. to find a tour from S to E.
- G9* (back) — It is possible to find a path that goes through eight of the ten doorways, but not more than eight. A tour is not possible from S to E.
- G9** (front) — It is possible to find a path that goes through eight of the ten doorways, but not more than eight. A tour is not possible from S to E.
- G9** (back) — To go through all of the doorways, the path must start in one of the rooms shaded gray in the next illustration and end in the other.



As you did with the floor plan on the board earlier in the lesson, discuss why a tour must start in one of two rooms (see the preceding illustration) and end in the other. The students should be becoming familiar with the type of argument:

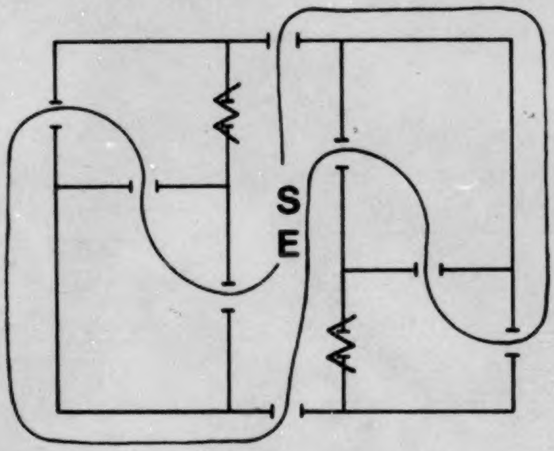
G9

- Consider one of the two rooms. Since it has exactly three doors, you can come in, go back out, and come in, but then you must stay there and so end the path there. You can also start inside the room and go out, come back in, and go out.

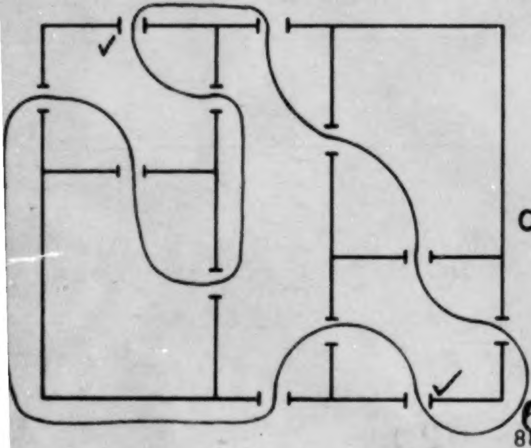
The other room has exactly three doors also and so the situation is the same.

If there is time remaining, ask if the floor plan could be modified by building or blocking doors so that it would be possible to start a tour anywhere in the picture. Let your students experiment until they find a solution; it need not be the most efficient solution but nevertheless a solution!

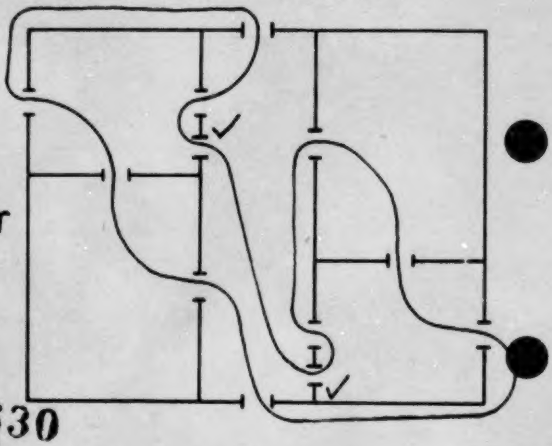
- Blocking doors — A student might suggest blocking the doors not used in his or her path on Worksheet G9**. For example,



- Building doors — there are many ways to build doors to accomplish the task at hand, but it is sufficient to build just two. For example,

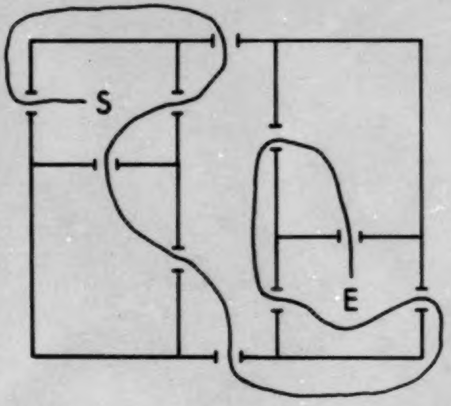


or



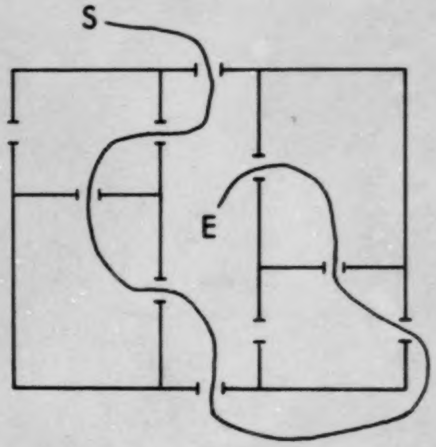
Name ______ G 9 *

Find a walk starting at S and ending at E. Use as many doors as you can, but do not use a door more than once.



Other solutions are possible.

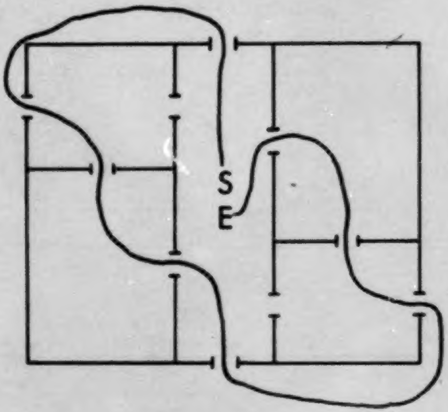
Find a walk starting at S and ending at E. Use as many doors as you can, but do not use a door more than once.



Other solutions are possible.

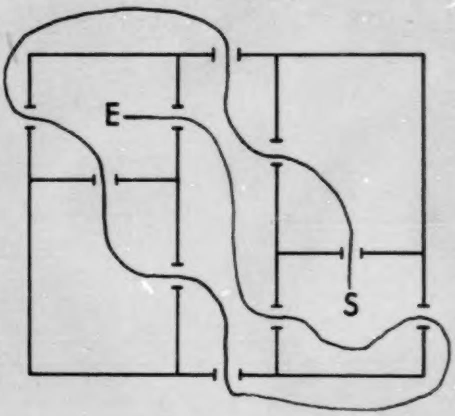
Name ______ G 9 **

Find a walk starting at S and ending at E. Use as many doors as you can, but do not use a door more than once.

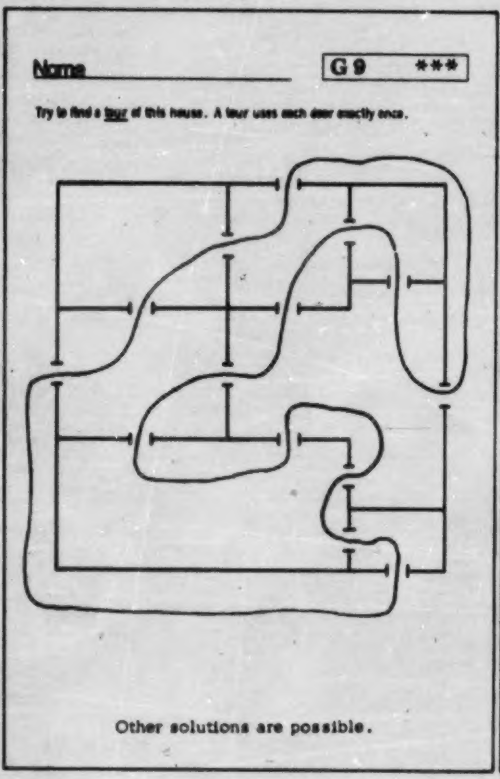


Other solutions are possible.

Find a walk using all the doors, but do not use a door more than once. Choose where to start and where to end yourself.



Other solutions are possible.

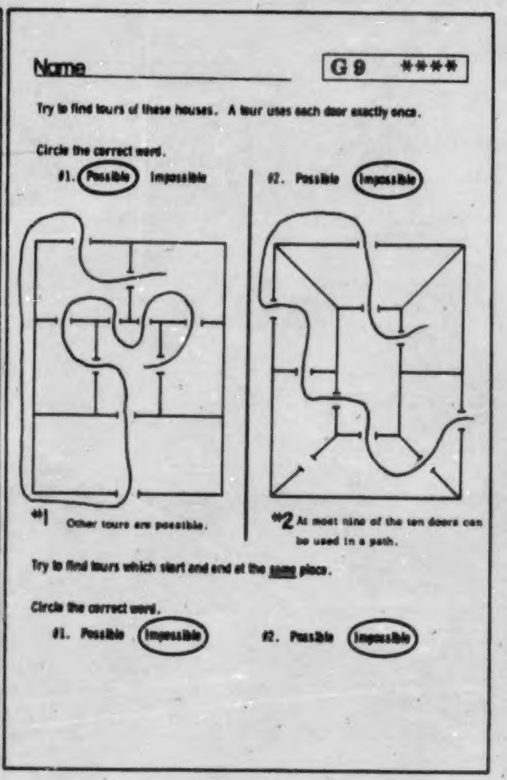


Name ______________________

G 9 ***

Try to find a tour of this house. A tour uses each door exactly once.

Other solutions are possible.



Name ______________________

G 9 ****

Try to find tours of these houses. A tour uses each door exactly once.

Circle the correct word.

#1. [✓ Possible] Impossible

#2. Possible [✓ Impossible]

**1 Other tours are possible.

**2 At most nine of the ten doors can be used in a path.

Try to find tours which start and end at the same place.

Circle the correct word.

#1. Possible [✓ Impossible]

#2. Possible [✓ Impossible]

G10 Rectangles of a Given Area

CAPSULE LESSON SUMMARY

Review the notion of a square corner and characterize rectangles as shapes with exactly four square corners and four straight sides. Find many rectangles with area 12 cm^2 and arrange them on a grid always with one corner being at a given point. Observe a pattern formed by the opposite corner points.

MATERIALS

Teacher: Geometry Posters #1, #2, and #3 for UPG-IV; tape; blue felt-tip marker; paper

Student: Paper; Worksheets G10* and **; colored pencils

NOTE: Before the lesson begins, cut out the rectangular regions outlined in blue on Geometry Poster #3. Cut along the underlying black borders for the regions even though the blue outlines may overlap them.

DESCRIPTION OF LESSON

T: What is a square corner? Where is there a square corner in this room?

Students may not be able to explain precisely what a square corner is, but most likely they will be able to point them out in the room.

S: The chalkboard has four square corners.

S: The window.

S: The ceiling tile.

S: The poster.

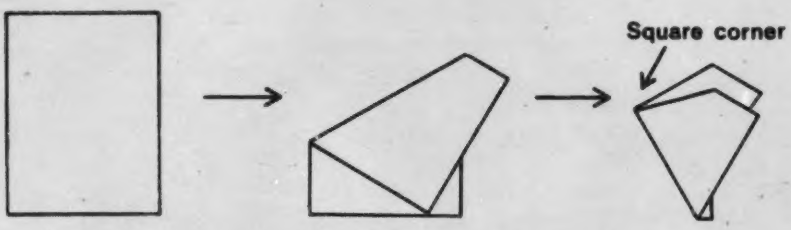
For each item suggested, ask the student to indicate specifically what they see as a square corner.

T: I'll show you how to make a square corner with a piece of paper.

G 10

ou and your students should each have a piece of paper. Demonstrate how to ake a square corner as in the next illustration.

: Fold the paper any way you want, . . . and then fold it again so that the folded edges line up.



OTE: It may impress your students if you use an rregular shaped piece of paper for your demonstration.

: We can use this paper model to help us check for square corners.

heck some of the corners in the room which were suggested as square corners.

: I am thinking of a shape which has four square corners and which has four straight edges. What shape am I thinking of?

: A rectangle.

: You could be thinking of a square.

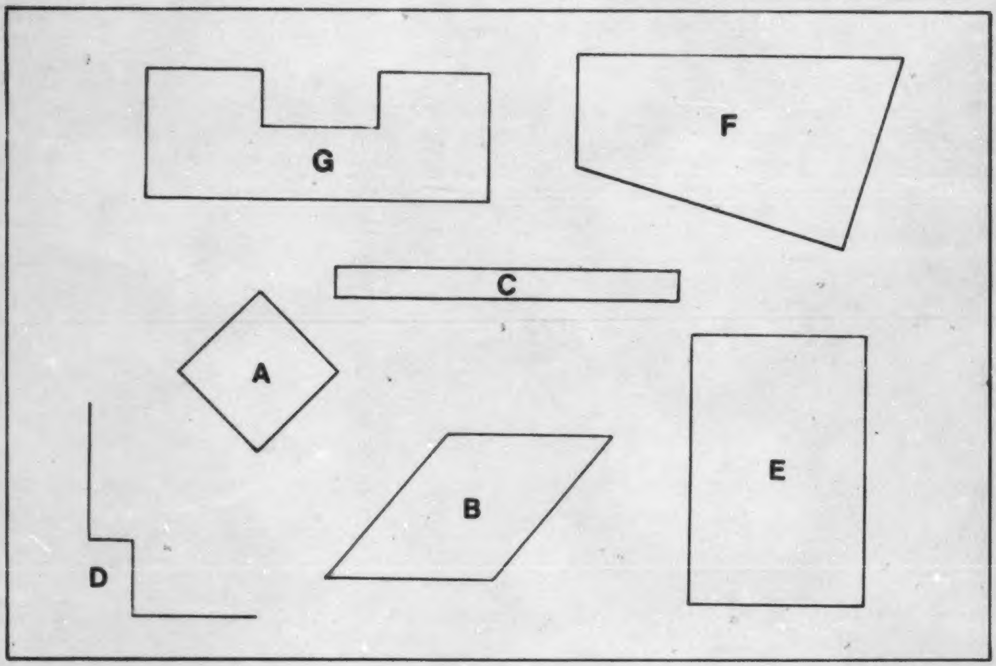
: Is a square a rectangle?

: Yes.

: A square is a special kind of rectangle in which all the sides have the same length.

Tape Geometry Poster #1 to the board.

Do not write the letters on the poster. They are here to facilitate the lesson description.



With the class check whether or not the shapes on the poster are rectangles by counting in each the number of edges and the number of square corners.

- A, C, and E are rectangles;
- B has no square corners;
- D has three square corners;
- F has two square corners;
- G has eight square corners.

Remove the poster and distribute copies of Worksheet G10*.

T (holding up a copy of Worksheet G10*): How long is each edge of one of these little squares?

S: 1 cm.

T: What is the area of one little square? Look at the blackened square.

S: 1 cm^2 (read: one square centimeter).

T: Color some rectangles each with an area of 12 cm^2. Use different colors for the different rectangles you find.

Write "12 cm^2" on the board.

12 cm^2

Show a few of the students' pictures to the class early during the individual work to help those students who are having trouble getting started. Allow about five minutes before calling the class's attention back to the board.

Hold up the cut-out which corresponds to one of your students' rectangles; for example,

T: This is a big copy of one of Elizabeth's rectangles.
What is the length of a short side? (1 cm)
What is the length of a long side? (12 cm)
I'll record the size of Elizabeth's rectangle like this.

1 cm by 12 cm

T: What number story could we write about this rectangle?

S: "1 × 12 = 12".

Record the number story on the board.

1 cm by 12 cm

$1 \times 12 = 12$

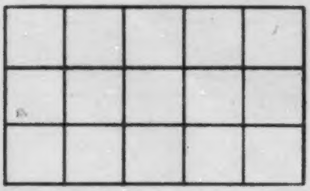
T: Who has a different rectangle?

Look at a volunteer's paper.

T: One side of William's rectangle is 3 cm. How long are the other sides?

S: Two of them are 4 cm each and there is another side 3 cm long.

Hold up the cut-out which corresponds to William's rectangle as you ask:



T: What number story could we write about this rectangle?

S: "3 x 4 = 12".

Add this information to the list forming on the board.

1 cm by 12 cm

$1 \times 12 = 12$

3 cm by 4 cm

$3 \times 4 = 12$

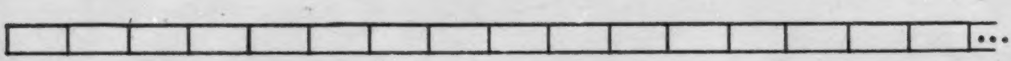
G 10

In the same manner continue describing and discussing rectangles of area 12 cm^2. Very likely your students will offer examples of all the rectangles listed below.

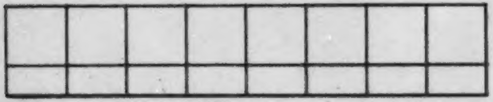
1 cm by 12 cm $1 \times 12 = 12$	3 cm by 4 cm $3 \times 4 = 12$	2 cm by 6 cm $2 \times 6 = 12$
12 cm by 1 cm $12 \times 1 = 12$	4 cm by 3 cm $4 \times 3 = 12$	6 cm by 2 cm $6 \times 2 = 12$

T: Did you find any other rectangles with area 12 cm^2?

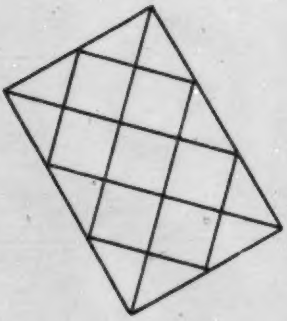
Perhaps students will suggest some of these rectangles.

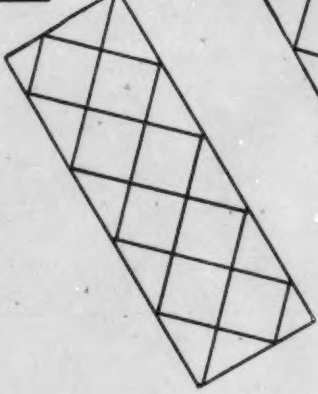


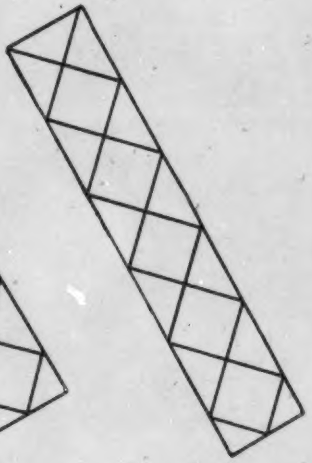
$\frac{1}{2}$ cm by 24 cm



$1\frac{1}{2}$ cm by 8 cm







NOTE: The dimensions of three of the rectangles are not given because they involve irrational numbers, but it is easy to decide their areas by counting squares, half-squares, and fourth-squares.

---

* The full picture of this rectangle is not given because of size limitations on the page. Also, a cut-out is not provided, although cut-outs are provided for the other four rectangles.

Tape Geometry Poster #2 to the board and distribute copies of Worksheet G10**.

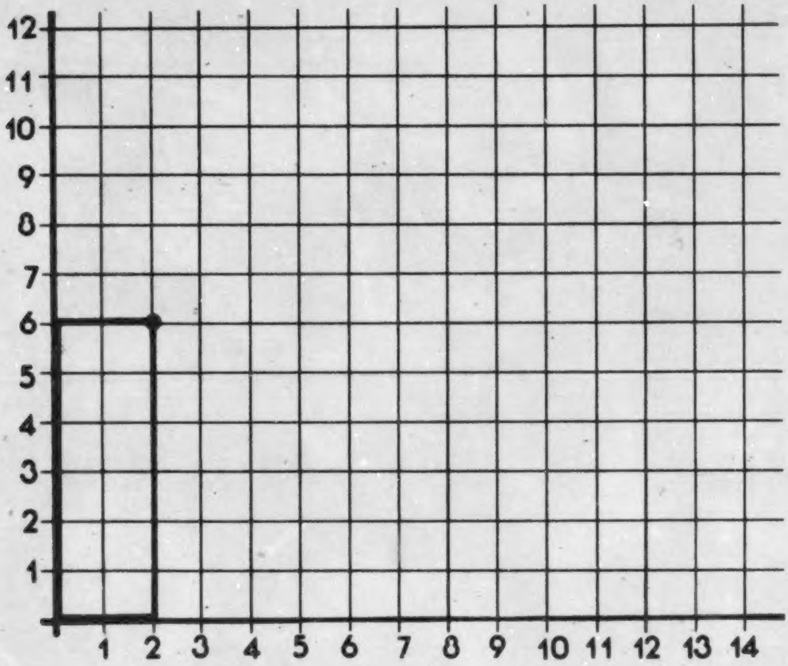
T: Let's show these rectangles in a picture. What we do on the board you should do on your worksheet as well.

One of the rectangles outlined in blue on your worksheet is 6 cm by 2 cm.

Hold up the cut-out of the same measurements.

T: We lay the rectangle on the grid and line up the sides like this (as in the next illustration), and then we trace it.

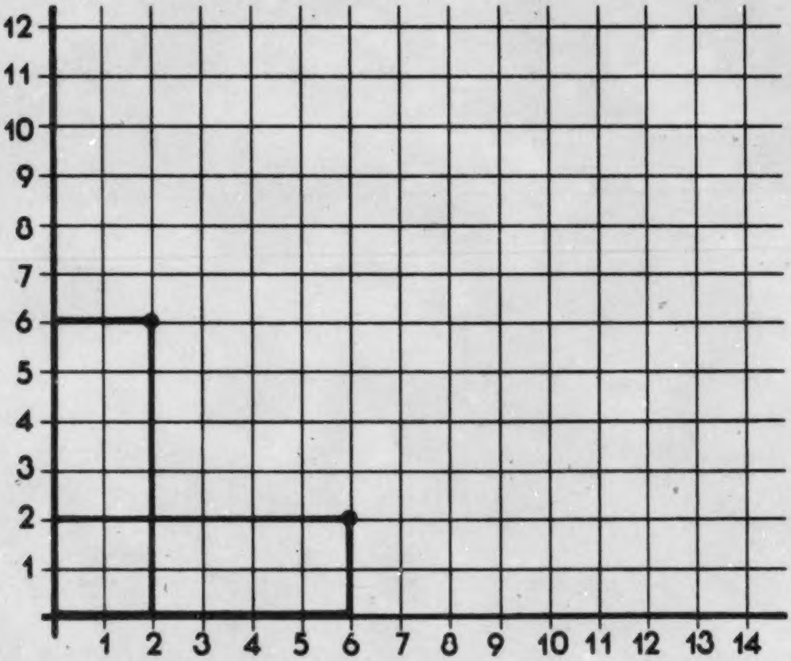
Trace the rectangle in blue. Take the cut-out off the grid and then draw a large blue dot at the upper right corner of the rectangle.



T: If we turn the rectangle, we can get a 2 cm by 6 cm rectangle.

G 10

Turn the rectangle and trace around it as in the next illustration. Remove the cut-out and draw a large dot at the upper right corner of the rectangle.

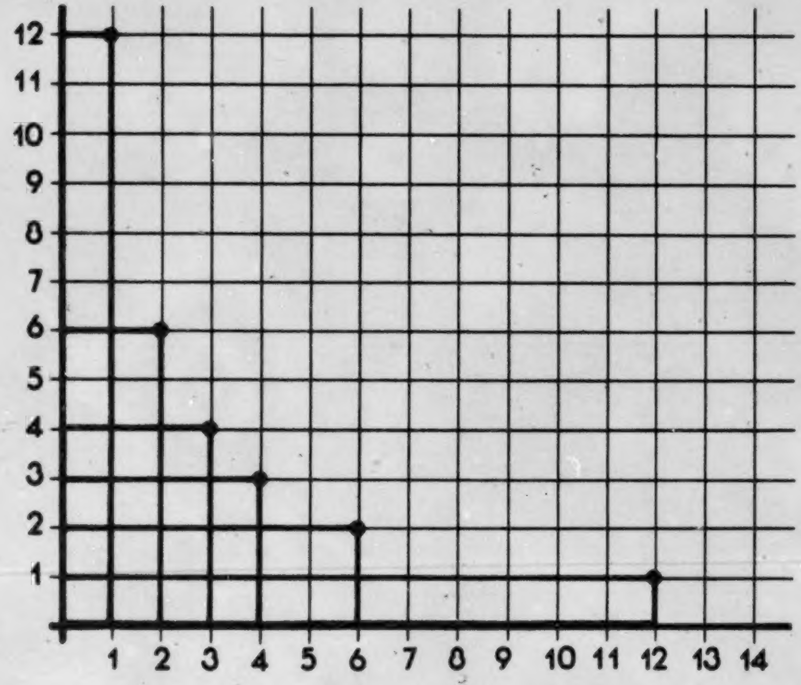

12
11
10
9
8
7
6
5
4
3
2
1
1 2 3 4 5 6 7 8 9 10 11 12 13 14

Circle "2 cm by 6 cm" and "6 cm by 2 cm" in the list on the board.

1 cm by 12 cm 1 × 12 = 12	12 cm by 1 cm 12 × 1 = 12
3 cm by 4 cm 3 × 4 = 12	4 cm by 3 cm 4 × 3 = 12
[✓ 2 cm by 6 cm 2 × 6 = 12]	[✓ 6 cm by 2 cm 6 × 2 = 12]

Ask the students to draw on their worksheets blue dots for other rectangles of area 12 cm^2. Help students who have difficulty getting started by referring then to the list on the board and asking them to draw the blue dots for those rectangles. As students are working individually invite some of them to draw rectangles and their corresponding blue dots on the poster. Give them the appropriate cut-outs to locate the dots. Students should convince the class that each suggested rectangle has area 12 cm^2.

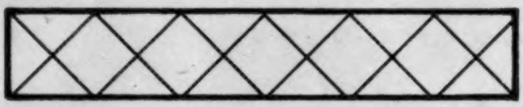
If necessary, prompt students to find rectangles other than the six with whole number dimensions (the six in the list) as in the dialogue which follows.



T: We have drawn the blue dots for the six rectangles in our list. Look at this picture (touch each blue dot) and think about where other blue dots might be. See if you can find other rectangles with area 12 cm^2.

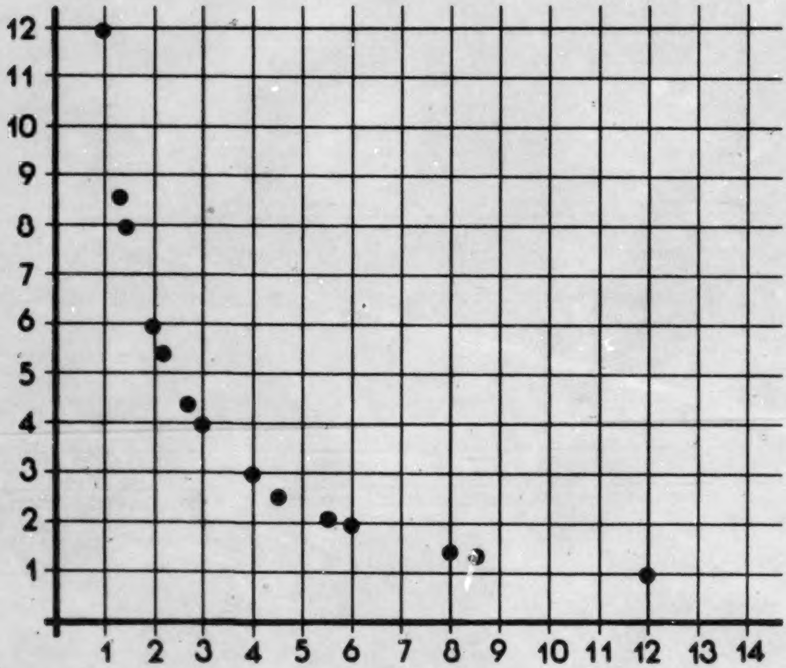
G 10

You may need to show the class a rectangle (cut-out) which involves half-squares; for example,

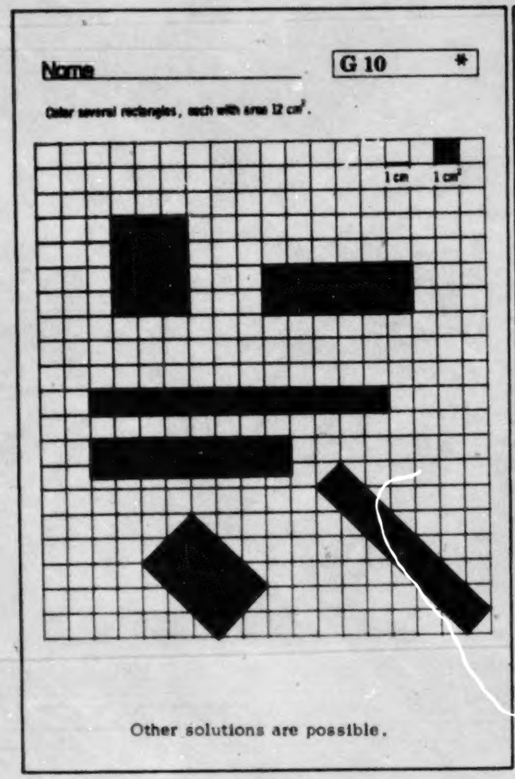


Let the class count the little squares to see that the area is 12 cm². On the graph plot the two corresponding blue dots for that rectangle. The new rectangle may lead some students to discover other rectangles of area 12 cm² (see those on page 96), but you need not insist that your class find more.

If all of the rectangles which were provided as cut-outs are found, the large blue dots on your graph will suggest a smooth (hyperbolic) curve.



Conclude the lesson by letting the students discuss the picture. If there is sufficient time, discuss whether or not rectangles of area 12 cm² all have borders of the same length, i.e. all have the same perimeter.

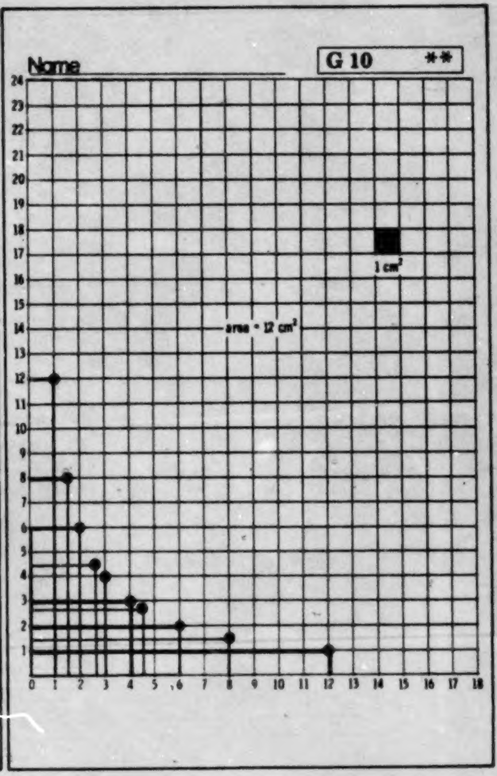


Name

G 10 *

Color several rectangles, each with area 12 cm².

Other solutions are possible.



Name

G 10 **

G 11

G11 Rectangles of a Given Perimeter

CAPSULE LESSON SUMMARY

Discuss the idea of perimeter of a rectangle and note that the perimeter is twice the sum of the lengths of two adjacent sides. Find many rectangles with perimeter 20 cm and arrange them on a grid always with one corner being at a given point. Observe a pattern formed by the opposite corner points.

MATERIALS

Teacher: Metric tape measure; colored chalk; Geometry Posters #4, #5, and #6 for UPG-IV; tape; red felt-tip marker

Student: Paper; metric tape measure; Worksheets G11* and **; red pencil

NOTE: Before the lesson begins, cut out the rectangular regions outlined in red from Geometry Posters #5 and #6. Cut along the underlying black borders for the regions even though the red outlines may overlap them.

DESCRIPTION OF LESSON

Exercise 1

T: What do you know about the rectangles?

S: They each have four straight sides and four square corners.

Ask that each student take out a sheet of paper. Distribute tape measures.

T: The length of the border of a shape is called its perimeter.

Write "perimeter" on the board.

T: Find the perimeter, the length of the border, of your piece of paper. Measure in centimeters. Next to each edge write its length.

perimeter **cm**

Students should measure to the nearest centimeter. After about five minutes discuss the results. A sample dialogue follows.

T: Patrick, what is the perimeter of your piece of paper?

S: 98 cm.

T: How did you find the perimeter?

S: I measured two of the sides and got 28 cm and 21 cm.
I multiplied "2 x 28" and "2 x 21" and then added the answers.

With the student assisting, record the calculations on the board.

$$\begin{array}{r} 2 \times 28 = 56 \\ 2 \times 21 = \underline{42} \\ 98 \end{array}$$

T: Nicole, how did you find the perimeter of your paper?

S: I measured two sides together and got 50 cm. Then I multiplied "2 x 50".

$$\begin{array}{rl} 2 \times 28 = 56 & \quad 2 \times 50 = 100 \\ 2 \times 21 = \underline{42} & \\ 98 & \end{array}$$

Choose a fairly large rectangular object in the room to measure. The following dialogue assumes a table top is being measured.

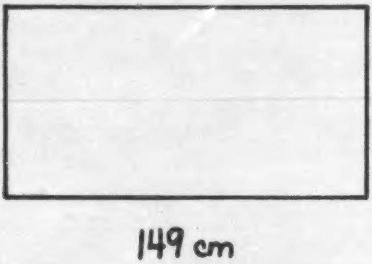
T: I would like two of you to help me find the perimeter of the top of this table.

G 11

S (measuring one side of the table): 149 cm.

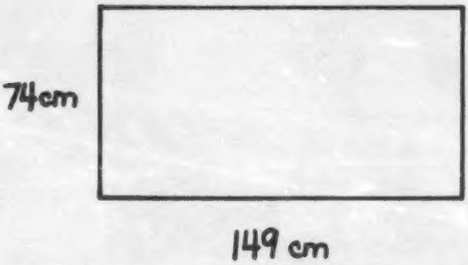
T: So a long side has length 149 cm.

Show this information in a diagram on the board.

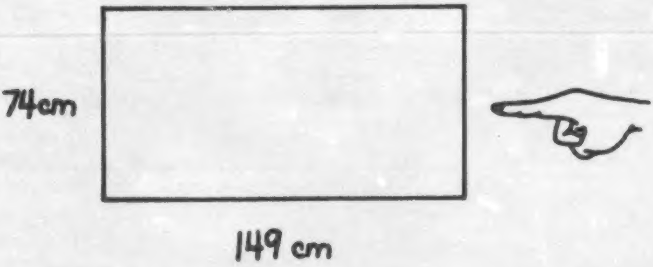


S (measuring another side of the table): 74 cm.

T: So a short side has length 74 cm.

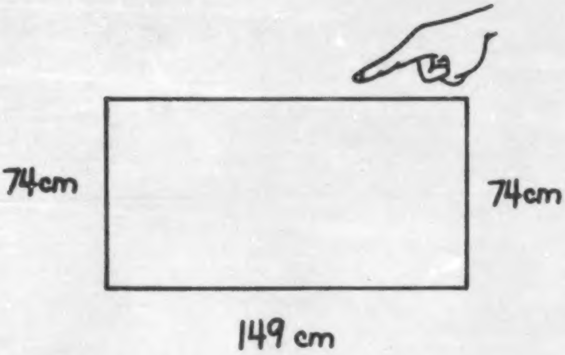


T: What is the length of this side?



S: 74 cm.

T: . . . and this side?



S: 149 cm.

T: How can we find the perimeter of the table top?

Follow the suggestions of your students; one method is described in the sample dialogue below.

S: Calculate "2 x 149" and "2 x 74".

Record these problems on the board.

$$2 \times 149 =$$
$$2 \times 74 =$$

S: 2 x 74 = 148.

$$2 \times 149 =$$
$$2 \times 74 = 148$$

T (pointing to "2 x 149"): What number is 2 x 150?

S: 300.

T: So what is 2 x 149?

S: 298.

$$\begin{aligned} 2 \times 149 &= 298 \\ 2 \times 74 &= 148 \end{aligned}$$

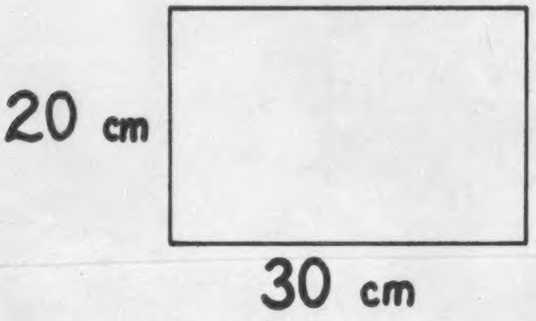
Ask for a volunteer to do the calculation "298 + 148" at the board.

$$\begin{array}{rcr} 2 \times 149 &=& 298 \\ 2 \times 74 &=& \underline{148} \\ && 446 \end{array}$$

Exercise 2

Erase the board and draw a rectangle.

T: I am thinking of a rectangle. A long side has length 30 cm; a short side has length 20 cm. What is the perimeter of the rectangle?



S: 100, because 20 + 20 = 40, 30 + 30 = 60, and 40 + 60 = 100.

S: 100, because 20 + 30 = 50 and 2 x 50 = 100.

Repeat this activity for rectangles with the following or similar dimensions. With each situation let students describe their methods of calculating the perimeter.

- A long side has length 34 cm and a short side has length 15 cm. (Perimeter is 98 cm)
- A long side has length 112 cm and a short side has length 51 cm. (Perimeter is 326 cm)

Exercise 3

If you have written dimensions near the sides of the rectangle on the board, erase them.

T: I'm thinking of a rectangle which has perimeter 20 cm. What could be the lengths of its sides?

Accept a couple of suggestions; for example,

S: The sides could all be 5 cm long.

T: Then it would be a special kind of rectangle — it would be a square.

Label each of the sides of the rectangle "5 cm".

T: Is the perimeter 20 cm?

S: Yes.

Begin a list on the board.

5 cm by 5 cm

G 11

S: A long side could be 6 cm long and a short side could be 4 cm long.

Label the sides of the rectangle appropriately.

T: Is the perimeter 20 cm?

S: Yes; 2 x 6 = 12, 2 x 4 = 8, and 12 + 8 = 20.

S: Yes; a long side and a short side together have a length of 10 cm and 2 x 10 = 20.

Distribute copies of Worksheet G11*.

T (holding up a copy of Worksheet G11*): On this worksheet I want you to draw some rectangles each with perimeter 20 cm.

As you observe students' work, announce to the class the size of various rectangles being found. Each time record the dimensions on the board and check that the perimeter is 20 cm. This should help students who have difficulty getting started. One of our development classes found the rectangles with the dimensions listed here.

5 cm by 5 cm
4 cm by 6 cm
7 cm by 3 cm
8 cm by 2 cm
1 cm by 9 cm
½ cm by 9½ cm
1½ cm by 8½ cm
2½ cm by 7½ cm

Tape Geometry Poster #4 to the board and distribute copies of Worksheet G11**.

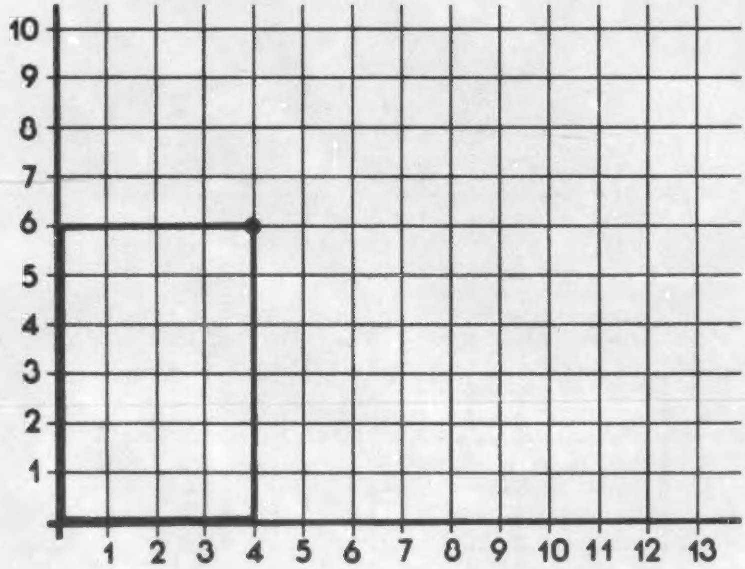
T: Let's show these rectangles in a picture. What we do on the board you should do on your worksheet as well.

One of the rectangles outlined in red on your worksheet is 6 cm by 4 cm.

Hold up the cut-out of the same dimensions.

T: We lay the rectangle on the grid and line up the sides like this (as in the next illustration) and then we trace it.

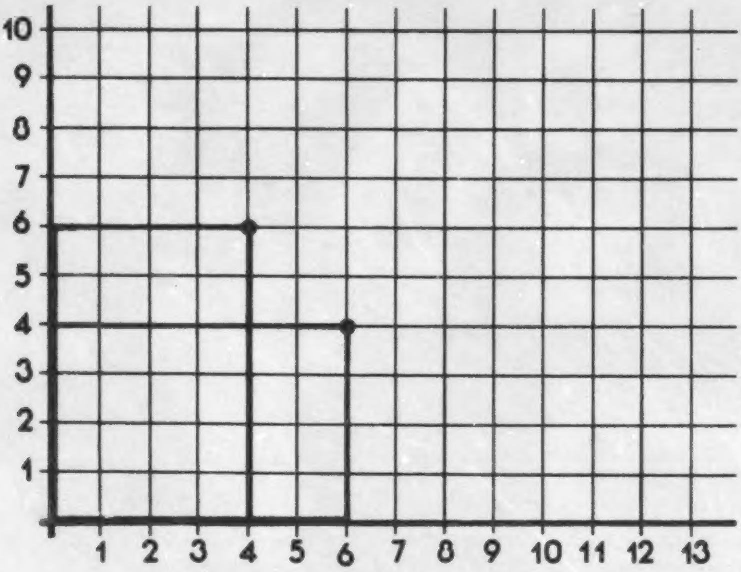
Trace the rectangle in red. Take the cut-out off the grid and then draw a large red dot at the upper right corner of the rectangle.



T: If we turn the rectangle, we can get a 4 cm by 6 cm rectangle.

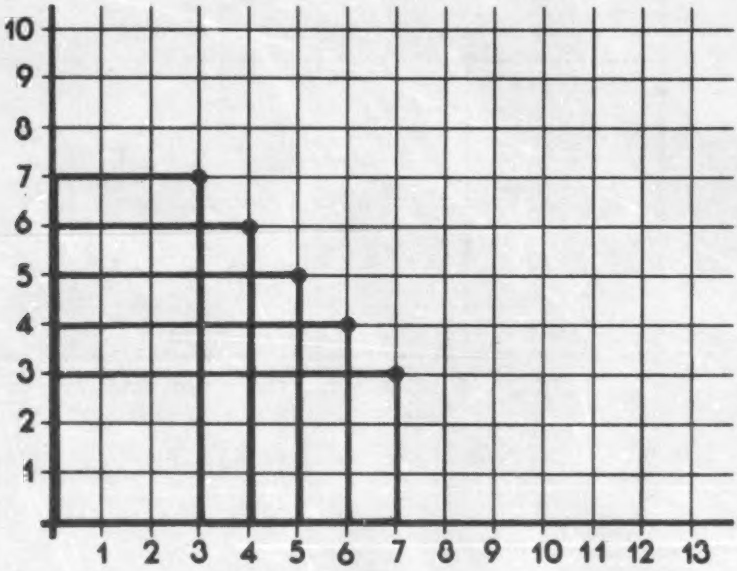
G 11

Turn the cut-out and trace around it as in the next illustration. Remove the cut-out and draw a large dot at the upper right corner of the rectangle.

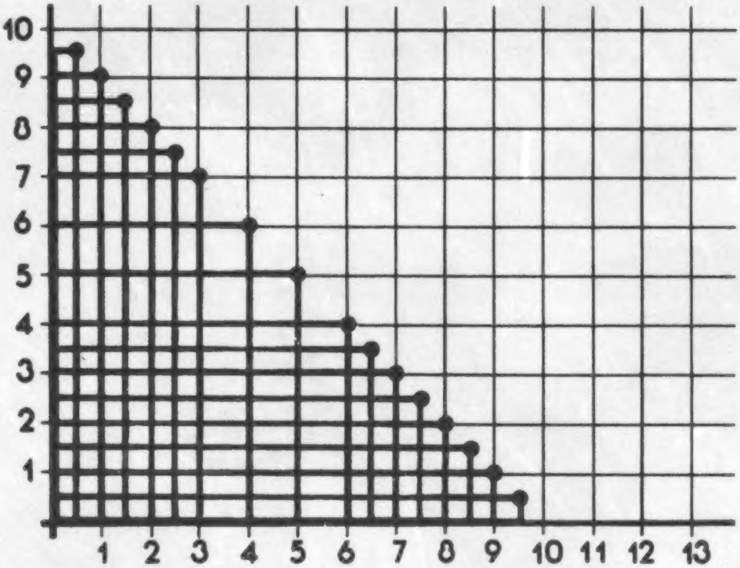

10
9
8
7
6
5
4
3
2
1
1 2 3 4 5 6 7 8 9 10 11 12 13

Circle "4 cm by 6 cm" (or "6 cm by 4 cm") in the list on the board.

Repeat this activity with two other rectangles described in your list. If you choose the 5 cm by 5 cm square and the 7 cm by 3 cm (or 3 cm by 7 cm) rectangle, your picture would look like this.


10
9
8
7
6
5
4
3
2
1
1 2 3 4 5 6 7 8 9 10 11 12 13

Ask the students to draw on their worksheets red dots for all the rectangles described in your list. As the students are working independently, invite some of them to draw rectangles and their corresponding red dots on the poster. The cut-outs should be available for their use. The next illustration shows what the picture would look like for a sample (partial) list.



T: Does the picture help us to find other rectangles with a perimeter of 20 cm?

S: Yes; $3\frac{1}{2}$ cm by $6\frac{1}{2}$ cm.

S: $4\frac{1}{2}$ cm by $5\frac{1}{2}$ cm.

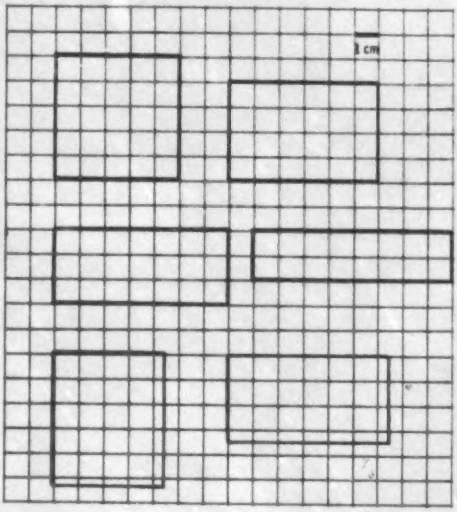
Students might suggest other points in line with the red dots (other than half-way points) which could be corners of such rectangles. Take suggestions as they are offered and draw the corresponding rectangles and red dots.

T: Do you see anything interesting about these dots (run your finger along the red dots)?

Let students describe what they see. If no one notices that the red dots lie in a straight line, point this out yourself.

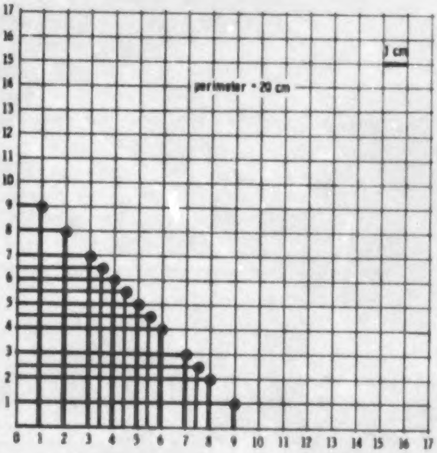
Name ______________________ **G 11 ***

Draw several rectangles each with perimeter 20 cm.



Other solutions are possible.

Name ______________________ **G 11 ****



G12 Touring Houses #3

CAPSULE LESSON SUMMARY

Review the idea of a tour of a house and decide whether or not certain houses have tours. Modify houses by building doors so that there are tours or so that there are special kinds of tours. Relate the idea of a tour for houses to the idea of hiking trails by associating rooms of houses with campsites and doors with paths between campsites.

MATERIALS

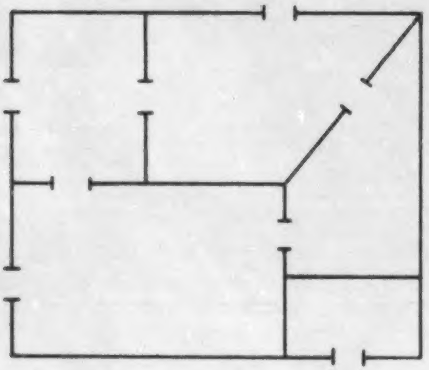
Teacher:	Colored chalk; (optional) overhead projector; Geometry transparency #3
Student:	Worksheets G12*, **, ***, and ****; tracing paper; colored pencils

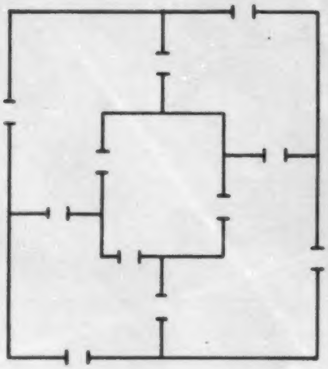
NOTE: A transparency is provided for this lesson. If you have access to an overhead projector, use the transparency to project on the board the house plan discussed in Exercise 2 of the lesson. Exercise 2 will proceed more smoothly in this manner.

DESCRIPTION OF LESSON

Exercise 1

Draw these pictures on the board.





T: What kind of pictures are these?

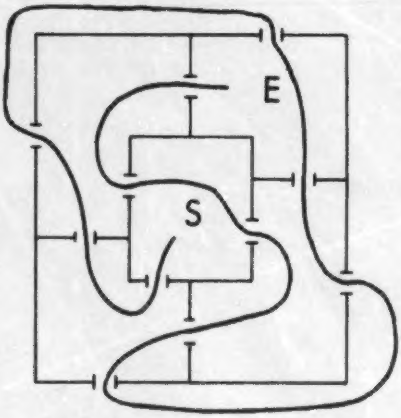
S: They are floor plans for houses.

Let the class recall situations from previous lessons on touring houses. Be sure to mention that the walks through the house may go through a door just once

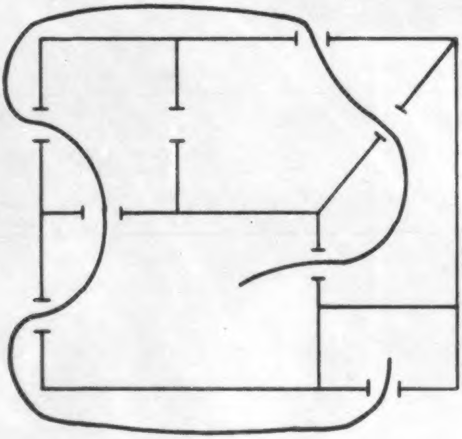
G 12

Review the idea that a tour is a path that goes through each door exactly once. Perhaps your students will volunteer the information that sometimes there is a tour of a house that starts and ends at the same place, sometimes there is a tour that starts in one place and ends in another, and sometimes there is no tour at all. If not, mention this fact yourself and ask the class to predict (before making any trials) whether or not these houses have tours.

S: This house (on the right) has a tour starting in the middle room and ending here (upper right room).



S: This house (on the left) does not have a tour. For a tour we must use all the doors and that includes the door of the room with just one door. We could start (end) there, but then we would end (start) in a room with three doors. There are several rooms with three doors and we can't end (start) in them all. So some of these doors would not be used.

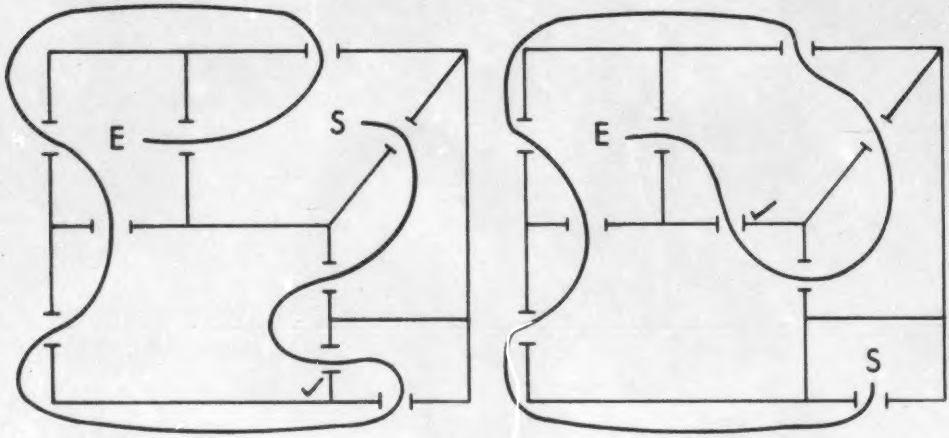


NOTE: Allowing the outside to be called a "room," a house has a tour if and only if each room has an even number of doors (tours in this case always start and end at the same place) or there are exactly two rooms, with odd numbers of doors (tours in this case start in one of the rooms with an odd number of doors and end in the other).

The discussion should review the roles played by rooms with odd numbers of doors and by rooms with even numbers of doors. Do not expect well-formed explanations; rephrase them as necessary for class understanding.

T (pointing to the house plan on the left): There is no way to tour this house. Could we build a door somewhere so that there would be a tour of the house?

Your class might suggest building several doors. Allow this possibility as long as it leads to a tour of the house. The most efficient solution is to build a door between two rooms with odd numbers of doors. For example,



(A check marks the new door.)

Exercise 2

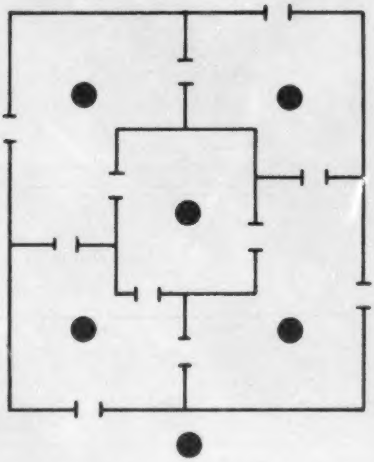
If you are using the overhead projector, erase the board and project Transparency #3 on the board. Otherwise, erase only the house plan on the left.

NOTE: This description assumes an overhead projector is used. You may need to adjust your actions if you do not have access to an overhead projector. For example, when we say "turn off the projector," you erase from the board the outline of the house plan.

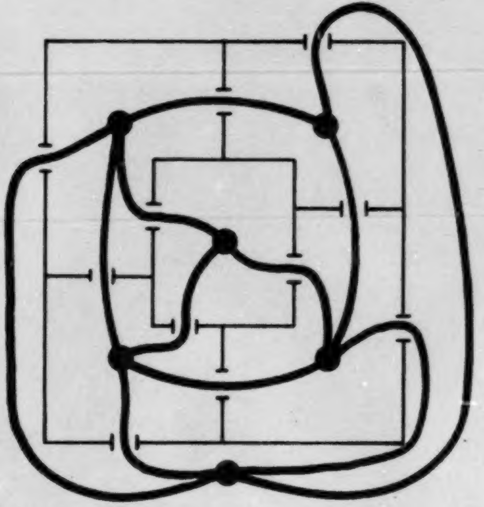
T: You predicted that this house has a tour. Who can show us a tour?

Let one or two students trace tours. On the transparency they could draw the paths and then you could easily erase them without spoiling the house plan. Observe that any tour either starts in the middle room and then ends in the upper right room, or vice versa.

T: Now, I'm going to draw a large dot in each room of the house and one dot outside.

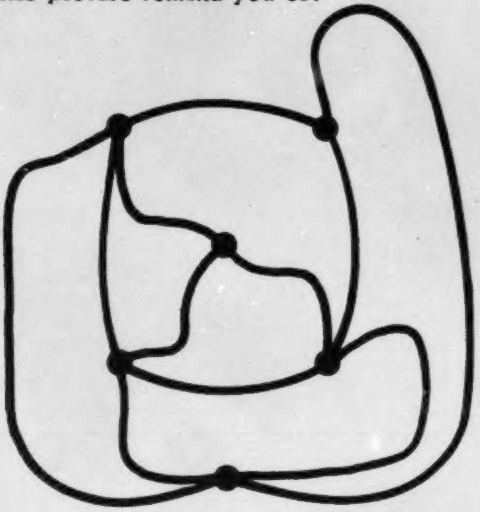


T: Then I'll connect the dots with a cord if there is a door between the rooms or between a room and the outside.



Turn off the overhead projector.

T: What does this picture remind you of?



S: Campsites and hiking trails.

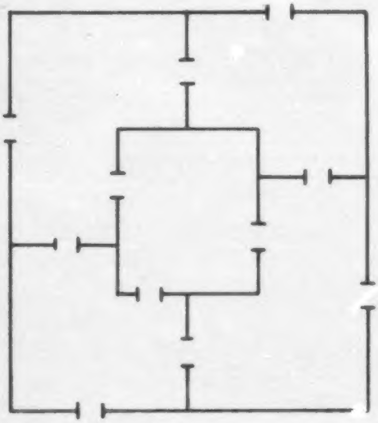
If necessary, recall the maps from the Hiking Trails lessons (UPG-III G12 and G13). Observe that the rooms of the house correspond to campsites and the doors correspond to trails between campsites.

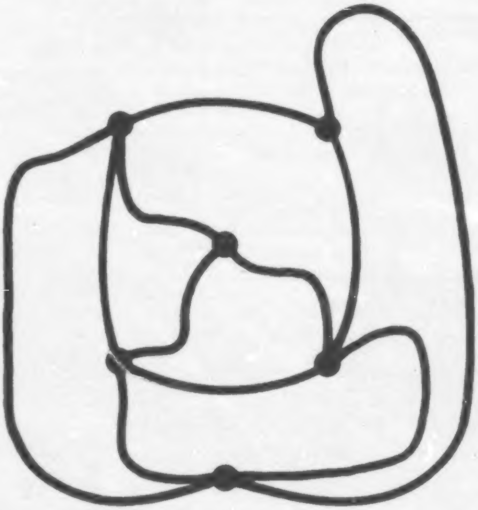
G 12

T: If this were the map of a campground, could we find a hiking trail which uses every path exactly once?

Let a student trace such a trail checking (√) each path as it is used so that no path is used more than once and so that all the paths are used. Observe where a complete hiking trail starts and ends.

Move the overhead projector so that you can project the house plan next to the campground map.





T: Let's see if we can follow a tour of the house with a hiking trail on the campground map.

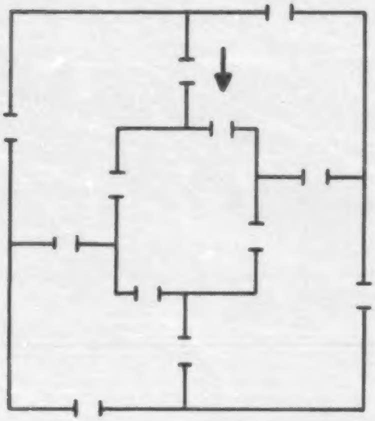
Call on two students, one to draw a tour on the transparency and one to trace a hiking trail. Ask the student who is drawing a tour of the house to pause after going through each room so that the student tracing a hiking trail can trace along the corresponding path.

Then erase the tour on the the transparency. Call on two students, one to trace a hiking trail and the other to draw a tour. Ask the student at the board to trace a hiking trail very slowly, pausing after each path so that the student at the projector can draw the part of the tour going through the corresponding door.

T: We can follow a tour of the house with a hiking trail on the campground map and we can follow a hiking trail on the campground map with a tour of the house.

Look at the house plan. Each time a tour started (ended) in the middle room and ended (started) in the upper right room. Could we build another door so that there would be a tour starting and ending at the same place?

There are many solutions to this problem if we allow building more than one door. Encourage the class to look for a solution where only one new door is built. Perhaps a student will observe that all one needs to do is to build a door between the middle room and the upper right room.

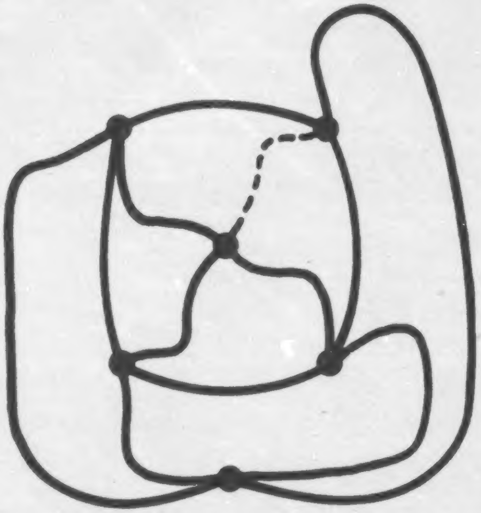


As a result each room, including the outside, has an even number of doors.

T: What addition should we make to the campground map so that it is the map corresponding to this house with an extra door?

G 12

Draw a path between the middle campsite and the one here (upper right).



T: What kind of hiking trail do we get when we follow a tour of the house now?

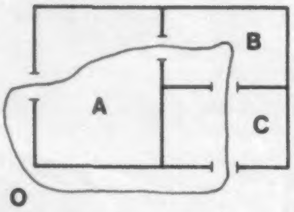
S: A round-trip trail — a trail that starts and ends at the same campsite.

You may need to actually demonstrate this fact.

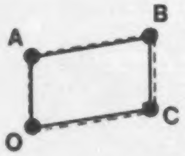
Worksheets G12*, **, ***, and **** are available for individual work. Supply tracing paper for students to draw their maps on and suggest that they show the corresponding hiking trails in color.

Name ____________________ **G 12 ***

Find a tour of this house. You may start and end where you like.

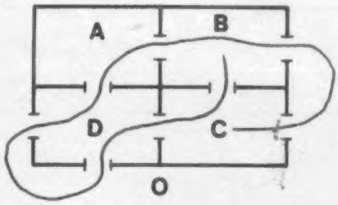


On tracing paper, draw a map of this house. Show a hiking trail corresponding to your tour.

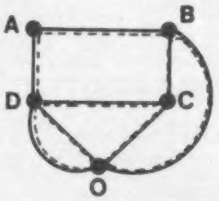


Name ____________________ **G 12 ****

Find a tour of this house. Choose where to start and where to end yourself.

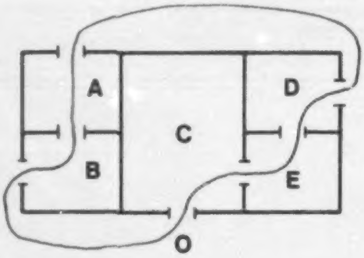


On tracing paper, draw a map of this house. Show a hiking trail corresponding to your tour.

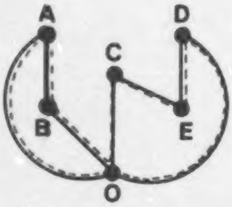


Name ____________________ **G 12 *****

Find a tour of this house. You may start and end where you like.

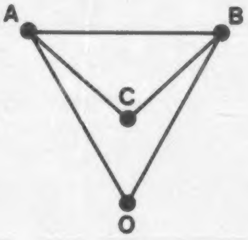


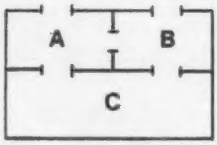
On tracing paper, draw a map of this house. Show a hiking trail corresponding to your tour.



Name ____________________ **G 12 ******

Draw a house plan for this map.





G 13

G13 Pencil Tracings

CAPSULE LESSON SUMMARY

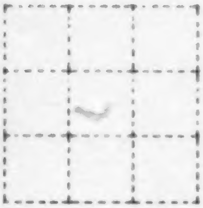
Play a game on rectangular arrays of small squares. The rules of the game are to start at one corner of a little square and make a tracing without going over any edge more than once and without lifting the pencil (chalk). The game score is the number of little squares that are surrounded by the tracing. Find the largest possible scores for various arrays and observe some patterns.

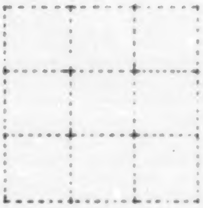
MATERIALS

Teacher: Colored chalk
Student: Worksheets G13*, **, ***, and ****; colored pencils

DESCRIPTION OF LESSON

Draw several three by three grids on the board using dotted lines.





T (tracing one of the large squares): What shape are these?

S: Squares.

T: A square is a special kind of . . . ?

S: Rectangle.

T: How many little squares are each of these squares divided into?

S: Nine.

T: How did you get nine?

S: I added "3 + 3 + 3".

S: I multiplied "3 x 3".

S: I counted them.

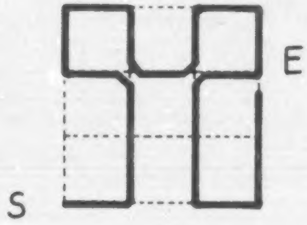
T: There are four large squares, so how many little squares are there altogether?

S: Thirty-six, because 4 x 9 = 36.

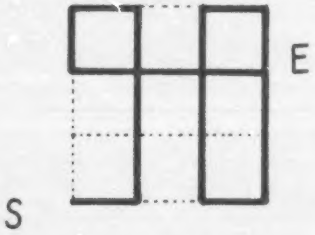
S: Thirty-six, because 9 + 9 = 18 and 18 + 18 = 36.

T: I'm going to start at a corner of one of the little squares and make a tracing. I'll try to surround as many squares as possible. The rules are that I can't go over any edge more than once and that I can't lift the chalk off the board until I'm finished.

Starting from a corner, make a tracing such as the following. Label your starting point "S" and your ending point "E".

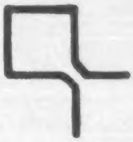


NOTE: The preceding illustration indicates how the tracing is executed. Your drawing would look more like this.

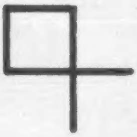


G 13

Throughout this lesson description we have used the technique of drawing



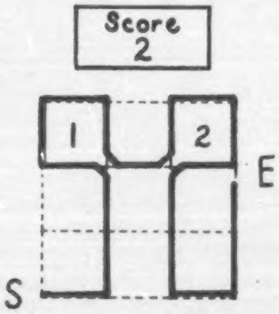
instead of drawing



to make the illustrations easier to follow.

T: How many squares did I surround?

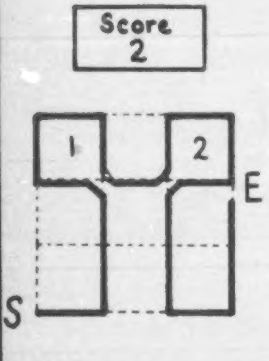
Invite a student to number the surrounded squares; indicate the total number as in the next illustration.

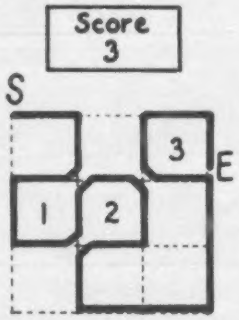

Score
2
1
2
E
S

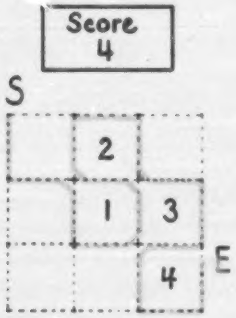
Distribute copies of Worksheet G13* as you say:

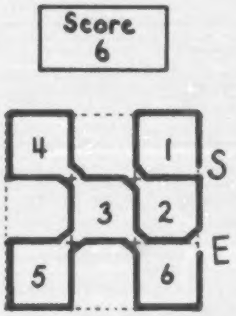
T: My score for this tracing is 2. Make your own tracings and see if you can get a higher score.

After about five minutes ask several students to put their tracings on the board. Be sure to include a student who has surrounded six squares (maximal score).

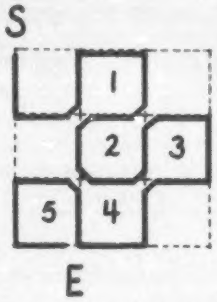

Score
2
1
2
E
S


Score
3
S
3
E
1
2

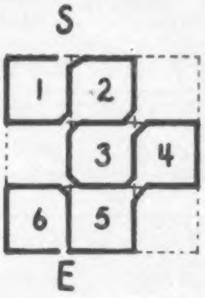

Score
4
S
2
1
3
4
E


Score
6
4
1
S
3
2
E
5
6

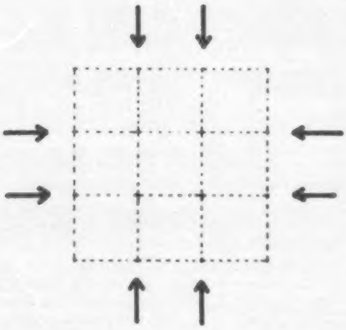
If the best score your class finds is 5, it is possible that such a tracing could be slightly modified to surround six squares. For example,



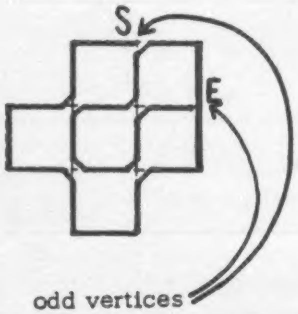
could be changed to

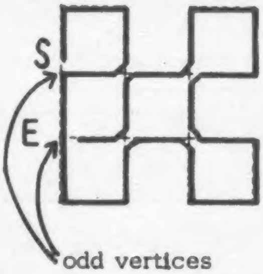


NOTE: One approach to this problem is to consider the number of odd vertices (an odd vertex has an odd number of edges meeting at it). A picture can be completely traced if there are exactly two odd vertices (a starting point and an ending point) or none at all. Because there are eight odd vertices in this picture,



. . . it is impossible to trace the entire grid and therefore impossible to surround all nine squares. The maximal number of squares which can be surrounded is six. In the next illustration we show two examples and delete edges not used in the tracings. Note the two odd vertices in each.





667

G 13

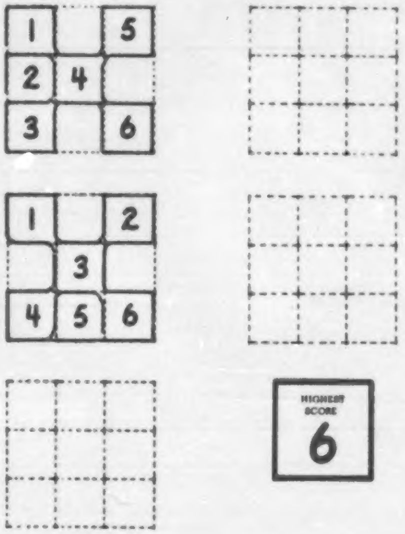
Worksheets G13**, ***, and **** are available for individual work. It is easy to check the results:

- on the *** worksheet, the highest scores start at 3 and increase by ones (3, 4, 5, 6); and
- on the **** worksheet the highest scores start at 2 and increase by twos (2, 4, 6, 8, 10).

Perhaps some of your students will notice these patterns, but you need not point them out yourself.

Name ____________________ G 13 *

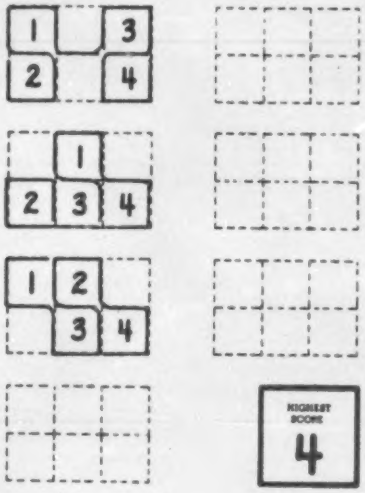
Try to find the highest score in the tracing game for this picture.


1
5
2
4
3
6
1
2
3
4
5
6
HIGHEST SCORE
6

Other solutions are possible.

Name ____________________ G 13 **

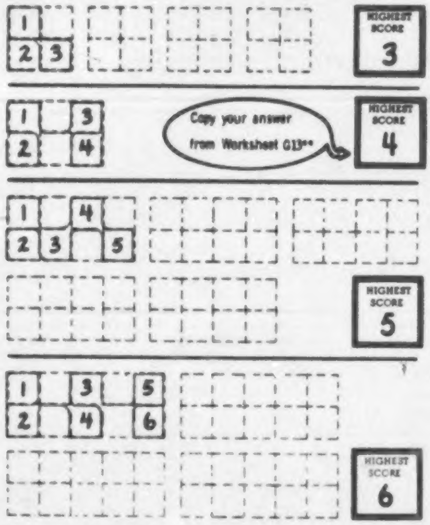
Try to find the highest score in the tracing game for this picture.


1
3
2
4
1
2
3
4
1
2
3
4
HIGHEST SCORE
4

Other solutions are possible.

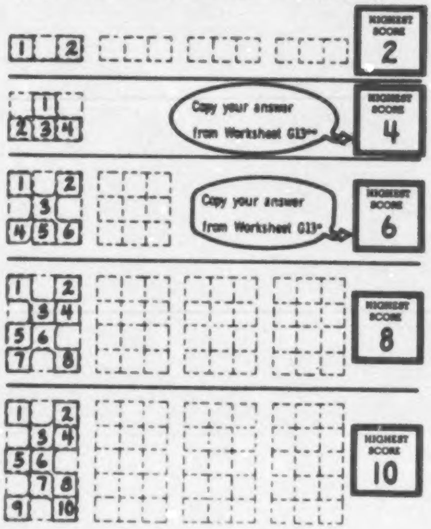
Name ____________________ G 13 ***

Try to find the highest score in the tracing game for these pictures.


1
2
3
HIGHEST SCORE
3
1
3
2
4
Copy your answer from Worksheet G13**
HIGHEST SCORE
4
1
4
2
3
5
HIGHEST SCORE
5
1
3
5
2
4
6
HIGHEST SCORE
6

Name ____________________ G 13 ****

Try to find the highest score in the tracing game for these pictures.


1
2
HIGHEST SCORE
2
1
2
3
4
Copy your answer from Worksheet G13**
HIGHEST SCORE
4
1
2
3
4
5
6
Copy your answer from Worksheet G13*
HIGHEST SCORE
6
1
2
3
4
5
6
7
8
HIGHEST SCORE
8
1
2
3
4
5
6
7
8
9
10
HIGHEST SCORE
10

WORKBOOKS

Comprehensive School Mathematics Program

CSMP Mathematics for the Upper Primary Grades

Part IV

Workbooks

CEMREL, Inc.

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WORKBOOKS

INTRODUCTION

There are many opportunities for the student to work individually during the course of the lessons described in the other content strands. In the Workbook strand, however, it is this individualized work which becomes the principal mode of the majority of lessons. It has been our goal in this strand to provide the students with opportunities:

- to review many of the ideas they have met in other content strands;
- to apply their acquired knowledge to new situations requiring various kinds of strategic thinking;
- to learn how to read and use mathematics workbooks.

The following workbooks are provided:

GALAXY OF PROBLEMS #1
GALAXY OF PROBLEMS #2
GALAXY OF PROBLEMS #3
GALAXY OF PROBLEMS #4
FISHING FOR NUMBERS

. . . and one story-workbook is provided:

TO PICTURE

The five workbooks contain problems of varying levels of difficulty all in one booklet. Approximately the first ten pages of each booklet have easy problems, the next ten to twelve pages have problems of average level difficulty, and the last ten pages have more challenging problems. For each workbook, we suggest that all students start work at the easiest level of difficulty (i.e. on page 2) and then work through as many pages as they can handle during the two lessons scheduled for that workbook. In an average class we estimate that about two-thirds of the students will finish the first ten pages, about one-third will finish the first twenty pages, and a few will finish all or most of the booklet. These proportions will vary from class to class.

This guide contains an answer key for each workbook. The key follows an introduction to the workbook and suggested collective lessons which present the workbook to the whole class or provide a warm-up activity on a problem similar to one found in the workbook.

Ten of the seventeen lessons in the strand have the individual work in the workbooks as their main activity. In the Story-workbook TO PICTURE, used in two lessons, the student locates eight numbers in an arrow picture and in a related arrow picture. The other five lessons present detective stories in which there are one to three secret numbers and several clues that lead to their discovery. The clues review many ideas from the other content strands as well as involve students in new situations involving patterns and counting techniques.

Use of the Workbook Strand for Evaluation Purposes*

The workbooks provide an excellent instrument to assess the progress of your students on a weekly basis. You may not feel that it is necessary to check every page and problem for each student. You should develop the sort of procedures for checking students' work with which you are most comfortable. These may include checking one or more specific pages, discussing some particular mistakes with individual students and letting them correct their work, or just looking carefully at the first few pages to be sure the students have understood the general idea of the problems in that particular book.

A progress chart is provided on pages viii and ix to help you keep track of

1) where each of your students is on any particular workbook day,
2) how much each student has completed in a workbook,

---

* See also Role and Use of Workbooks in Section 1 of the General Introduction.

3) your personal degree of satisfaction with the student's work in each workbook,

4) which students are showing improvement from one workbook to the next,

and

5) which students are having difficulty with even the easiest problems and would benefit from more individual attention during the lessons.

Here is a list of some of the important points to keep in mind for workbooks.

- All students start at the beginning of each workbook and progress as far as they can.
- All students begin a workbook on the same day, even if some students have not finished work on the preceding workbook.
- Not all students are expected to complete a given workbook. Only the faster students will reach the most challenging problems. The slower students may succeed only in doing the easiest problems, although you should not assume that this will automatically be the case -- surprises are not at all uncommon in this area.
- You should read the introductory material for each workbook and give the short collective lesson(s) described.

Workbook Progress Chart

STUDENT'S NAME	GALAXY OF PROBLEMS #1	GALAXY OF PROBLEMS #2	TO PICTURE

Workbook Progress Chart

GALAXY OF PROBLEMS #3	GALAXY OF PROBLEMS #4	FISHING FOR NUMBERS

W1 GALAXY OF PROBLEMS #1 (Lesson 1)

CAPSULE LESSON SUMMARY

Fill in a subtraction table. Begin the Workbook GALAXY OF PROBLEMS #1.

MATERIALS

Teacher: None

Student: Workbook GALAXY OF PROBLEMS #1 ; colored pencils

DESCRIPTION OF LESSON

Draw this table on the board.

−	9	6	3	0
9				
7				
5				

T: The minus sign here (point to "-" in the upper left corner) tells us that this is a subtraction table. When this table is complete, there will be a number in each of the squares. To find out which numbers are in the table, we start with a number in the left column (point to "9", "7", and "5") and then subtract one of the numbers in the top row (point to "9", "6", "3", and "0"). For example, one of the numbers in this table is 7 - 6.

W1

Point to "7", "-", and "6" in the table.

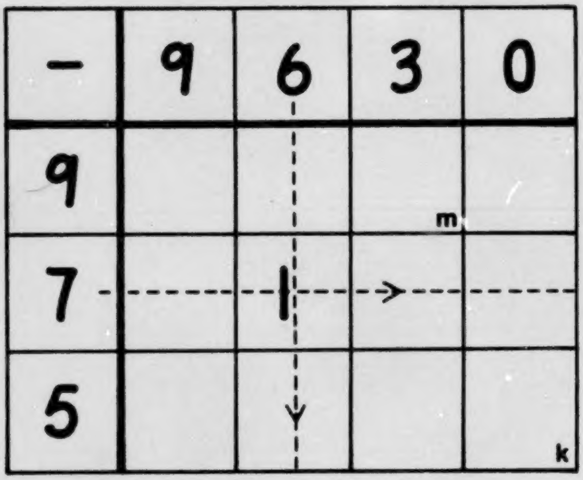
T: What number is 7 - 6? (1)

Does anyone know where we should write "1" in this table?

Some of your students may know how to complete an operation table like this. Let a student show where "1" should be written or, if necessary, do so yourself.

T: "1" goes in the same row as "7" and in the same column as "6".

Simultaneously trace the "7 row" with your left index finger and the "6 column" with your right index finger. Write a "1" in the sqaure where your fingers cross. Emphasize that the "1" is in the same row as "7" and in the same column as "6".



-	9	6	3	0
9			m	
7		1		
5				k

Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T (pointing to m): Which number belongs in this square? What calculation do we need to make?

S: 9 - 3 = 6, so we should write "6" in that square.

T (pointing to k): Which number goes in this square? Tell me your answer with a whisper.

Allow many students to whisper the answer to you before letting a student write the answer in the table.

−	9	6	3	0
9			6	
7		1		
5				5

T: Who can fill in another square of this table?

Continue inviting students to fill a square of the table until it is complete. Emphasize that only the numbers in the column under "-" and in the row to the right of "-" are used to decide which numbers go in the table. Frequently trace the appropriate row and column and say the subtraction calculation corresponding to the number a student has entered in the table.

W1

The completed table is shown below.

−	9	6	3	0
9	0	3	6	9
7	$\hat{2}$	1	4	7
5	$\hat{4}$	$\hat{1}$	2	5

Distribute copies of the Workbook GALAXY OF PROBLEMS #1. Emphasize that the instructions on each page should be read carefully. Read the instructions on pages 2 and 3 collectively before the students begin working individually.

At the end of the class period, collect the workbooks. Check them and have them ready for use in Lesson W2.

W2 GALAXY OF PROBLEMS #1 (Lesson 2)

CAPSULE LESSON SUMMARY

Continue working in the Workbook GALAXY OF PROBLEMS #1. (This is the second of two lessons using this workbook.) Play the Number Line Game.

MATERIALS

Teacher:	None
Student:	Workbook GALAXY OF PROBLEMS #1; colored pencils

DESCRIPTION OF LESSON

Your review of the students' workbooks may have indicated that a short collective discussion is needed about a particular page in the workbook. If appropriate, collectively solve a problem similar to that problem in the workbook before distributing the students' copies of the Workbook GALAXY OF PROBLEMS #1.

Allow the students to work on the workbook for about 30 minutes. Play the Number Line Game for the remaining time allotted for the lesson.

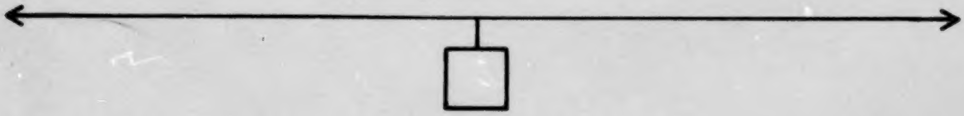
The Number Line Game

The Number Line Game may be played frequently throughout the year whenever you finish a lesson early or have some extra time. Each game takes approximately five to ten minutes. Several variations of this game are given at the end of this lesson. Use these variations when your class is ready for them.

T: Today we are going to play the Number Line Game. I'm thinking of a number between 900 and 1,100. I'll show you where it is on the number line; my number is hidden in the box.

W2

Draw a line on the board and position a box for the secret number near the center of your picture.



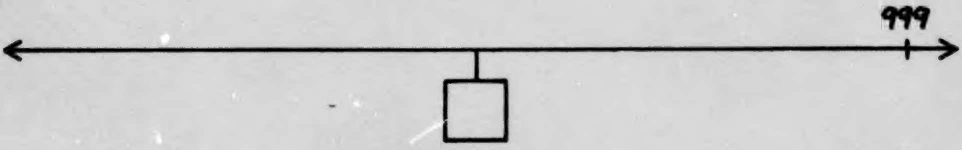
T: Guess what my secret number is.

If someone suggests a number which is not between 900 and 1,100, do not write anything. Just remind the class that the secret number is between 900 and 1,100.

This is a description of a possible game. Suppose the secret number is 927.

S: 999.

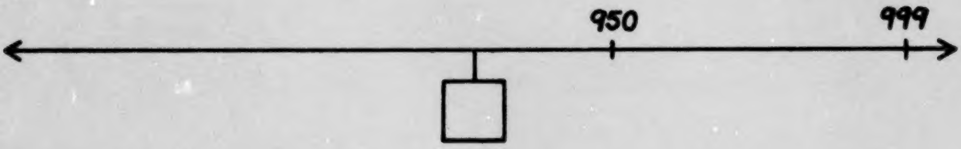
T: I'll show you where 999 is on this number line. It might not be in exactly the right place, but I'll try to locate it as closely as possible.



T: 999 is more than my secret number.

Ask another student to guess the secret number.

S: 950.

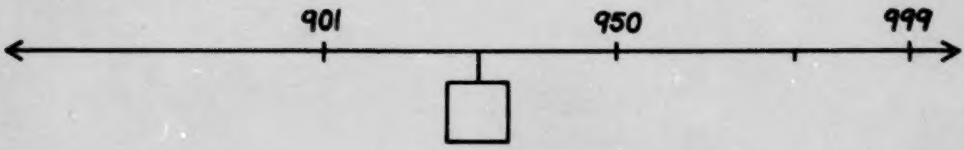


T: 950 is more than my secret number.

S: 900.

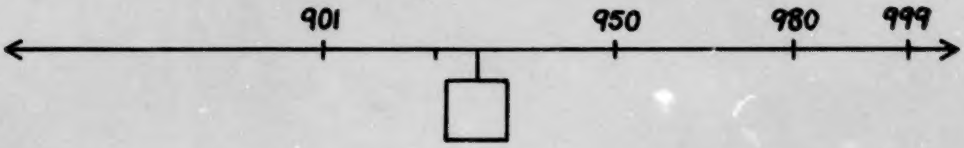
T: My number is between 900 and 1,100. 900 cannot be my secret number.

S: 901.



T: 901 is less than my secret number.

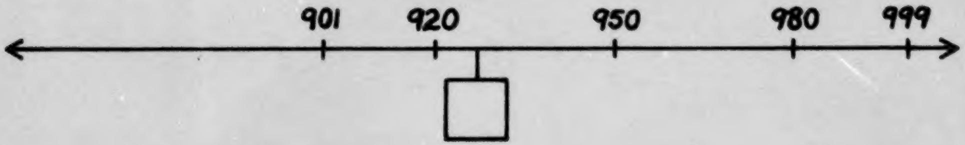
S: 980.


901
950
980
999

T: 980 is more than my secret number.

NOTE: Although a guess may give no new information, such as 980 in this sample game, do not criticize or comment except to say that it is more or less than your number.

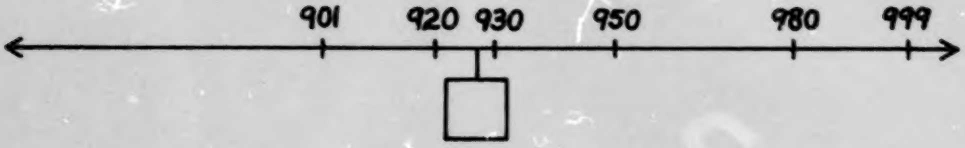
S: 920.


901
920
950
980
999

T: 920 is less than my secret number.

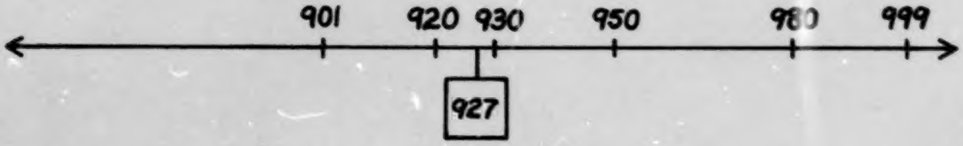
W 2

S: 930.


901
920
930
950
980
999

T: 930 is more than my secret number.

S: 927.


901
920
930
950
980
999
927

T: 927 is my secret number.

Play the game a few more times. You may wish to let the student who guessed your number choose the secret number for the next game and whisper it to you. In this case, ask that student to keep the number a secret and not to play in that game.

VARIATIONS OF THE NUMBER LINE GAME

Variation 1

As your class becomes familiar with the Number Line Game, you can expand the range of possible numbers. Some possibilities are:

- numbers between 1,000 and 1,500;
- numbers between $\widehat{100}$ and 100;
- numbers between 1,900 and 2,100.

Variation 2

As the students become familiar with numbers such as $3\frac{1}{2}$, occasionally you can choose such a non-integer as your secret number. In this case, it is best to limit the range of possible numbers to those between 0 and 20.

Variation 3

As the students become familiar with numbers such as 0.58, occasionally you can choose a non-integer decimal number as your secret number. Again, it is best to limit the range of possible numbers to those between 0 and 2.

Variation 4

When you are using the whole numbers only for your range of possible numbers, ask a student to choose a secret number and to judge the guesses of the other students. The student who chose the secret number tells the class whether the guess is more or less than the secret number. You label a mark on the number line for each guess and, if necessary, help the student to judge. Remind the student judges not to give additional clues.

Variation 5

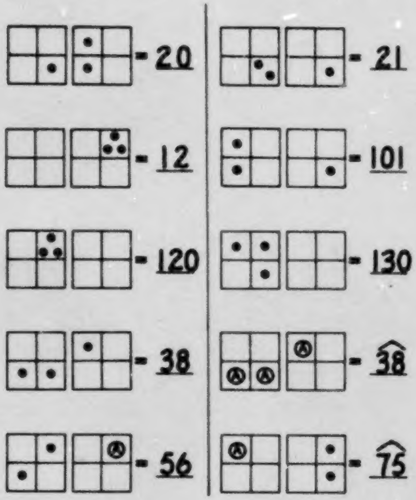
Divide the class into two teams. The teams alternate turns and the members of each team take turns guessing what the secret number is.

GALAXY OF PROBLEMS

#1

Galaxy of Problems #1

Put any numbers you wish on the Minicomputer using exactly three checkers for each.

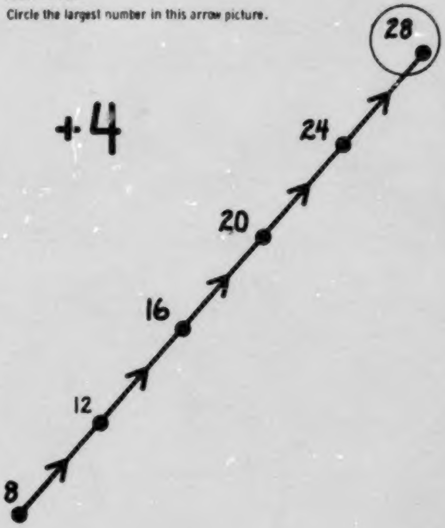

= 20
= 21
= 12
= 101
= 120
= 130
= 38
= $\widehat{38}$
= 56
= $\widehat{75}$

2

Many solutions are possible.

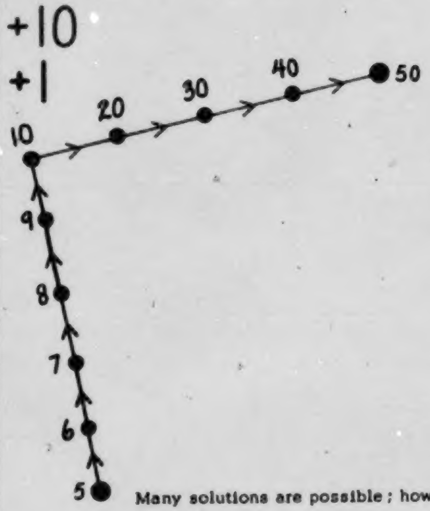
Label the dots.

Circle the largest number in this arrow picture.


+4
8
12
16
20
24
28

3

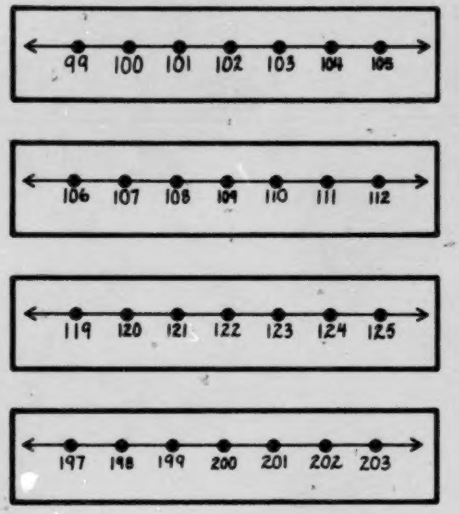
Build a blue-red road from 5 to 50.


+10
+1
5
6
7
8
9
10
20
30
40
50

Many solutions are possible; however a shortest road has five red arrows and four blue arrows.

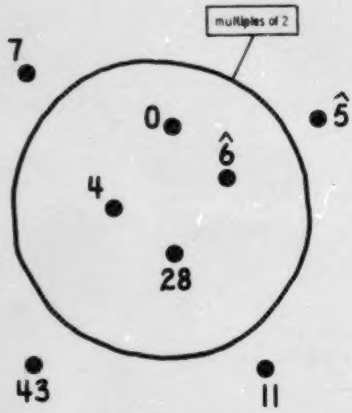
4

Label the dots on these number lines.


99 100 101 102 103 104 105
106 107 108 109 110 111 112
119 120 121 122 123 124 125
197 198 199 200 201 202 203

5

Label the dots in this string picture.
Many answers are possible.

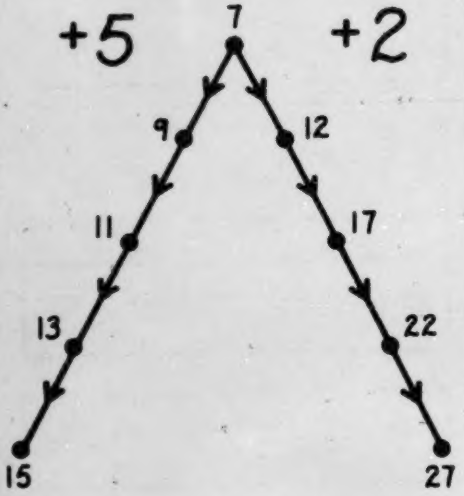

multiples of 2
7
0
$\hat{6}$
$\hat{5}$
4
28
43
11

Many solutions are possible.

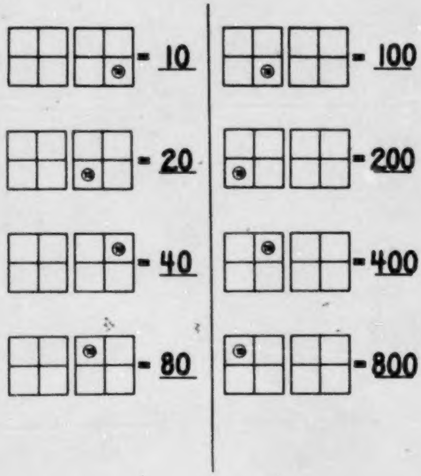
Complete this addition table.

+	10	9	7	8
10	20	19	17	18
8	18	17	15	16
9	19	18	16	17
7	17	16	14	15

7 is the smallest number in this arrow picture.
Label the dots.

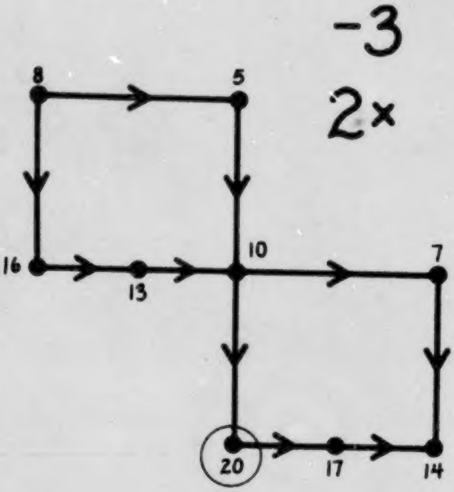

+5
+2
7
9
12
11
17
13
22
15
27

What number is on the Minicomputer?


= 10
= 20
= 40
= 80
= 100
= 200
= 400
= 800

Label the dots.

Circle the largest number in this arrow picture.


-3
2×
8
5
16
13
10
7
20
17
14

Complete.

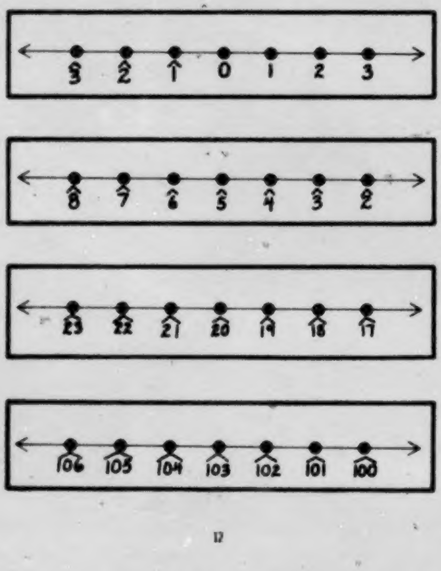
$$\begin{array}{r}16\\16\\+16\\\hline 48\end{array}$$

$3 \times 16 = \underline{48}$

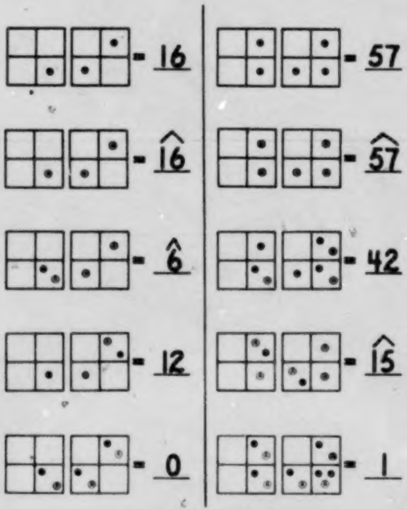
$3 \times 18 = \underline{54}$

$$\begin{array}{r}18\\18\\+18\\\hline 54\end{array}$$

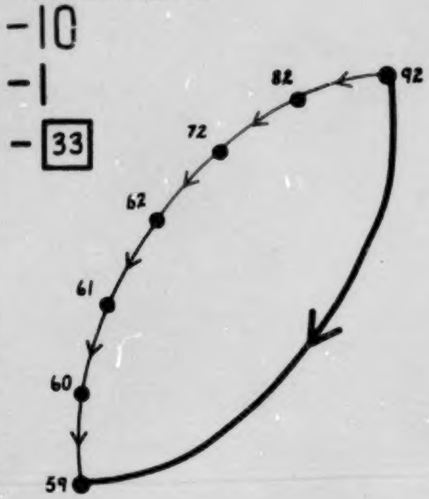
Label the dots on these number lines.


$\hat{3}$ $\hat{2}$ $\hat{1}$ 0 1 2 3
$\hat{8}$ $\hat{7}$ $\hat{6}$ $\hat{5}$ $\hat{4}$ $\hat{3}$ $\hat{2}$
$\widehat{23}$ $\widehat{22}$ $\widehat{21}$ $\widehat{20}$ $\widehat{19}$ $\widehat{18}$ $\widehat{17}$
$\widehat{106}$ $\widehat{105}$ $\widehat{104}$ $\widehat{103}$ $\widehat{102}$ $\widehat{101}$ $\widehat{100}$

What number is on the Minicomputer?

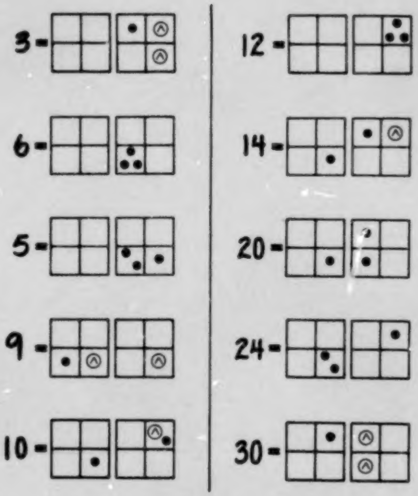

= 16
= 57
= $\widehat{16}$
= $\widehat{57}$
= $\hat{6}$
= 42
= 12
= $\widehat{15}$
= 0
= 1

Build a blue-red road from 92 to 59.
Fill in the box for the green arrow.


-10
-1
- 33
92
82
72
62
61
60
59

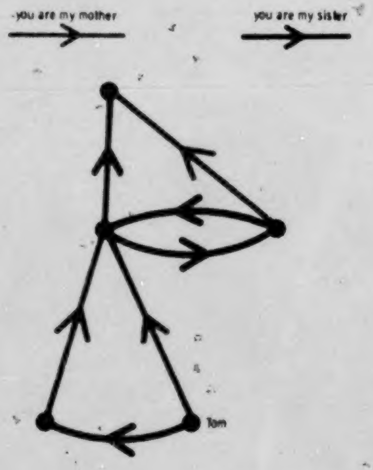
Other solutions are possible; however, a shortest road has three blue and three red arrows.

Put these numbers on the Minicomputer using exactly three checkers (positive or negative).


3
6
5
9
10
12
14
20
24
30

Many solutions are possible.

This picture shows some people in Tom's family. Draw the missing red and blue arrows.


you are my mother
you are my sister
Tom

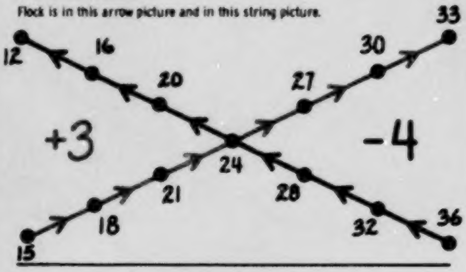
Complete these subtraction tables.

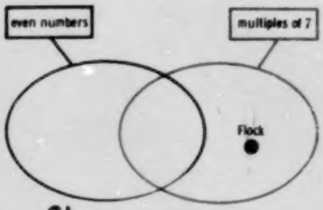
–	5	7	3	8
10	5	3	7	2
8	3	1	5	0

–	6	4	10	9
12	6	8	2	3
15	9	11	5	6

Flock is a secret number.

Flock is in this arrow picture and in this string picture.


12
16
20
27
30
33
+3
-4
24
21
28
18
32
36
15

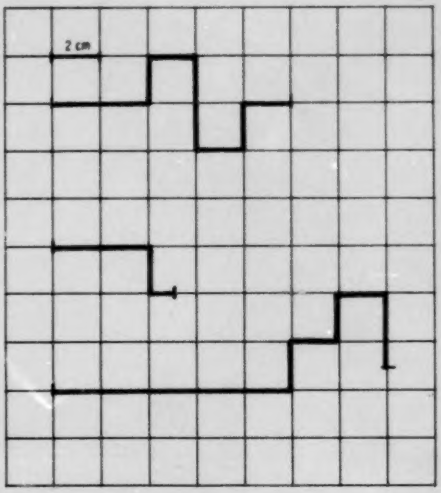

even numbers
multiples of 7
Flock

Who is Flock? 21

Draw a red zigzag path which is 18 cm long.

Draw a blue zigzag path which is 7 cm long.

Draw a green zigzag path which is 21 cm long.



Many solutions are possible.

Complete these number stories.

$(6 + 5) \times 2 =$ 22

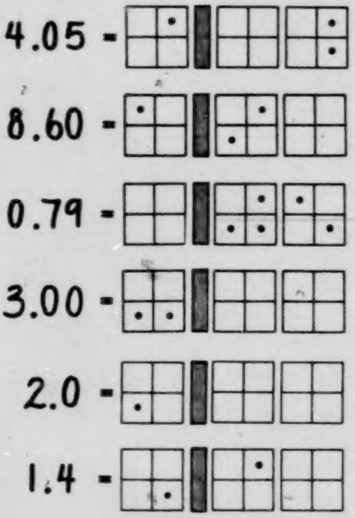
$6 + (5 \times 2) =$ 16

$(7 - 2) \times 3 =$ 15

$7 - (2 \times 3) =$ 1

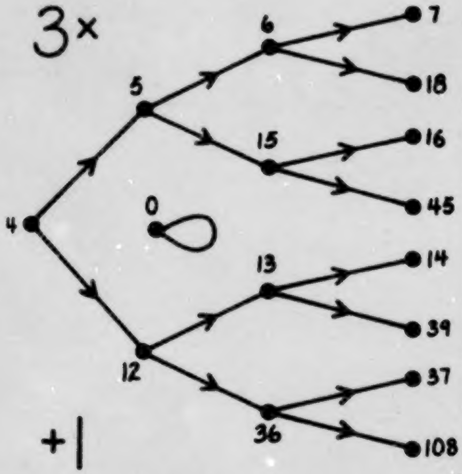
$(7 - 3) \times 2 =$ 8

Put these numbers on the Minicomputer.


4.05 =
8.60 =
0.79 =
3.00 =
2.0 =
1.4 =

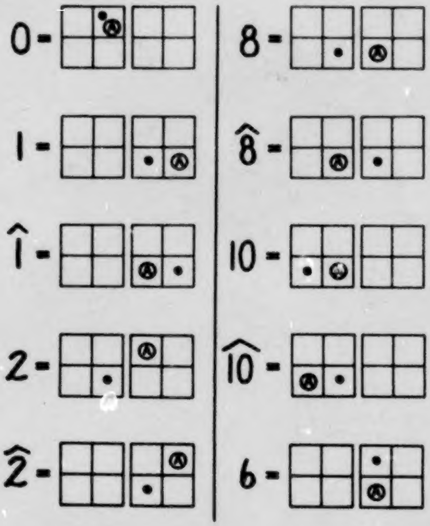
The solutions here use standard configurations; however, many other solutions are possible.

Label the dots.


3×
+1
4
5
12
6
15
0
13
36
7
18
16
45
14
39
37
108

22

Put these numbers on the Minicomputer using exactly one regular checker and one negative checker.

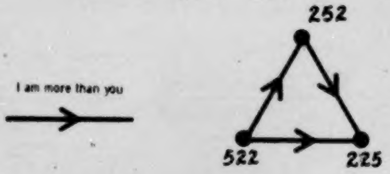

0 =
1 =
$\hat{1}$ =
2 =
$\hat{2}$ =
8 =
$\hat{8}$ =
10 =
$\widehat{10}$ =
6 =

23

Other solutions are possible.

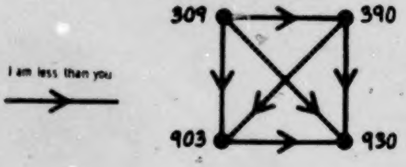
Put these numbers in the arrow picture.

225 252 522


I am more than you
252
522
225

Put these numbers in the arrow picture.

390 930 309 903


I am less than you
309
390
903
930

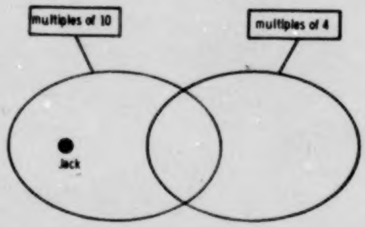
24

Jack is a secret number.

Jack can be put on this Minicomputer with five regular checkers on the same square.

Jack could be 5, 10, 20, or 40

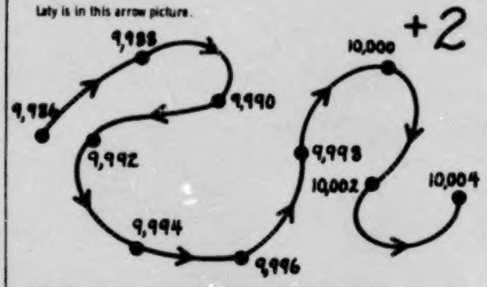
Jack is in this string picture.


multiples of 10
multiples of 4
Jack

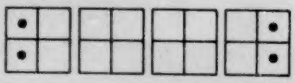
Who is Jack? 10

25

Laty is a secret number.
Laty is in this arrow picture.

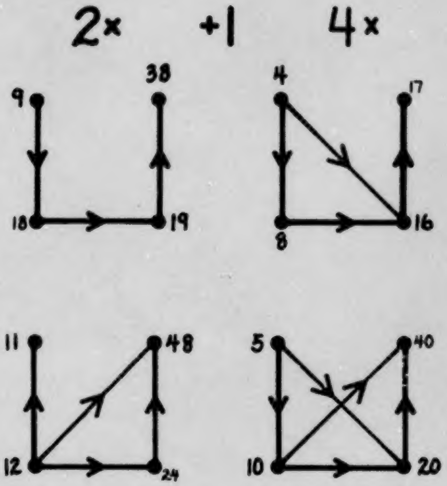

+2
9,986
9,988
9,990
9,992
9,994
9,996
9,998
10,000
10,002
10,004

Laty can be put on this Minicomputer by moving exactly one checker.



Who is Laty? 10,002

Label the dots.
Draw all the possible blue arrows.


2x
+1
4x
9
38
18
19
4
17
8
16
11
48
12
24
5
40
10
20

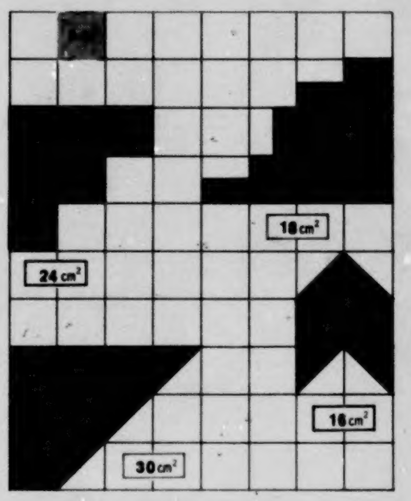
You should have four blue arrows.

Complete.

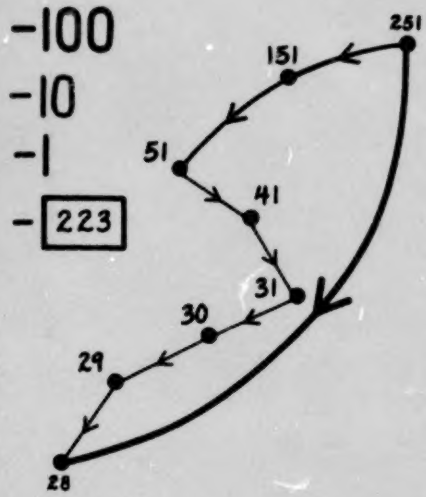
$$\begin{array}{r} 27 \\ +\ 33 \\ \hline 60 \end{array} \qquad \begin{array}{r} 154 \\ +\ 37 \\ \hline 191 \end{array}$$

$$\begin{array}{r} 408 \\ +\ 192 \\ \hline 600 \end{array} \qquad \begin{array}{r} 3{,}275 \\ +\ 725 \\ \hline 4{,}000 \end{array}$$

Find the area of each shape.

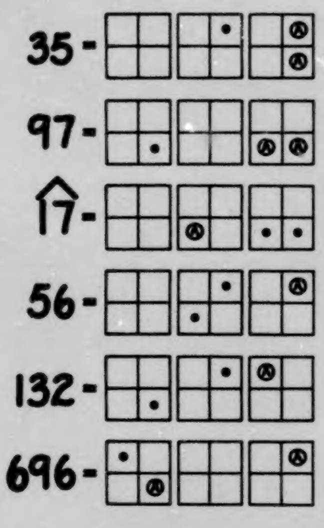

18 cm²
24 cm²
16 cm²
30 cm²

Build a black-blue-red road from 251 to 28.
Fill in the box for the green arrow.


-100
-10
-1
-223
251
151
51
41
31
30
29
28

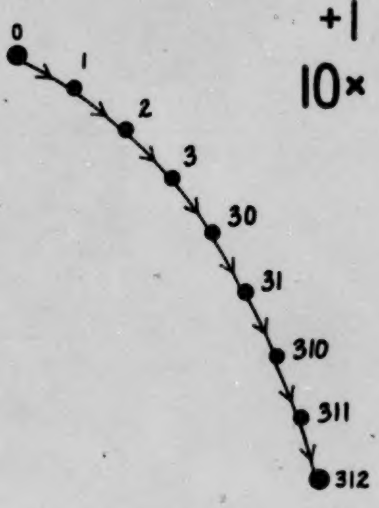
Other solutions are possible; however, a shortest road has two black, two blue, and three red arrows.

Put these numbers on the Minicomputer using exactly three checkers (positive or negative).


35 -
97 -
$\hat{1}7$ -
56 -
132 -
696 -

Other solutions are possible.

Build a red-blue road from 0 to 312.


+1
10×
0
1
2
3
30
31
310
311
312

Other solutions are possible; however, this is a shortest road.

W3 Who is Ply?

CAPSULE LESSON SUMMARY

Solve a detective story in which the clues involve a string picture, The Minicomputer, and an arrow picture.

MATERIALS

Teacher: Colored chalk; Minicomputer set; ⑩-checkers
Student: Worksheet W3*, **, ***, and ****; pencil; paper

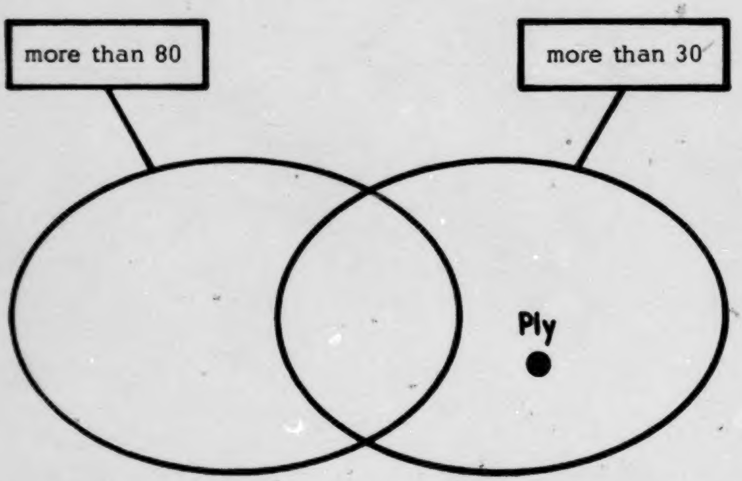
DESCRIPTION OF LESSON

T: Today you will have another opportunity to be detectives. The name of the secret number is Ply. Ply is a whole number. Follow the clues carefully and you will be able to discover who Ply is.

First clue

T: I'll show you where Ply belongs in a string picture.

Draw this string picture on the board.



T: What does this string picture tell us about Ply?

W 3

Encourage several students to comment. A possible dialogue is given here.

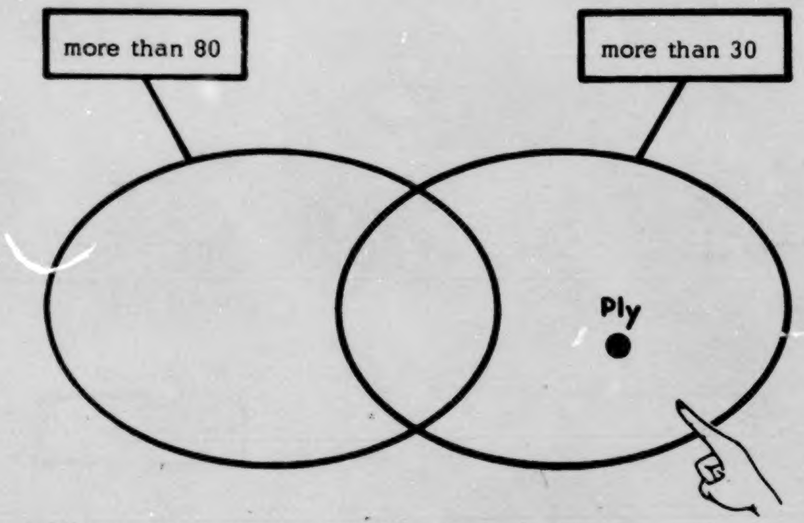
S: Ply is more than 30.

S: Ply is less than 80.

S: Ply could be 80.

T: Where does 80 belong in this string picture?

Ask a student to point to the region where 80 belongs.



T: 80 is more than 30, so 80 is inside the red string.
80 is not more than 80, so 80 is outside the blue string.
What is the largest number Ply could be?

S: 80.

T: Could Ply be 81? (No)
Why not?

S: 81 is more than 80, so 81 belongs inside the blue string.

Record this information on the board.

80 is the largest number Ply could be.

T: What is the smallest number Ply could be?

S: 30.

S: Ply can't be 30! 30 is not more than 30.

T: Where does 30 belong in this string picture?

S: 30 belongs outside both strings.

T: What is the smallest number Ply could be?

S: 31.

If necessary, remind the students that Ply is a whole number.

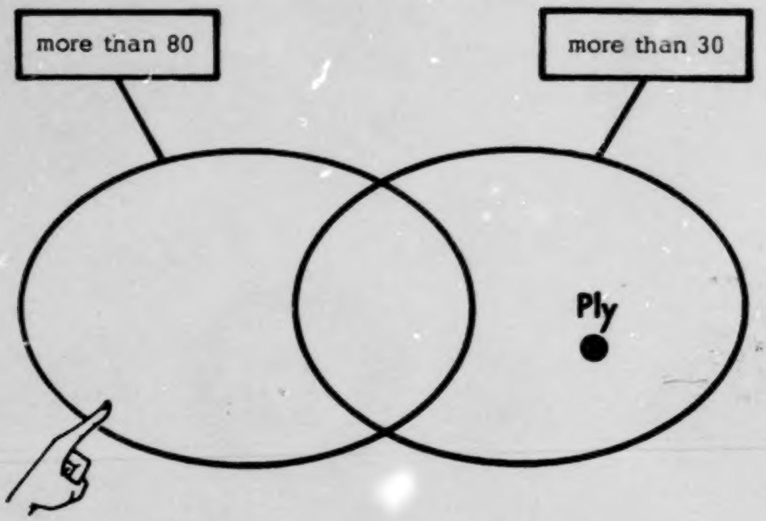
Record this information on the board.

80 is the largest number Ply could be.

31 is the smallest number Ply could be.

W 3

Point to the region on the left.



T: Which numbers belong in this part of the string picture?

That region is empty; however, a student might suggest some number belongs there, for example, 100.

S: 100.

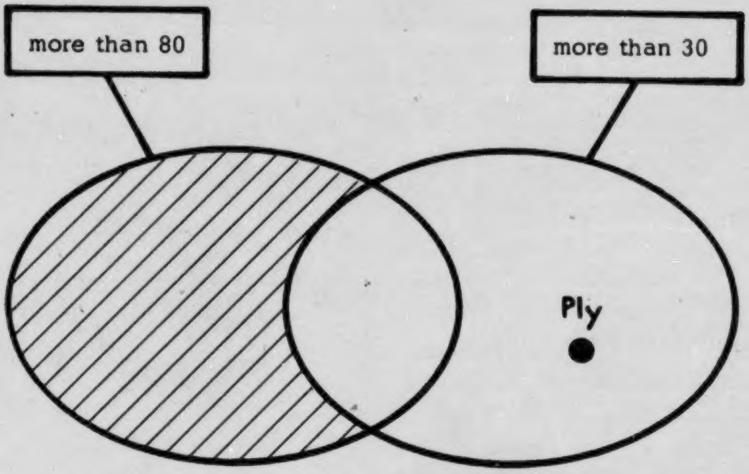
T: Where does 100 belong in this string picture?

S: 100 belongs in the middle, because 100 is more than 80 and more than 30.

S: There aren't any numbers in that part of the string picture, because all the numbers that are more than 80 are also more than 30.

S: We can hatch that part of the string picture.

Hatch the region on the left to show that it is empty.



T: What are some numbers that would be outside both of these strings?

S: 30.

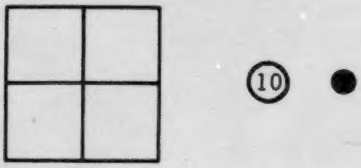
S: 0.

S: $\widehat{100}$.

T: We know that the largest number Ply could be is 80 and that the smallest number Ply could be is 31. Ply could also be any of the whole numbers between 31 and 80.

Second clue

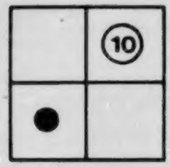
Display one Minicomputer board and hold up one regular checker and one ⑩ -checker.



W 3

T: Ply can be put on this Minicomputer using exactly these two checkers. Which numbers could Ply be?

Invite students to put numbers on the Minicomputer using the two given checkers. When a student puts a number on the Minicomputer, ask another student to identify which number it is. For example, suppose a student has put this configuration on the Minicomputer.



T (pointing to the ⑩-checker): 10 × 4 = . . . ?

S: 40.

T (pointing to the regular checker): . . . plus 2 more? What number is on the Minicomputer?

S: 42.

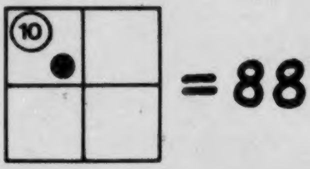
Record "42" on the board. Continue this activity until three or four possible numbers have been found.

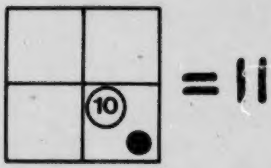
T: How many different numbers could we put on this Minicomputer using exactly these two checkers?

Allow several students to predict the number of possibilities.

T: What is the largest such number? (88)
What is the smallest such number? (11)

If either of these numbers has not already been put on the Minicomputer, ask for a volunteer to put it on the Minicomputer at this time.





Continue asking students to put numbers on the Minicomputer until your list is complete.

11 12 14 18
21 22 24 28
41 42 44 48
81 82 84 88

You may wish to commend any students who predicted there would be sixteen numbers that could be put on this Minicomputer using exactly one regular and one ⑩-checker.

Students may have already commented during this activity that some of the numbers in the list cannot be Ply because they are either less than 31 or more than 80. Accept such comments if they are offered, but encourage the students to wait until the list is complete before telling you which numbers to cross out.

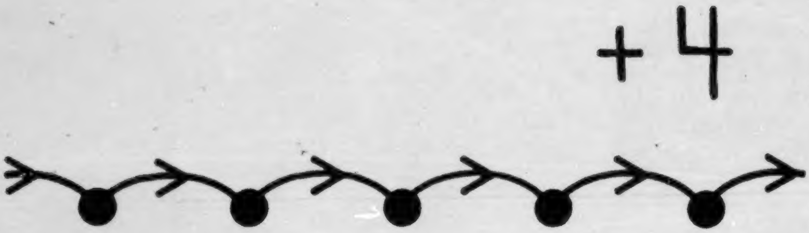
T: Which of these numbers could Ply be?

Check whatever suggestions are given to ensure that the numbers fit both of the clues. Cross out the numbers which cannot be Ply. Continue until your class concludes that Ply is 41, 42, 44, or 48. Erase everything on the board except these four numerals.

W 3

Third clue

Draw this arrow picture on the board.



T: What do the unfinished arrows tell us?

S: The arrow picture continues in both directions.

T: What can we say about all the numbers on this +4 road?

S: All of the numbers on this arrow road could be multiples of 4.

T: If one of the numbers on this arrow road is a multiple of 4, then all the numbers are multiples of 4. What can we say if we know that one of these numbers is not a multiple of 4?

S: That none of them are multiples of 4.

T: What can we say if one of these numbers is an odd number?

S: That all of them are odd numbers.

T: . . . and if one of them is an even number?

S: That all of them are even numbers.

T: Ply is one of the numbers on this arrow road. Now I will tell you one of the numbers in this arrow picture. When you know which number Ply is, write it on a piece of paper.

Label any one of the dots in the arrow picture "6". Look at many of the students' answers before asking a student to reveal Ply's identity. Discuss with the class why 42 is the only number in the list that would also be on this +4 arrow road.

T: Let's check the other numbers before we agree that Ply is 42. Could 41 be the secret number?

S: No, because 41 is an odd number.

T: Could 44 be Ply?

S: 44 is a multiple of 4 and 6 and is not a multiple of 4, so they wouldn't be on the same +4 arrow road.

T: Could Ply be 48?

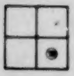
S: 48 is also a multiple of 4.

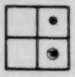
Conclude that Ply is 42.

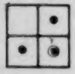
Worksheets W3*, **, ***, and **** are available for individual work.

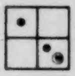
Name ____________ **W 3 ***

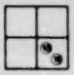
What number is on the Minicomputer?

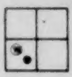
 = 10

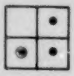
 = 14

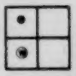
 = 16

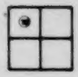
 = 19

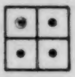
 = 20

 = 22

 = 25

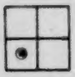
 = 28

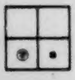
 = 80

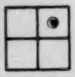
 = 87

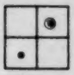
Name ____________ **W 3 ****

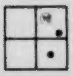
Put these numbers on the Minicomputer. Use at least one ⑩ -checker for each number.

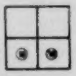
20 = 

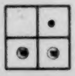
21 = 

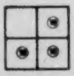
40 = 

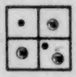
42 = 

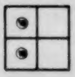
45 = 

30 = 

34 = 

70 = 

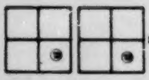
79 = 

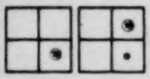
100 = 

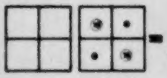
Many solutions are possible.

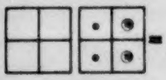
Name ____________ **W 3 *****

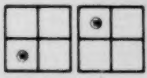
Put numbers you like on the Minicomputer. Use at least two ⑩ -checkers for each number.

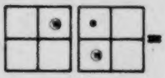
 = 110

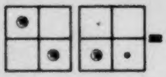
 = 141

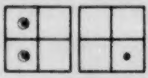
 = 96

 = 60

 = 280

 = 428

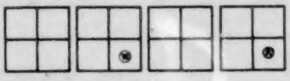
 = 921

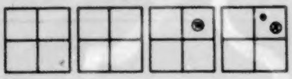
 = 1,001

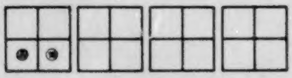
Many solutions are possible.

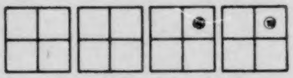
Name ____________ **W 3 ******

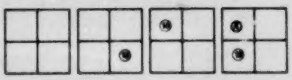
Put numbers you like on the Minicomputer. Use at least one ⑩ -checker and one negative checker for each number.

 = 999

 = 400

 = 8,000

 = 0

 = 1,812

Many solutions are possible.

W4 GALAXY OF PROBLEMS #2 (Lesson 1)

CAPSULE LESSON SUMMARY

Begin the Workbook GALAXY OF PROBLEMS #2. Play the Number Line Game.

MATERIALS

Teacher:	None
Student:	Workbook GALAXY OF PROBLEMS #2; colored pencils

DESCRIPTION OF LESSON

Distribute copies of the Workbook GALAXY OF PROBLEMS #2 and let the students begin working individually. If the questions you are receiving indicate that many of your students are having difficulty with a particular page, hold a short collective discussion of the problem presented on that page.

Allow 30 to 40 minutes of the lesson for the workbook and then collect them for your review. Spend any remaining time playing the Number Line Game, as described in Lesson W2, with secret numbers between 900 and 1,100.

W5 GALAXY OF PROBLEMS #2 (Lesson 2)

CAPSULE LESSON SUMMARY

Display some decimals on the Minicomputer and look at different names for the same number, for example, 2.5 and 2.50. Decide how to share $10 equally among two, among four, and among eight people. Continue working in the Workbook GALAXY OF PROBLEMS #2. (This is the second of two lessons using this workbook.)

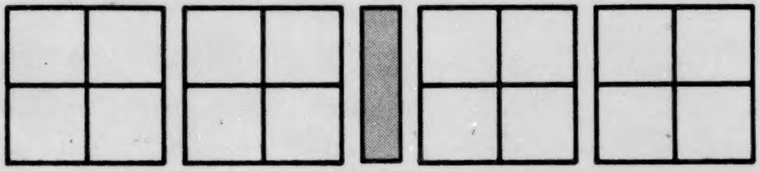
MATERIALS

Teacher: Minicomputer set; colored chalk
Student: Workbook GALAXY OF PROBLEMS #2; pencil
colored pencils

DESCRIPTION OF LESSON

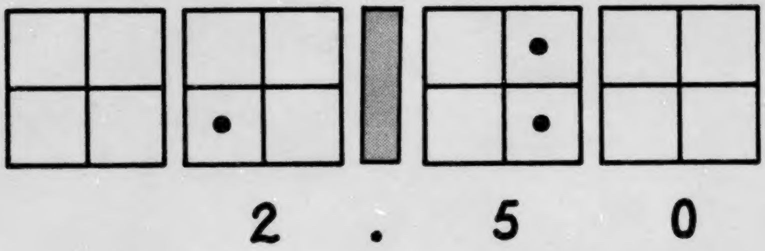
Exercise 1

Display four Minicomputer boards. Draw a green bar between the second and third Minicomputer boards.



T: Who can put 2.5 (read: two point five) on the Minicomputer? Who can write this number below the Minicomputer?

Perhaps the volunteer will write "2.50" below the Minicomputer. If the student writes "2.5", ask what could be written below the pennies' board and write the "0" yourself.



T: I wanted someone to put 2.5 on the Minicomputer.
Is this number 2.5?

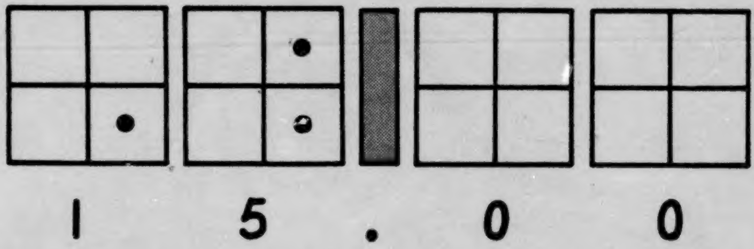
S: Yes, we could just erase the "0".

T: "2.5" and "2.50" are different names for the same number.

Remove the checkers and erase the board.

T: Who can put 15 on the Minicomputer?
Who can write this number below the Minicomputer?

Perhaps the volunteer will write "15.00" below the Minicomputer.



T: Is this number 15? (Yes)
What other names does this Minicomputer suggest for 15?

S: "15.00" (read: fifteen point zero zero).

Record the equality expression on the board.

$$15 = 15.00$$

S: "15.0".

$$15 = 15.00 = 15.0$$

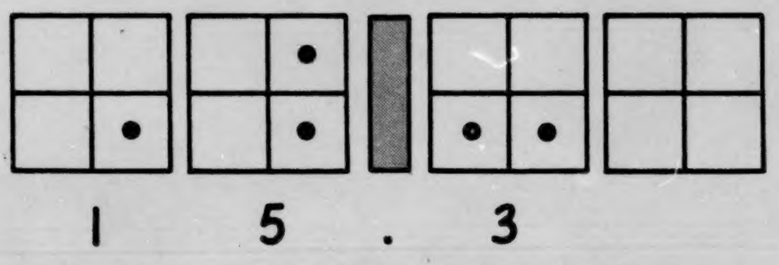
W 5

T: "15", "15.0" and "15.00" are all names for the same number.

Erase the board.

T: The number on the Minicomputer is 15. Who can put 15.3 on the Minicomputer, using the checkers that are already there and some others?

Write "15.3" below the Minicomputer.



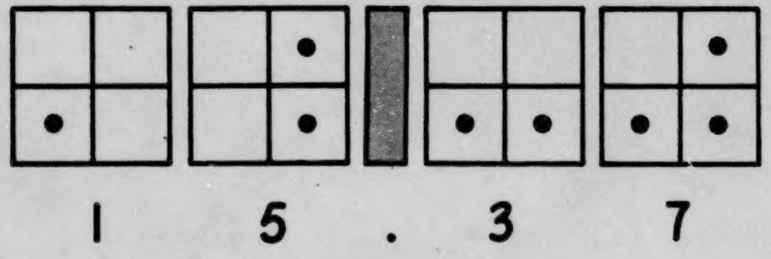
T: What is another name for 15.3?

S: "15.30".

T: This "3" (point to "3" in "15.3") is for three dimes, and three dimes is the same as 30¢.

Erase the board.

T: The number on the Minicomputer is 15.3. Who can put 15.37 on the Minicomputer by just adding some more checkers?
Who can write this number below the Minicomputer?



Remove the checkers and erase the board. Ask students to put these numbers on the Minicomputer and to write the appropriate numerals below the Minicomputer.

26　　**2.06**　　**20.06**

2.6　　**0.26**　　**20.6**

Remove the checkers and erase the board before going on to Exercise 2.

Exercise 2

T: Imagine that we have \$10 and want to share it equally among two people. How much money should each person receive?

S: They each should get \$5 because $\frac{1}{2} \times 10 = 5$.

T: Suppose we want to share \$10 equally among four people.

S: They each should get \$2.50 because \$2.50 + \$2.50 = \$5 and \$2.50 + \$2.50 = \$5 and \$5 + \$5 = \$10.

T: Who can write "\$2.50" on the board?

\$2.50

T: The dot separates the dollars and the cents.
Now suppose we want to share \$10 equally among eight people.
How much money would we give each person?

Your students might suggest several incorrect answers. Perhaps someone will suggest \$1.50. A sample dialogue is given here.

W 5

S: $1.50.

T: To give $1.50 to eight people how much money do we need?

Write the multiplication problem on the board.

$$8 \times \$1.50$$

T: Suppose we give each person one $1-bill and one 50¢-piece.
How many $1-bills do we need? (Eight)
How much monev is eight $1-bills? ($8)

$$8 \times \$1.50 = \$8 +$$

T: How many 50¢-pieces do we need? (Eight)
How much money is eight 50¢-pieces?

S: $4, because two 50¢-pieces make $1.

$$8 \times \$1.50 = \$8 + \$4$$

T: To give $1.50 to eight people how much money do we need?

S: $12.

Complete the number story on the board.

$$8 \times \$1.50 = \$8 + \$4 = \$12$$

T: $1.50 is too much to give each person because we have just $10 to share.

S: Give each of them $1.25.

T: How did you decide that each person should be given $1.25?

S: To share $10 among four people, we give each person $2.50. So I divided each of four shares in half. $1 is one-half of $2 and 25¢ is one-half of 50¢.

Write the multiplication problem on the board.

$$8 \times \$1.25$$

T: Suppose we give each person one $1-bill and one quarter.
How many $1-bills do we need? (Eight)
How much money is eight $1-bills? ($8)
How many quarters do we need? (Eight)
How many dollars are eight quarters?

S: Four quarters are $1, so eight quarters are $2.

Add this information to the number story forming on the board.

$$8 \times \$1.25 = \$8 + \$2$$

T: To give $1.25 to eight people, how much money would we need?

S: $10.

$$8 \times \$1.25 = \$8 + \$2 = \$10$$

Conclude that if $10 is shared equally among eight people, each person receives $1.25.

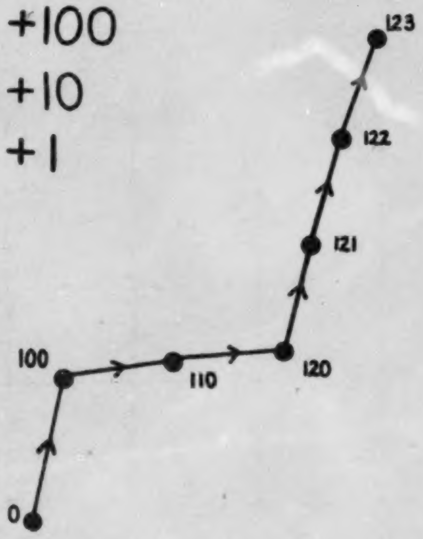
Distribute the students' copies of the Workbook GALAXY OF PROBLEMS #2. At the end of the lesson, collect the workbooks for your review.

GALAXY OF PROBLEMS

#2

Galaxy of Problems #2

Build a red-blue-green road from 0 to 123.


+100
+10
+1
123
122
121
100
110
120
0

2

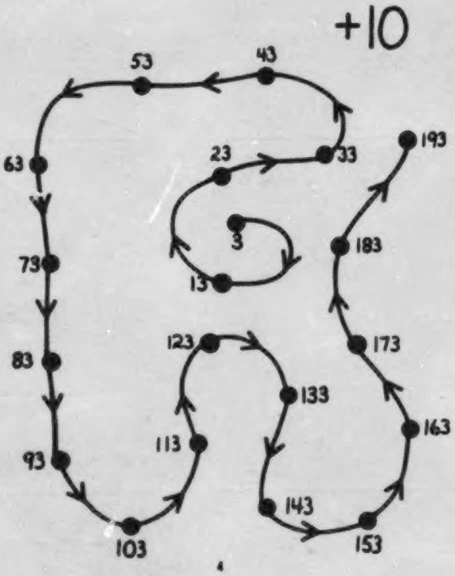
Other solutions are possible; however, a shortest solution has one red, two blue, and three green arrows.

Complete this addition table.

+	10	20	30	40
10	20	30	40	50
9	19	29	39	49
8	18	28	38	48
7	17	27	37	47

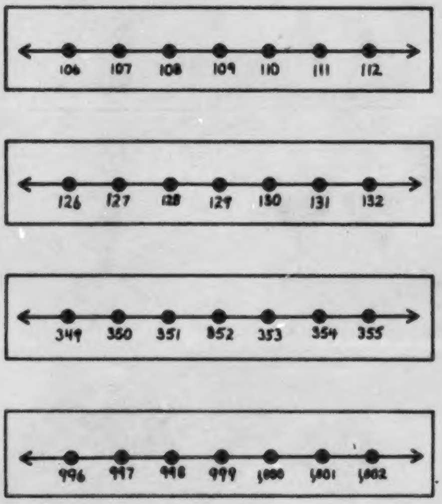
3

Label the dots.


+10
3
13
23
33
43
53
63
73
83
93
103
113
123
133
143
153
163
173
183
193

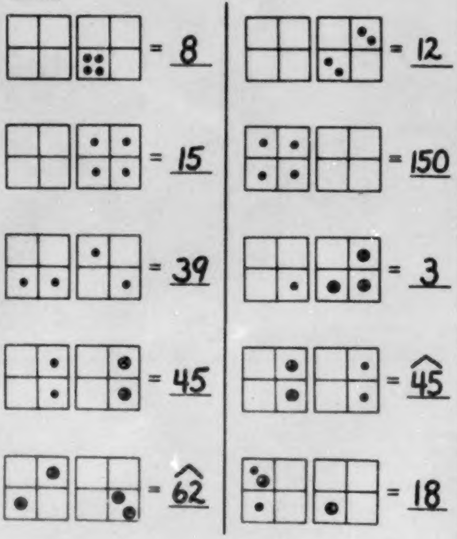
4

Label the dots on these number lines.


106
107
108
109
110
111
112
126
127
128
129
130
131
132
349
350
351
352
353
354
355
996
997
998
999
1,000
1,001
1,002

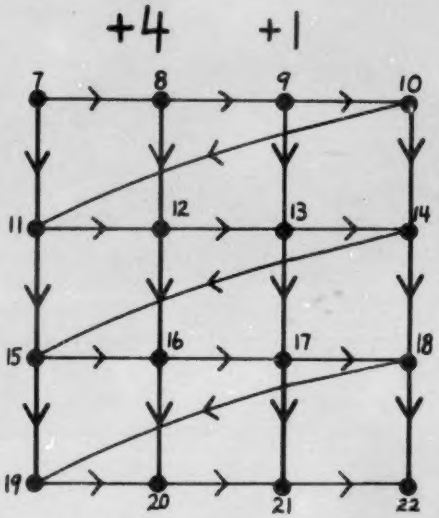
5

Put any numbers you wish on the Minicomputer using exactly four checkers for each.

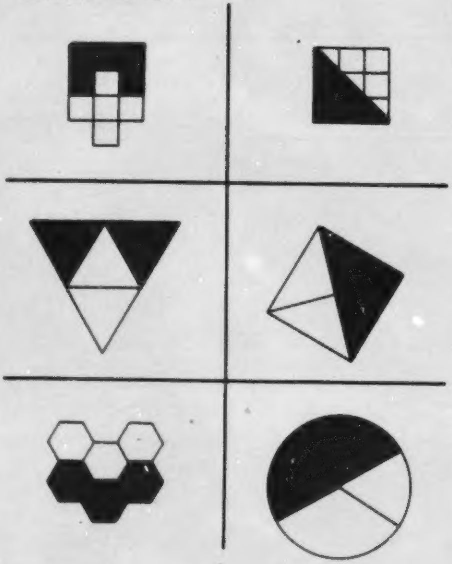

= 8
= 12
= 15
= 150
= 39
= 3
= 45
= $\widehat{45}$
= $\widehat{62}$
= 18

Many solutions are possible.

Label the dots.
Draw all the missing blue arrows.


+4
+1
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22

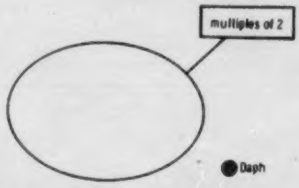
Color one-half of each shape red.



Other colorings are possible.

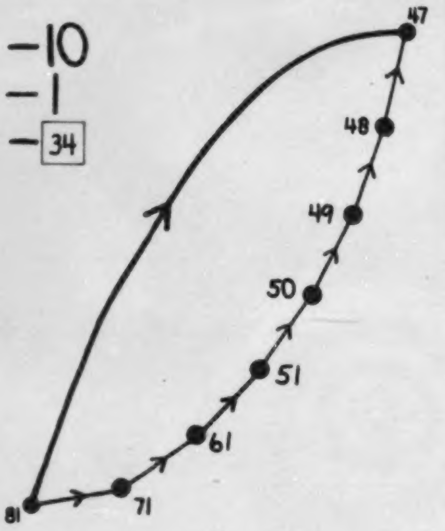
Daph is a secret number.
Daph can be put on this Minicomputer with three regular checkers on the same square.

Daph could be 3, 6, 12, or 24.


multiples of 2
Daph

Who is Daph? 3

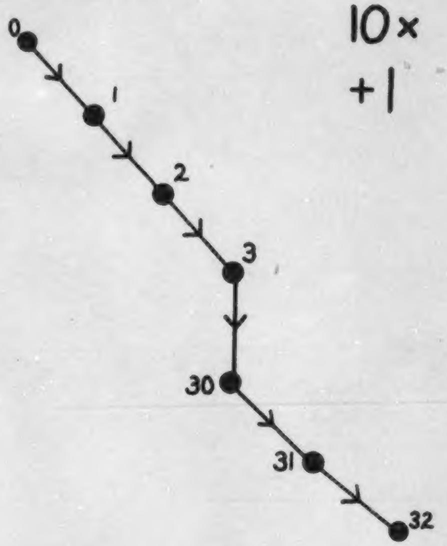
Build a red-blue road from 81 to 47.
Fill in the box for the green arrow.


−10
−1
−$\boxed{34}$
47
48
49
50
51
61
71
81

10

Other solutions are possible; however, a shortest road has three red and four blue arrows.

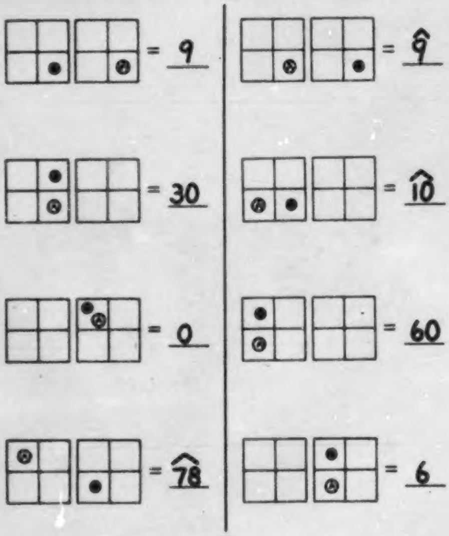
Build a red-blue road from 0 to 32.


10×
+1
0
1
2
3
30
31
32

11

Other solutions are possible; however, a shortest solution has one red and five blue arrows.

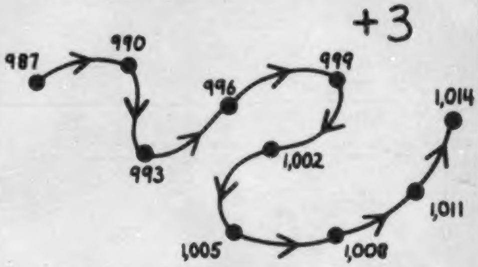
Put any numbers you wish on the Minicomputer using exactly one regular checker and one negative checker for each.


= 9
= $\hat{9}$
= 30
= $\hat{10}$
= 0
= 60
= $\hat{78}$
= 6

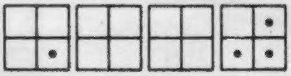
12

Many solutions are possible.

Ling is a secret number.
Ling is in this arrow picture.


+3
987
990
993
996
999
1,002
1,005
1,008
1,011
1,014

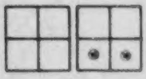
Ling can be put on this Minicomputer by taking off exactly one checker.

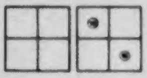


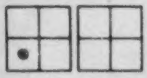
Who is Ling? 1,005

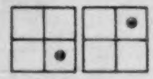
13

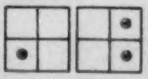
Complete.


= 30 10 × 3 = 30

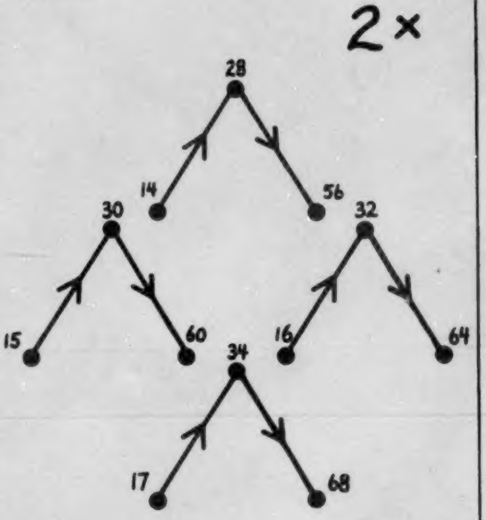

= 90 10 × 9 = 90


= 200 10 × 20 = 200


= 140 10 × 14 = 140


= 250 10 × 25 = 250

Label the dots.

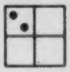

2×
28
14
56
30
32
15
60
34
16
64
17
68

Show a way to have each amount of money in change.
Many answers are possible. One is done for you.

25¢	10¢	5¢	1¢	TOTAL AMOUNT
0	4	1	1	46¢
0	1	1	2	17¢
2	1	1	0	65¢
3	2	0	5	$1.00
1	0	2	4	$0.39
2	4	0	1	$0.91

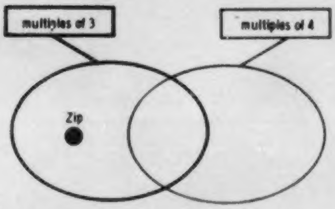
Many solutions are possible.

Zip is a secret number.
Zip can be put on this Minicomputer by adding one more regular checker.



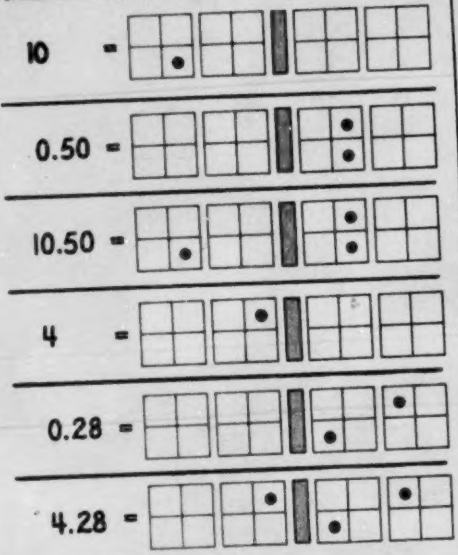
Zip could be 17, 18, 20, or 24.

Zip is in this string picture.


multiples of 3
multiples of 4
Zip

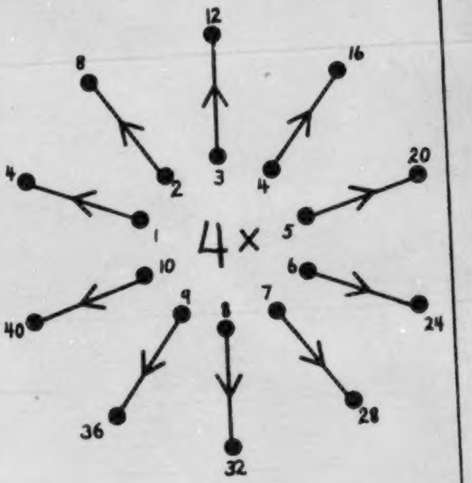
Who is Zip? 18

Put these numbers on the Minicomputer.


10 =
0.50 =
10.50 =
4 =
0.28 =
4.28 =

Standard configurations are given here although there are many other possibilities.

Label the dots.

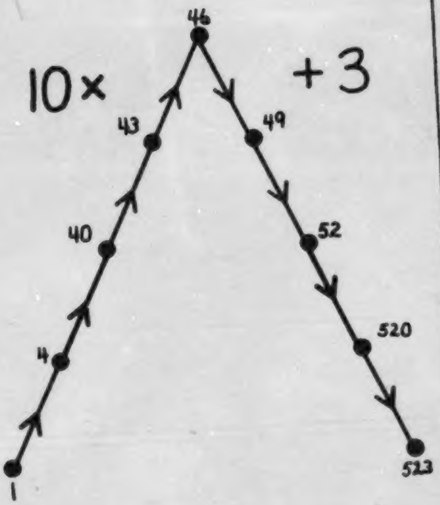

4×
12
8
16
3
2
4
20
4
1
5
6
10
9
8
7
40
24
36
28
32

NOTE: There are two dots in this picture for 4 and for 8. This situation occurs here in order not to detract from the patterns in the picture.

Complete this addition table.

+	5	0	$\hat{2}$	10
$\hat{1}$	4	$\hat{1}$	$\hat{3}$	9
$\hat{5}$	0	$\hat{5}$	$\hat{7}$	5
2	7	2	0	12
$\hat{3}$	2	$\hat{3}$	$\hat{5}$	7

Label the dots.


10×
+3
46
43
49
40
52
4
520
1
523

Complete these number stories.

$(7 + 8) \times 2 = \underline{30}$

$7 + (8 \times 2) = \underline{23}$

$8 + (7 \times 2) = \underline{22}$

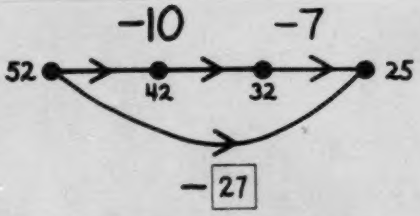
$(3 \times 5) - 4 = \underline{11}$

$3 \times (5 - 4) = \underline{3}$

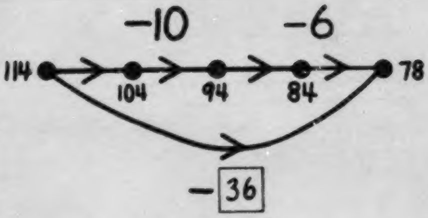
$(3 \times 4) - 5 = \underline{7}$

Label the dots.

In each picture fill in the box for the green arrow.



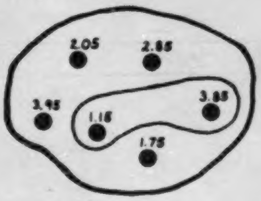
−10
−7
52
42
32
25
−27



−10
−6
114
104
94
84
78
−36

Lisa buys two flags and spends exactly $5.

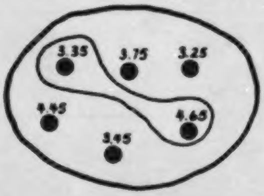
Draw one string around the prices of these two flags.



2.05
2.85
3.95
1.15
3.85
1.75

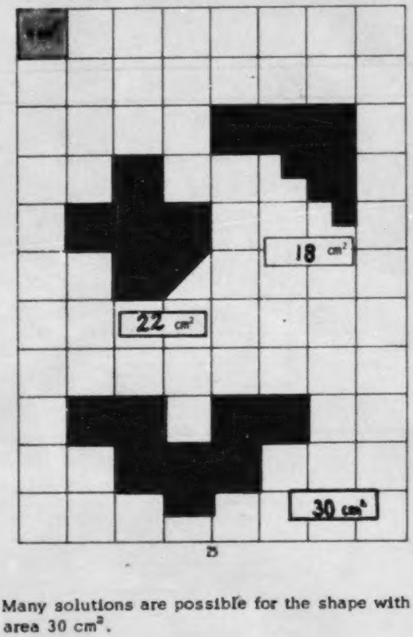
Charles buys two candles and spends exactly $8.

Draw one string around the prices of these two candles.



3.35
3.75
3.25
4.45
4.65
3.45

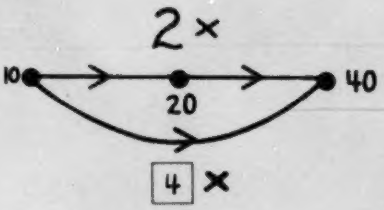
Find the area of each square. Color a shape whose area is 30 cm².

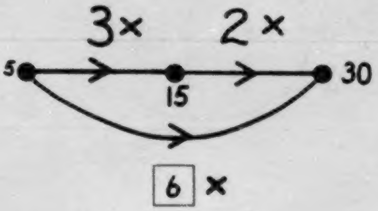


18 cm²
22 cm²
30 cm²

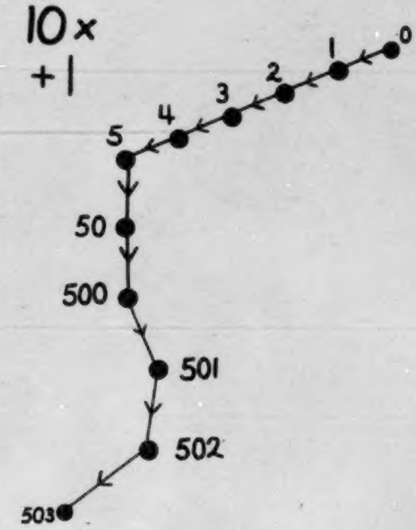
Many solutions are possible for the shape with area 30 cm².

Label the dots.
In each picture fill in the box for the green arrow.



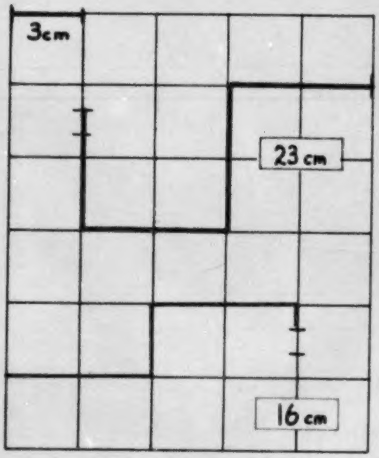


Build a red-blue road from 0 to 503.


10×
+1
0
1
2
3
4
5
50
500
501
502
503

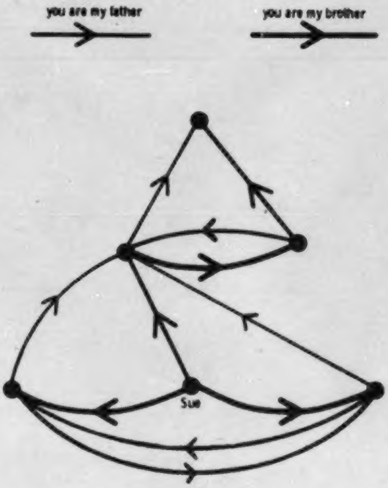
Other solutions are possible; however, a shortest road has two red and eight blue arrows.

Find the length of the red zigzag path.
Draw a blue zigzag whose length is 16 cm.

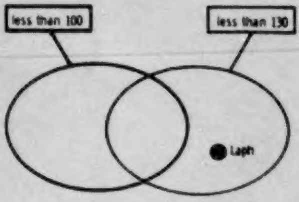

3cm
23 cm
16 cm

Many solutions are possible for a zigzag path of length 16 cm.

This picture shows some people in Sue's family.
Draw the missing red and blue arrows.

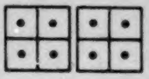

you are my father
you are my brother
Sue

Laph is a secret number.
Laph is in this string picture.



Which is the largest whole number Laph could be? 129
Which is the smallest whole number Laph could be? 100

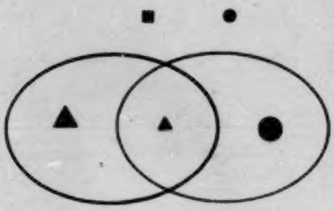
Laph can be put on this Minicomputer by taking off exactly one checker.



Laph is 125.

These are the possible string labels.

RED	YELLOW	BLUE	GREEN
NOT RED	NOT YELLOW	NOT BLUE	NOT GREEN
○	□	△	BIG
NOT ○	NOT □	NOT △	LITTLE



The red string is for △.

The blue string is for Green.

Put in parentheses to get as small a number as possible.

5 +(3 × 2) = 11

(4 × 4) + 6 = 22

7 ×(2 − 1) = 7

8 −(4 × 2) = 0

9 +(3 × 5) = 24

W6 Who are Kit and Kat?

CAPSULE LESSON SUMMARY

Name some even numbers smaller than 5,000; include the largest such number. Solve a detective story in which the clues involve a string picture and two arrow pictures.

MATERIALS

Teacher: Colored chalk
Student : None

DESCRIPTION OF LESSON

Exercise 1

Ask the class to name many even numbers. A possible dialogue is given here.

T: What are some even numbers?

S: 124.

S: 16.

S: 100.

T: What are some small even numbers?

S: 2.

S: 4.

S: 0.

T: What are some large even numbers?

S: 1,006.

S: 990.

S: 2,004.

T: What is an even number larger than 2,004?

S: 4,004.

T: What is an even number larger than 4,004 but less than 5,000?

S: 4,098.

Begin a sequence of inequalities on the board.

$$4{,}004 < 4{,}098$$

T: 4,098 is more than 4,004. What is an even number which is more than 4,098 but less than 5,000?

S: 4,910.

$$4{,}004 < 4{,}098 < 4{,}910$$

T: 4,910 is more than 4,098 and less than 5,000. What is the largest even number less than 5,000?

Possibly the first number suggested will not be the largest, but record it anyway; for example,

S: 4,980.

$$4{,}004 < 4{,}098 < 4{,}910 < 4{,}980$$

T: 4,980 is more than 4,910 and less than 5,000, but it isn't the largest even number less than 5,000.

S: 4,998 is the largest.

Record "4,998" and "5,000" in the list on the board.

$$4{,}004 < 4{,}098 < 4{,}910 < 4{,}980 < 4{,}998 < 5{,}000$$

T: 4,998 is the largest even number less than 5,000. Is 5,000 an even number?

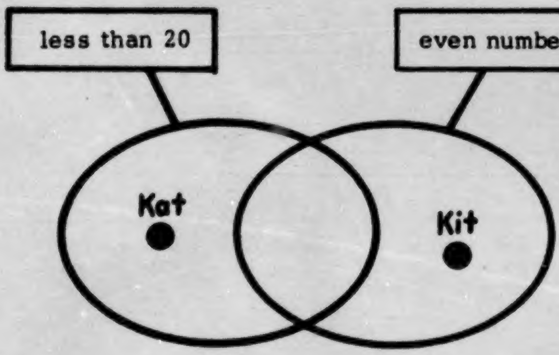
S: Yes, because it ends in 0.

Exercise 2: Detective Story

T: Today I have two secret numbers for you to discover. The names of the secret numbers are "Kit" and "Kat". I'll give you some clues.

First clue

Draw this string picture on the board.



W6

T: What can you tell me about the numbers in the blue string?

S: They are even numbers.

S: They are multiples of 2.

T: Kit is in the blue string, but is not in the red string. So what do we know about Kit?

S: Kit is an even number.

S: Kit is more than 20.

T: Is that correct? Must Kit be more than 20? (No)
What is the smallest number Kit could be?

S: 20, 20 is not less than 20.

If the answer is not forthcoming, prompt the students by asking if Kit could be 20.

T: What are some numbers Kit could be?

Encourage students to respond. Kit could be any even number larger than 19.

T: What can you tell me about Kat?

S: Kat is less than 20.

S: Kat is not an even number.

S: Kat is an odd number.

T: What are some numbers Kat could be?

Encourage students to respond. Kat could be any odd number less than 20.

T: What is the largest number Kat could be?

S: 19.

T: What is the smallest number Kat could be?

A possible dialogue is given here.

S: Kat could be $\widehat{19}$.

T: $\widehat{19}$ is an odd number and less than 20. Is $\widehat{19}$ the smallest number Kat could be?

S: Kat could be $\widehat{4,009}$.

T: $\widehat{4,009}$ is an odd number, but is $\widehat{4,009}$ less than 20?

S: $\widehat{4,009}$ is less than 20. In fact, all the negative numbers are less than 20.

T: Is $\widehat{4,009}$ the smallest number Kat could be?

S: No.

Conclude that there is no smallest number Kat could be, because no matter what number is suggested, someone else can always find an even smaller odd number.

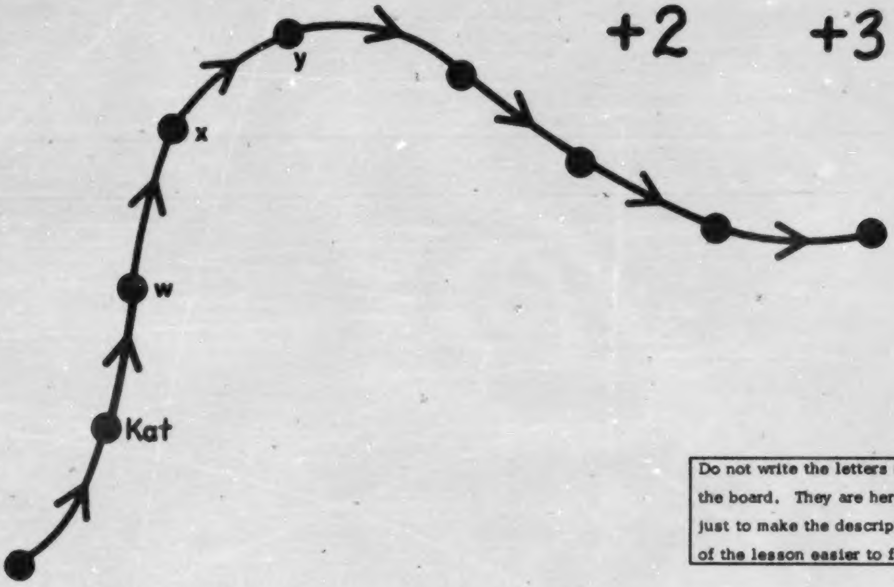
T: Which number is larger, Kat or Kit?

S: Kit.

W 6

Second clue

Draw this arrow picture on another part of the board. Do not erase the string picture.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

T: Kat and Kit are in this arrow picture. Are there any odd numbers in this arrow picture?

S: Yes, Kat is an odd number.

Point to the dot for Kat. A possible dialogue is given here.

T: We know that Kat is an odd number, so we know that this number (point to Kat) is an odd number.

Write "O" near the dot for Kat.

T (pointing to w): Is this dot for an even number or for an odd number?

S: An odd number, because if we start counting by twos from an odd number we always get odd numbers.

Write "O" near w.

T (pointing to x): Is this number even or odd?

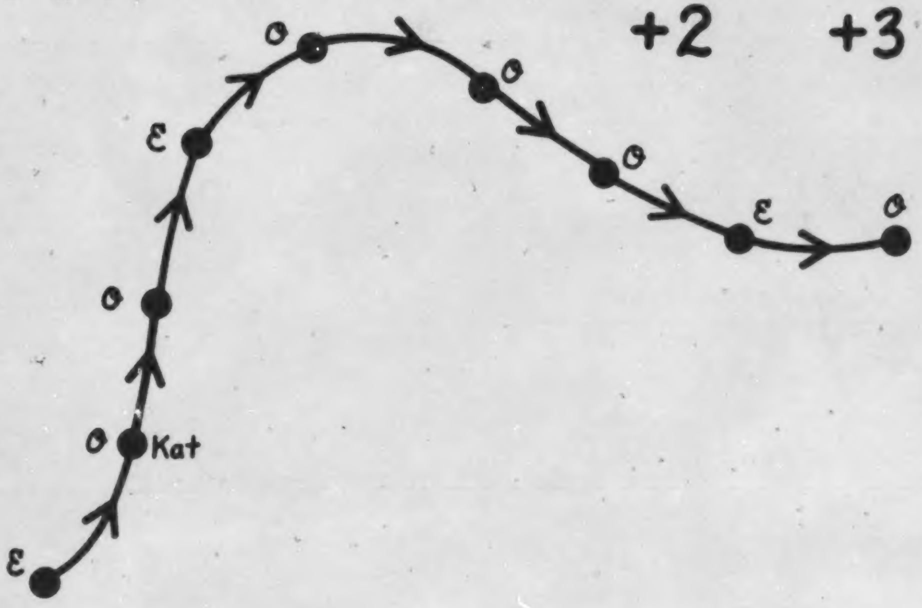
S: If this dot (pointing to w) were for 11, then this dot (x) would be for 14. 14 is an even number.

Ask the students to repeat this activity several times by assigning various odd numbers to w and deciding if the number at x is an odd or an even number. Conclude that the number at x is even. Write "E" near x. Point to y.

T: Is this number even or odd?

S: Odd, because an even number plus 3 equals an odd number.

Write "O" near y. Continue this activity until all the dots are labeled either "O" or "E".



W6

T: Kit is in this arrow picture. Where could Kit be?

A student should point to the three dots labeled "ε".

T: How do you know that Kit is one of those numbers?

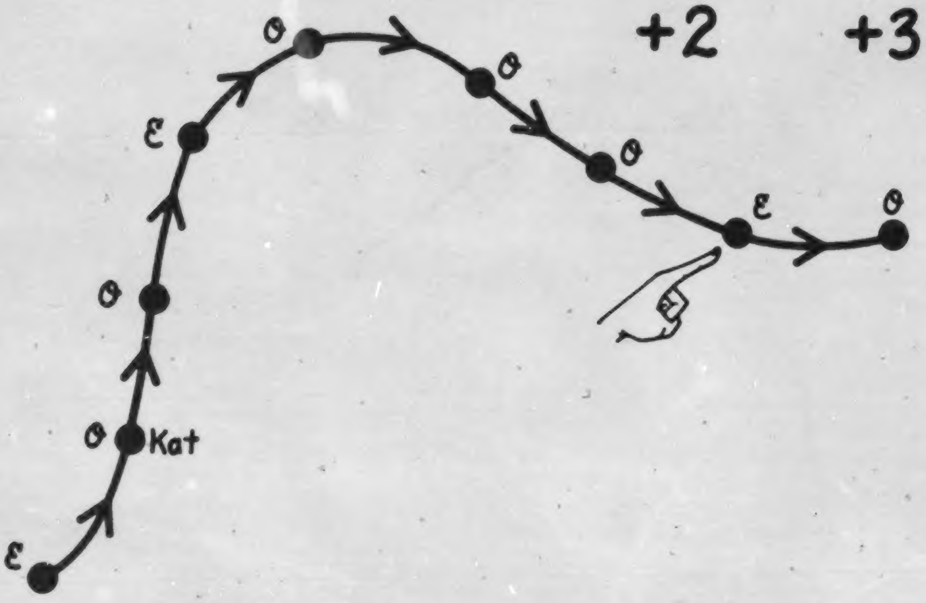
S: Kit is an even number.

Perhaps someone will notice that Kit cannot be the first number in the arrow road because Kit is larger than Kat and the blue arrow adds 3.

Third clue

T: Here is another clue. Kit is the largest even number in this arrow picture. Which of these dots is for Kit?

Ask a student to point to the dot for Kit.



Write "Kit" near that dot (see the above illustration).

730

T: How do you know that this is the largest even number in this arrow picture?

S: All the arrows are for +2 or +3, so the numbers get larger as you move to the right.

Draw this table near the arrow picture.

Kat	Kit

T: Let's try to find some pairs of numbers that could be Kit and Kat. Give me a number that Kat could be.

Accept any odd number less than 20. A possible dialogue is given here.

S: Kat could be 5.

Write "5" near the dot for Kat in the arrow picture. Point to the dot for Kit in the arrow picture.

T: If Kit were 5, what number would Kit be?

Allow several students to whisper their answers to you before asking a student to answer aloud.

S: 20.

W 6

Write "20" near the dot for Kit in the arrow picture.

T: How did you know that Kit would be 20 if Kat were 5?

S: 5 + 2 = 7; 7 + 3 = 10; 10 + 3 = 13; 13 + 2 = 15; 15 + 2 = 17; and 17 + 3 = 20.

T: This arrow picture tells us that if Kat were 5, then Kit would be 20. (Point to the string picture.) Could Kit be 20?

S: Yes, 20 is not less than 20 and 20 is an even number.

Record "5" and "20" in the table on the board.

Kat	Kit
5	20

T: What is another number Kat could be?

Let students make suggestions and collectively check each one. If a student makes an incorrect suggestion, ask what number Kit would be and where that number belongs in the string picture. For example, suppose a student suggests that Kat could be $\hat{1}$.

S: $\hat{1}$.

T: If Kat were $\hat{1}$, what would Kit be?

When the students determine that Kit would be 14, refer to the string picture.

T: Could Kit be 14?

S: No, because 14 is less than 20 and belongs in the red string.

T: If Kit can't be 14, can Kat be $\hat{1}$?

S: No.

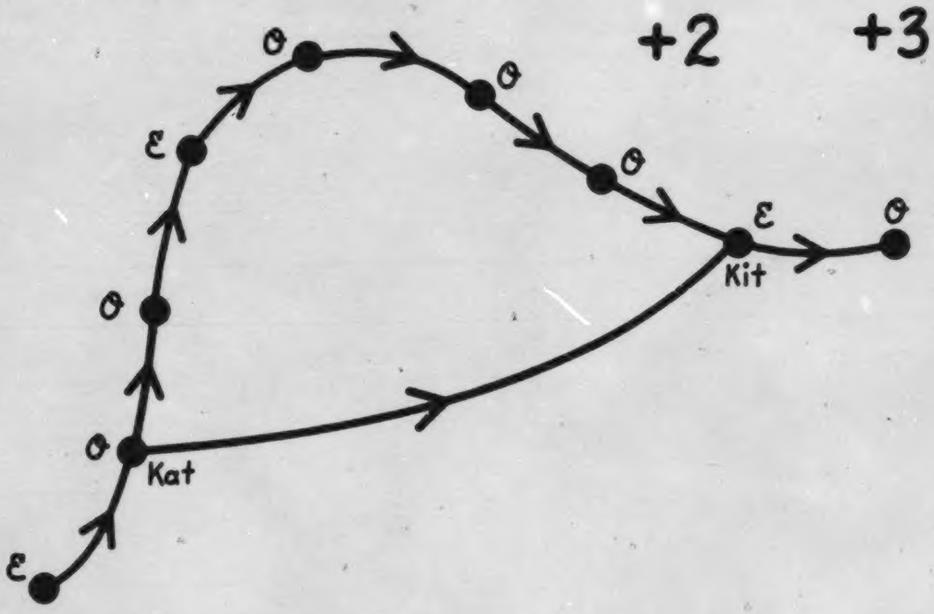
Do not record any incorrect pair, such as $\hat{1}$ and 14, in the table.

Continue this activity until several pairs of numbers have been recorded in the table. All of the possible pairs are recorded in the table below for your information.

Kat	Kit
5	20
7	22
9	24
11	26
13	28
15	30
17	32
19	34

W 6

Draw a green arrow from Kat to Kit in the arrow picture.


θ
+2
+3
θ
ε
θ
ε
θ
Kit
θ
θ
Kat
ε

T: What could this arrow from Kat to Kit be for?

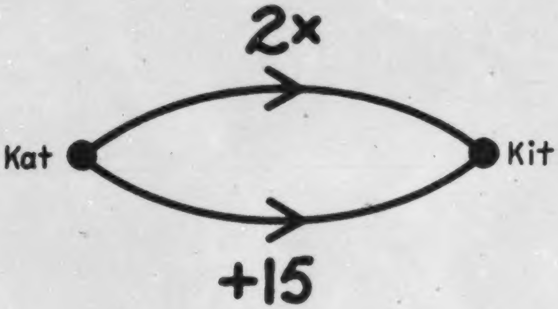
S: +15.

Label the green arrow "+15".

Fourth clue

T: We know that Kit is 15 more than Kat. Here is a new clue: Kit is twice as much as Kat.

Draw this arrow picture on the board.


2x
Kat
Kit
+15

T: 2 × Kat = Kit and Kat + 15 = Kit.
Which numbers are Kat and Kit?

Allow the students to whisper their answers to you. Do not indicate if the students are answering correctly or incorrectly. Perhaps a student will incorrectly identify Kat as 9. A possible dialogue is given here.

T: Someone told me that Kat is 9. Is that possible?

Write "9" near the dot for Kat.

T (tracing the orange arrow): If Kat is 9, then Kit would be 2 × 9. What number is 2 × 9?

S: 18.

Write "18" near the dot for Kit.

T (tracing the green arrow): Does 9 + 15 = 18?

S: No.

Erase "9" and "18" from the arrow picture.

T: This arrow picture tells us that 2 × Kat is the same number as Kat + 15. 2 × 9 is not the same number as 9 + 15, so Kat cannot be 9.

If 9 and 24 are recorded in your table as a pair of numbers which could be Kat and Kit, cross them out. Explain that if Kat cannot be 9, then Kit cannot be 24.

T: Which numbers are Kat and Kit?

W 6

Continue this activity until someone suggests that Kat is 15 and Kit is 30. Write "15" near the dot for Kat.

T: If Kat is 15, then 2 × 15 is the same number as 15 + 15. What number is 2 × 15?

S: 30.

Label the dot for Kit "30".

T: Is 15 + 15 = 30?

S: Yes.

Conclude that Kat is 15 and Kit is 30.

W7 TO PICTURE (Lesson 1)

CAPSULE LESSON SUMMARY

In the Story-workbook TO PICTURE, find where eight numbers belong in an arrow picture. Discover that a second arrow picture contains a disguise of the first picture and use this to help label the dots.

MATERIALS

Teacher : Story-workbook TO PICTURE; colored chalk
Student : Story-workbook TO PICTURE; pencil

DESCRIPTION OF LESSON

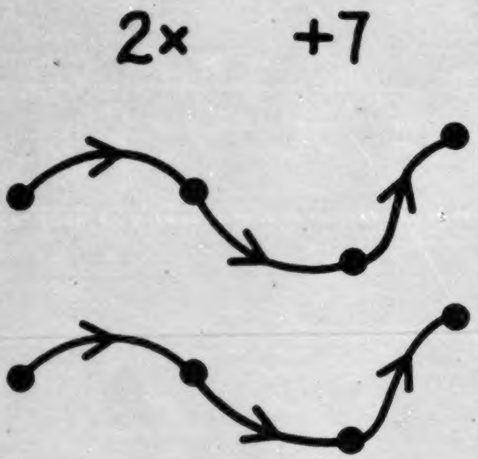
Distribute copies of the Story-workbook TO PICTURE to the students. Ask them to follow along as you, or students, read the story and not to read ahead individually.

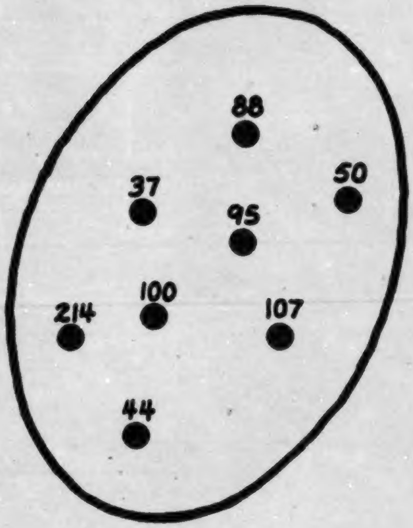
Encourage your class to discuss the problems confronted by the number friends and to explain the solutions that they propose. Additional comments on specific pages of the Story-workbook TO PICTURE are provided here.

Pages 3, 4, and 5

Each of the numbers on page 5 belongs in the arrow picture on page 4. This problem is similar to problems the students have solved in "Fishing for Numbers" activities. Ask the students to label the dots on page 4, but emphasize that they should only use the numbers on page 5.

While the students are working, copy the arrow picture from page 4 on the board. Near the arrow picture draw a string and put the eight numbers in it as on page 3.





Pages 6 - 10

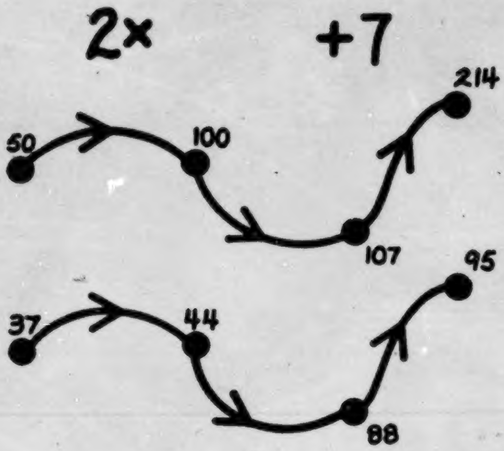
Encourage the students to comment about the errors on page 7.

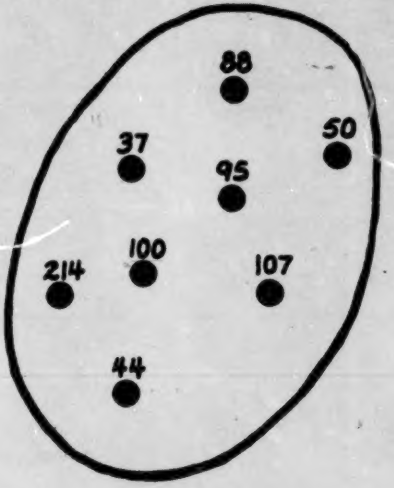
After page 10 has been read, ask the students where the smallest number is in each arrow road and why. Students very likely will tell you that the starting dot of each arrow road is for the smallest number because multiplying a number by 2 gives a larger number and adding 7 to a number gives a larger number.

NOTE: Doubling a number does not necessarily result in a larger number; for example, $2 \times \hat{5} = \widehat{10}$ and $\hat{5} > \widehat{10}$. In this arrow picture, we know that all the numbers are positive, so the starting numbers of the red arrows must be smaller than their ending numbers. This is for your information only.

Pages 11 and 12

Allow a few minutes for the students to label the dots on page 12. After a few minutes invite a student to label the dots in the arrow picture on the board. Check that each of the numbers in the string picture is in the arrow picture.

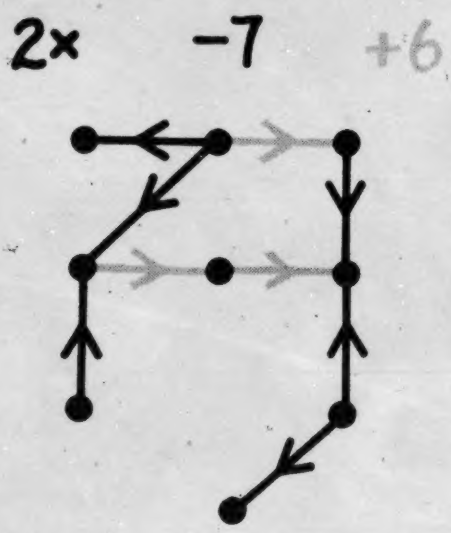

2×
+7
214
50
100
107
95
37
44
88

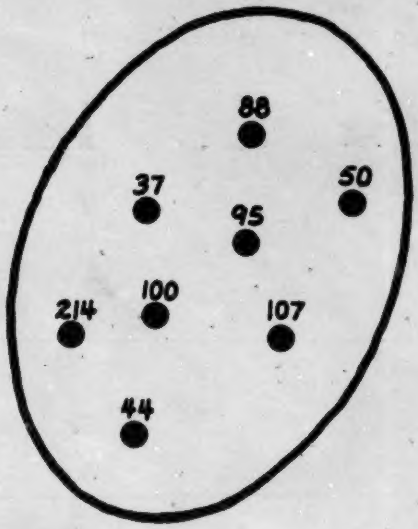

88
37
50
95
100
214
107
44

Ask the students if this is the only way that the eight number friends in the story can arrange themselves in the arrow picture. Consider any different arrangements that are proposed by the students and conclude that there is only one correct way to label the dots in the arrow picture with the eight numbers in the story.

Pages 13, 14, and 15

While the students are labeling the dots on page 14, erase the first arrow picture and copy the arrow picture from page 14 on the board near the string picture. Use white chalk to draw the arrows shown in gray below.




88
37
50
95
100
214
107
44

W 7

Pages 16 and 17

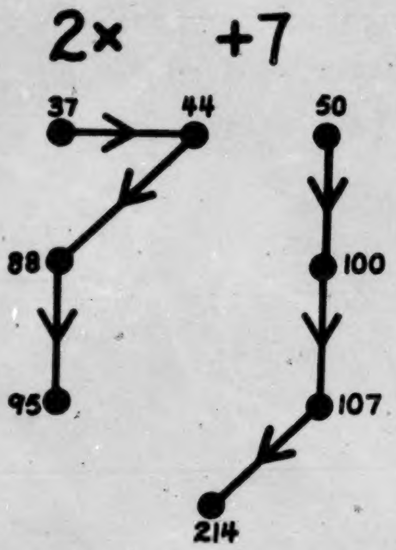
Erase the gray (white) arrows in the picture on the board. Trace one of the green arrows.

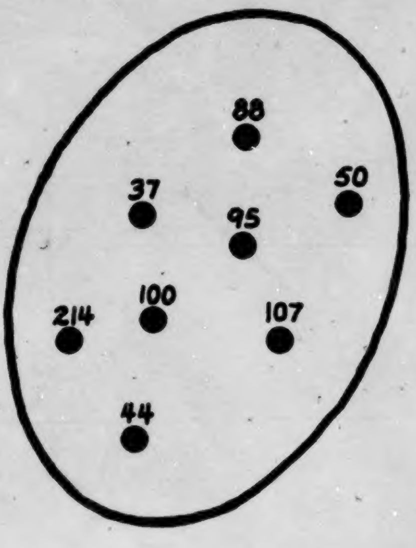
T: If this - 7 arrow points this way, which way should the + 7 arrow point?

S: The opposite way.

Erase the - 7 arrow and replace it with a blue + 7 arrow. Then, upon class agreement, replace all the - 7 arrows with + 7 arrows. The arrow picture on the board should look like the arrow picture on page 17. Ask the students to compare the arrow picture from page 4 with the arrow picture on the board. Emphasize that both arrow pictures have an arrow road with a blue arrow between two red arrows and an arrow road with a red arrow between two blue arrows.

The students can use page 4 to help them label the dots on page 17. After a few minutes invite a student to label the dots in the arrow picture on the board.





If the students had difficulty labeling the dots on page 14, ask them to return to the preceding page and label the dots before continuing.

Pages 18 - 21

Page 21 is a good place to end the lesson. Collect the story-workbooks and keep them for use in Lesson W8.

W8 TO PICTURE (Lesson 2)

CAPSULE LESSON SUMMARY

Finish the Story-workbook TO PICTURE. Discover that making one backward play and one forward play does not alter the number of checkers in a configuration on the Minicomputer and, of course, does not change the number displayed.

MATERIALS

Teacher :	Story-workbook TO PICTURE; Minicomputer set
Student :	Story-workbook TO PICTURE; paper; pencil

DESCRIPTION OF LESSON

Distribute the students' copies of the Story-workbook TO PICTURE. Ask the students to turn to page 22.

Pages 22 and 23

Read or ask students to read pages 22 and 23 aloud; then ask the students to determine which number is on each Minicomputer. Students who finish quickly should be encouraged to continue until they complete page 25.

When most of the students have completed pages 22 and 23, ask everyone to compare the numbers on these pages with the numbers in the arrow picture on page 20.

Pages 24 and 25

Read or ask students to read pages 24 and 25 aloud.

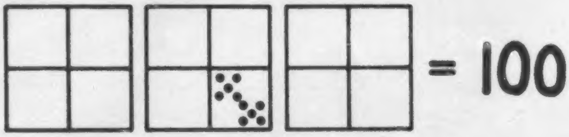
You may need to emphasize that the students are to put 100 on the Minicomputer using exactly ten regular checkers. Allow about five minutes for individual work on page 25. Students who finish quickly should try to find other configurations for 100 with ten checkers and to record them on a separate piece of paper.

Invite students one at a time to put 100 on the demonstration Minicomputer using exactly ten checkers. Continue this activity until four or five configurations have been shown.

Pages 26 and 27

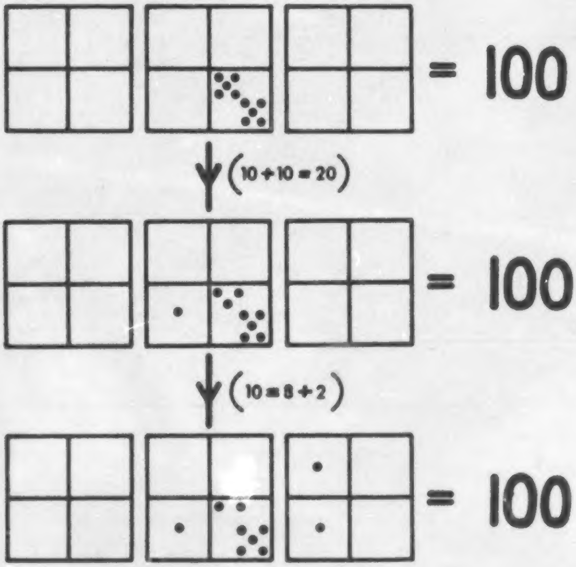
Ask the students to look carefully at pages 26 and 27 to see if they can determine what game 100 is playing.

Put this configuration on the Minicomputer.



T: Who can make some plays so our Minicomputer will have checkers on the same squares as the second Minicomputer on page 26?

The volunteer should make one forward play and one backward play. Either play may be made first.



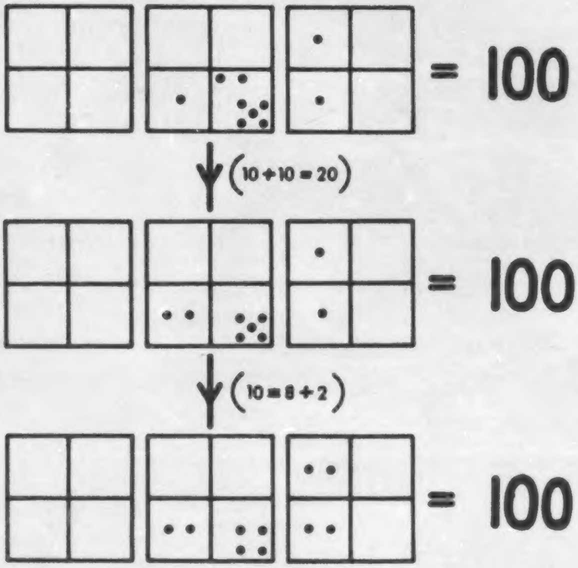
W 8

T: 100 made one forward play and one backward play. Are there still ten checkers on the Minicomputer?

S: Yes.

T: Who can make some plays so our Minicomputer will have checkers on the same squares as the third Minicomputer on page 26?

The volunteer should make one forward and one backward play.



T: So 100 made one forward play and one backward play. Are there still ten checkers on the Minicomputer?

S: Yes.

Continue this activity with the configurations on pages 26 and 27.

T: Do you understand 100's game?

S: 100 always makes one forward play and one backward play so that there are still ten checkers on the Minicomputer.

T: Could we continue this game?

S: Yes.

Invite students to play 100's game several times on the Minicomputer. Each student should make one forward and one backward play.

Ask the students to find several ways to put 88 on the Minicomputer using exactly twelve regular checkers and to show them on page 27. Students who finish quickly should draw other configurations with twelve checkers for 88 on a separate piece of paper. Allow a few minutes for individual work and then invite students to show some of these configurations on the demonstration Minicomputer. Continue this activity until four or five configurations have been shown.

Page 28

88 is playing 100's game. Put the first configuration on the demonstration Minicomputer and invite a student to make plays so that the demonstration Minicomputer has checkers on the same squares as the second Minicomputer on page 28. The student should make a "10 + 10 = 20" play and a "2 = 1 + 1" play.

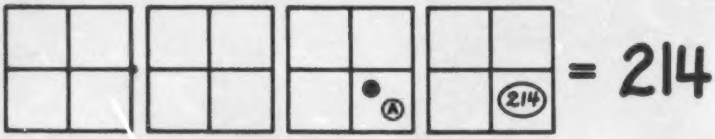
Continue this activity for the other configurations on page 28. If your students understand and enjoy 100's game, they may wish to continue playing the game a few more times.

W8

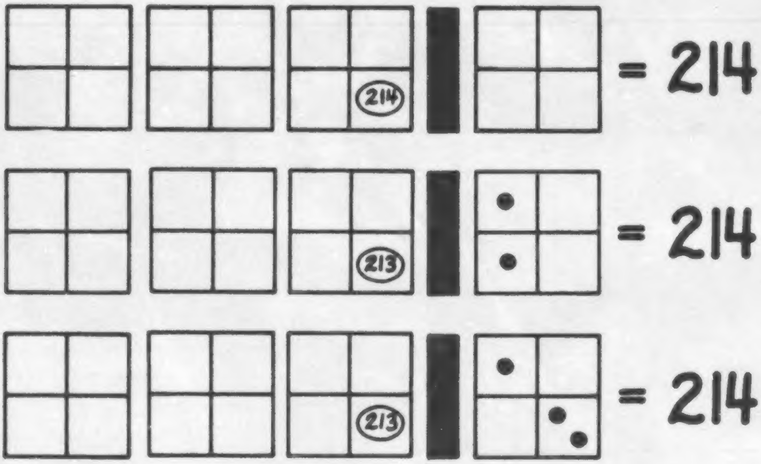
Pages 29 - 32

Ask the students to read and complete pages 29 - 32 independently. If your students are interested, you may wish to explore the possibility of putting 214 on the Minicomputer with more than 214 checkers.

NOTE: There are two ways which your students might suggest to use more than 214 checkers to put 214 on the Minicomputer. One way is to use negative checkers such as was done in the Story-workbook ROLLERSKATING 37 (see lessons W11 and W12 in UPG-III). For example,

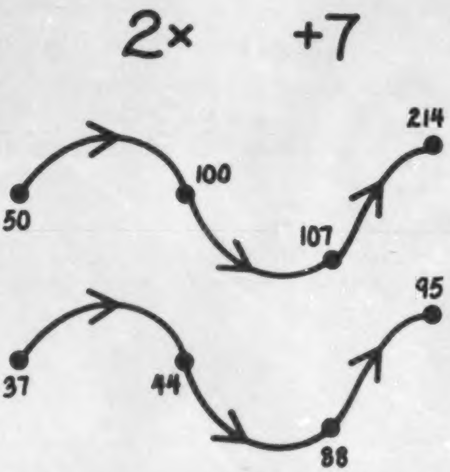


The other way is to draw a green bar, add boards to the right of it, and make backward plays. For example, 214 can be shown as 213 + 0.8 + 0.2 and as 213 + 0.8 + 0.1 + 0.1.



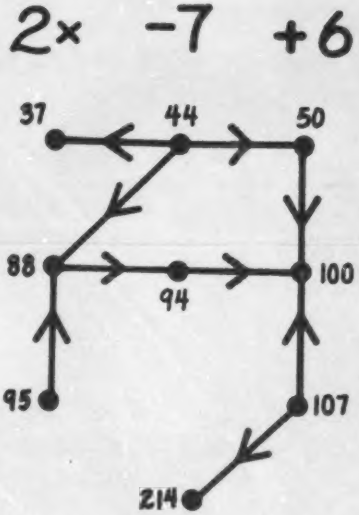
TO PICTURE

There was a strange picture drawn on the ground.


2×
+7
50
100
107
214
37
44
88
95

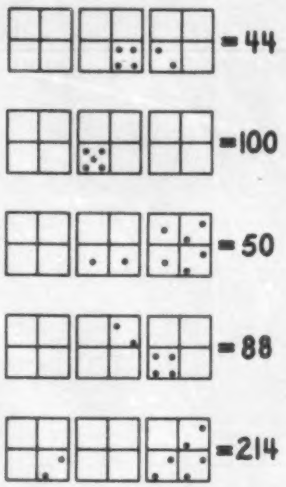
4

Compare page 12 with this page. The same arrow picture with dots to be labeled appears there.


2×
−7
+6
37
44
50
88
94
100
95
107
214

14

Compare page 20 to this page.


= 44
=100
= 50
= 88
=214

DO YOU RECOGNIZE EACH OF OUR FRIENDS? WRITE THEIR NAMES TO THE RIGHT OF THE MINICOMPUTERS.

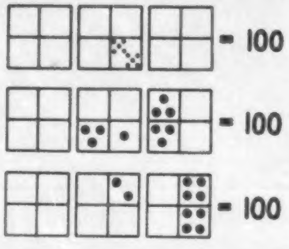
23

"Crying doesn't help," advised 100. "Let us try to find a solution."

100 thought for a while.

"There are ten checkers on my Minicomputer. I can easily use them all to show myself."

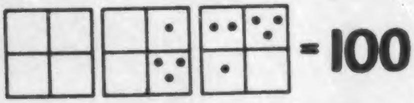
SOLVE THE PROBLEM FOR 100. BE SURE TO USE ALL TEN CHECKERS. THERE ARE MANY WAYS OF DOING IT. SHOW SOME OF THEM BELOW.



25

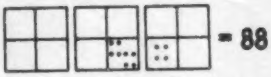
Many solutions are possible.

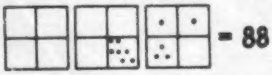
"And again."



"Your game is fun," said 88. "I will play it with my twelve checkers."

PLAY THE GAME FOR 88. BE SURE TO USE EXACTLY TWELVE CHECKERS. WRITE YOUR ANSWERS BELOW.





Many solutions are possible.

The other numbers cheered for 100 and 88.

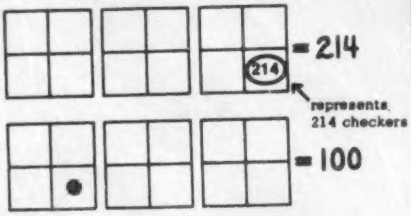
"Let's see who can use the most checkers to sit on the Minicomputer," said 95.

CAN YOU GUESS WHICH OF THESE NINE NUMBERS CAN USE THE MOST CHECKERS?

"Let's see who can use the fewest checkers to sit on the Minicomputer," suggested 37.

CAN YOU GUESS WHICH OF THESE NINE NUMBERS CAN USE THE FEWEST CHECKERS?

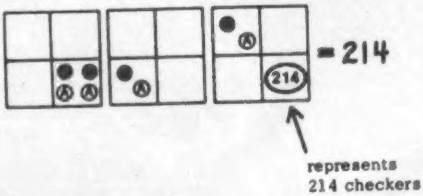
SHOW YOUR ANSWERS BELOW.



There is no largest number of checkers that can be used if you allow negative checkers. See page 31.

WHAT DO YOU THINK OF THIS PROBLEM?

DO YOU THINK IT IS POSSIBLE FOR ANY OF THESE NUMBERS TO USE EVEN MORE THAN 214 CHECKERS TO SHOW THEMSELVES ON THE MINI-COMPUTER? IF SO, DRAW YOUR SOLUTION BELOW.



Many solutions are possible, including solutions for the other numbers.

This is the end of the story of our nine friends who are happy to sign their book.

$8 \times 11 = 88$ $20 \times 5 = 100$

$\frac{1}{2} \times 428 = 214$ $50 - 6 = 44$

$(5 \times 7) + 2 = 37$ $2 \times 47 = 94$

$120 - 13 = 107$ $\frac{1}{3} \times 150 = 50$

$(2 \times 50) - 5 = 95$

DO YOU RECOGNIZE THEM? WRITE THEIR USUAL NAMES NEXT TO THEIR SIGNATURES. CAN YOU INVENT NEW SIGNATURES FOR EACH NUMBER? WRITE THEM ON A SEPARATE PIECE OF PAPER.

W9 GALAXY OF PROBLEMS #3 (Lesson 1)

CAPSULE LESSON SUMMARY

Find how much spending money Stanley brings from home to the zoo if he has 61¢ after his grandfather doubles the original amount and his grandmother adds 9¢. Begin the Workbook GALAXY OF PROBLEMS #3.

MATERIALS

Teacher : Colored chalk
Student : Workbook GALAXY OF PROBLEMS #3; colored pencils; centimeter tape; straightedge

DESCRIPTION OF LESSON

Tell the following or a similar story to your class. Choose one of your students to star in this story. Substitute that student's name for "Stanley" throughout the story.

T: Last Sunday Stanley's grandparents took him to the zoo. Stanley brought some spending money from home, and both of his grandparents gave Stanley some money to spend at the zoo. After his grandfather gave Stanley some money, Stanley had twice as much as he came with, and then Stanley's grandmother gave Stanley nine pennies more. Stanley had a total of 61¢ to spend at the zoo.

T: How much money do you think Stanley brought with him from home?

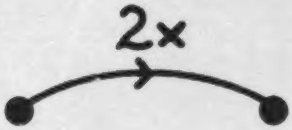
Let many students whisper their guesses to you.

T: How could we solve this problem with the help of an arrow picture?

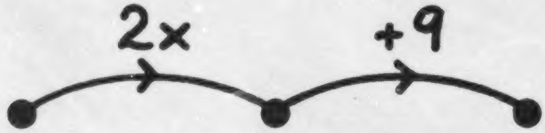
Encourage the students to tell you which arrows to draw.

T (drawing a dot on the board): This dot is for the money Stanley brought from home. Stanley's grandfather gave him some money and doubled this amount.

T: We can show this with a 2 × arrow.



T: Then Stanley's grandmother gave him nine pennies.
How much money is nine pennies? (9¢)
We can show this with a +9 arrow.



T: How much money did Stanley have after his grandmother gave him nine pennies?

S: 61¢.

Label the ending dot "61".

T: How could this arrow picture help us find out how much money Stanley brought from home?

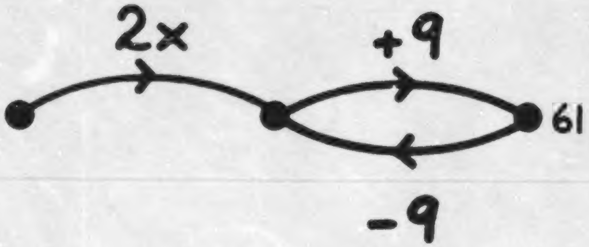
S: We need to draw the return arrows and then we can label the dots.

Draw the return arrow for the +9 arrow.

T: If the red arrow is for +9, what is the return arrow for?

S: -9.

Label the - 9 arrow.

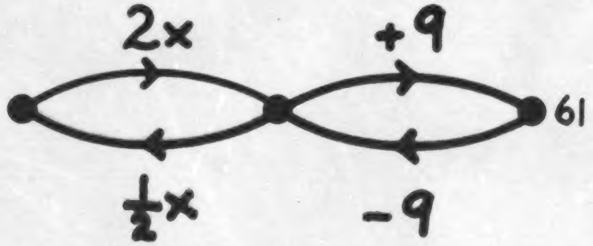


Draw the return arrow for the 2 × arrow.

T: If the blue arrow is for 2 ×, what is the return arrow for?

S: $\frac{1}{2}$ × (+2).

Label the return arrow "$\frac{1}{2}$ ×".



T: How much money did Stanley take with him from home?

Allow several students to whisper their answers to you before labeling the dots collectively.

T (tracing the green arrow): What number is 61 - 9?

S: 52.

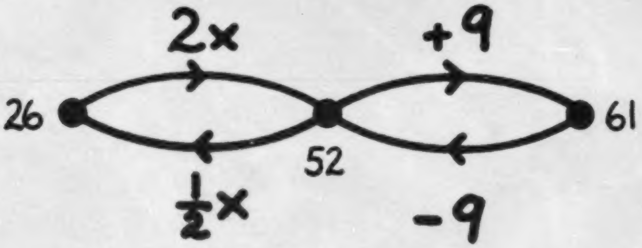
Label the dot for 52 and trace the orange arrow.

W 9

T: What number is $\frac{1}{2} \times 52$? How do you know?

S: 26. $\frac{1}{2} \times 50 = 25$ and $\frac{1}{2} \times 2 = 1$, so $\frac{1}{2} \times 52 = 26$.

Label the dot for 26.



T: How much money did Stanley take with him from home? (26¢)
How much money did Stanley's grandfather give him? (26¢)

Distribute copies of the Workbook GALAXY OF PROBLEMS #3 and let the students begin working individually. At the end of the class period, collect the workbooks for your review and for use in Lesson W10.

W10 GALAXY OF PROBLEMS #3 (Lesson 2)

CAPSULE LESSON SUMMARY

Continue working in the Workbook GALAXY OF PROBLEMS #3. (This is the second of two lessons using this workbook.) Play Minicomputer Golf.

MATERIALS

Teacher :	Minicomputer set; colored chalk
Student :	Workbook GALAXY OF PROBLEMS #3; colored pencils; centimeter tape; straightedge

DESCRIPTION OF LESSON

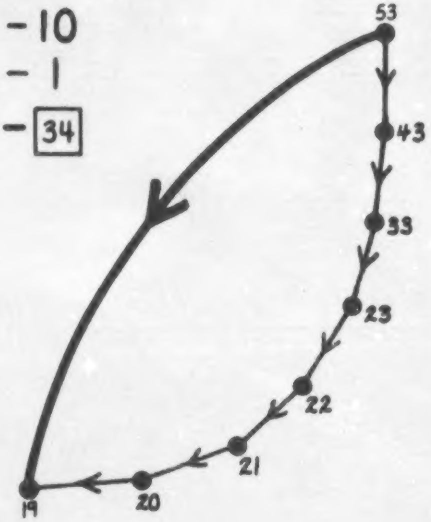
Your review of the students' workbooks may have indicated that a short collective discussion is needed about a particular page in the workbook. If appropriate, collectively solve a problem similar to the problem on that page in the workbook before distributing the students' copies of the Workbook GALAXY OF PROBLEMS #3.

Allow the students to work on the workbook for 30 to 40 minutes. Spend the remaining time playing Minicomputer Golf. For an explanation of this game, see Lesson N15.

GALAXY OF PROBLEMS
#3

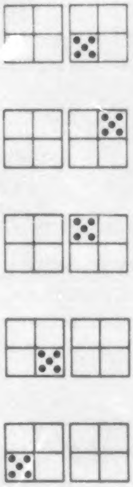
Galaxy of Problems #3

Build a red-blue road from 53 to 19.
Fill in the box for the green arrow.


-10
-1
-34
53
43
33
23
22
21
20
19

Other solutions are possible; however, a shortest road has three red and four blue arrows.

Complete.



- 10
- 20
- 40
- 50
- 100

$5 \times 2 - 10$

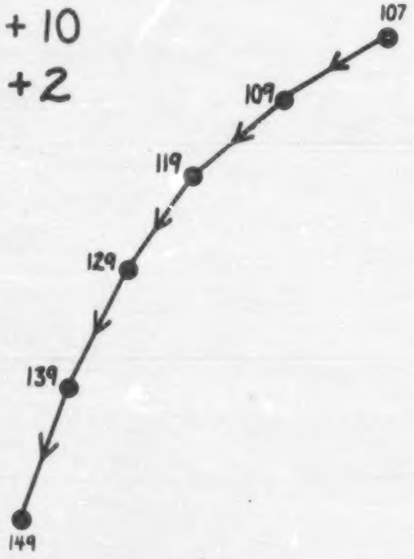
$5 \times 4 - 20$

$5 \times 8 - 40$

$5 \times 10 - 50$

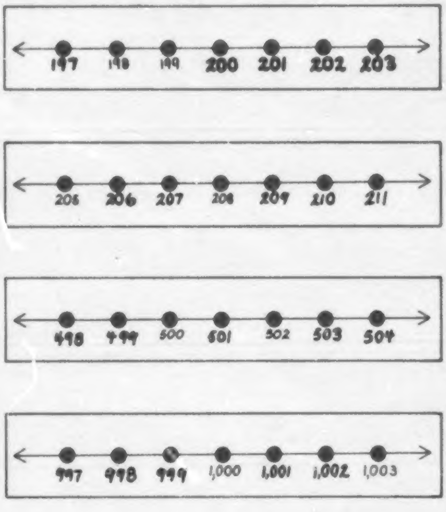
$5 \times 20 - 100$

build a blue-red road from 107 to 149

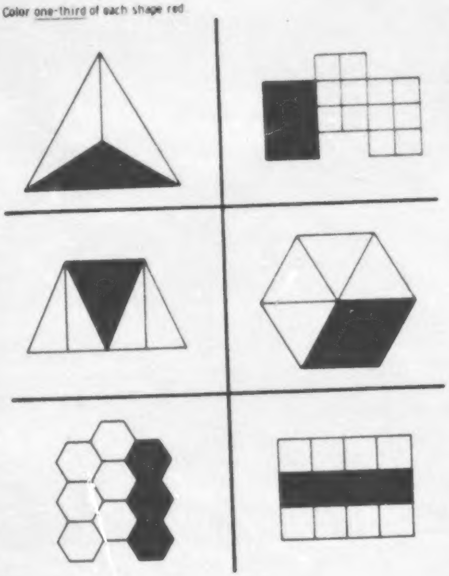

+10
+2
107
109
119
129
139
149

Other solutions are possible; however, a shortest road has one red and four blue arrows.

Label the dots on these number lines.

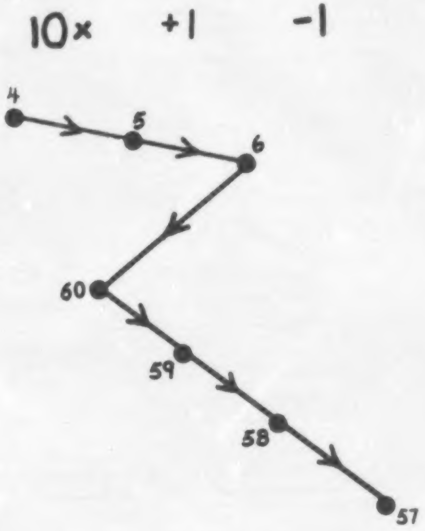

197 198 199 200 201 202 203
205 206 207 208 209 210 211
498 499 500 501 502 503 504
997 998 999 1,000 1,001 1,002 1,003

Color one-third of each shape red.



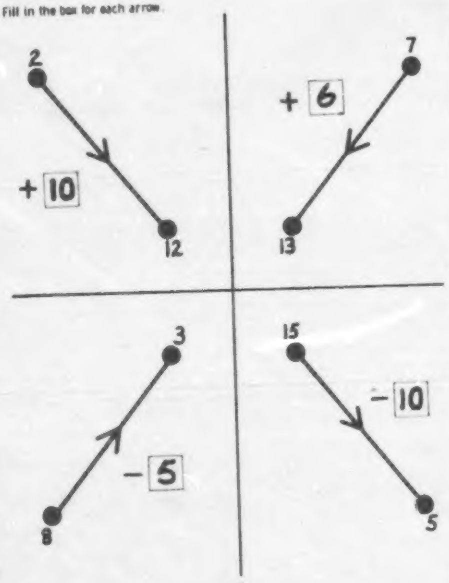
Other colorings are possible.

Build a red-blue-green road between 4 and 57. Try to use as few arrows as possible.



Other solutions are possible; however, this is a shortest road.

Fill in the box for each arrow.



Complete.

$10 - 5 = \boxed{5}$

$9 - 4 = \boxed{5}$

$8 - \boxed{3} = 5$

$\boxed{7} - 2 = 5$

$5 - 4 = \boxed{1}$

$6 - 4 = \boxed{2}$

$7 - \boxed{4} = 3$

$\boxed{8} - 4 = 4$

$15 - 5 = \boxed{10}$

$15 - 6 = \boxed{9}$

$\boxed{15} - 7 = 8$

$15 - \boxed{8} = 7$

$12 - 2 = \boxed{10}$

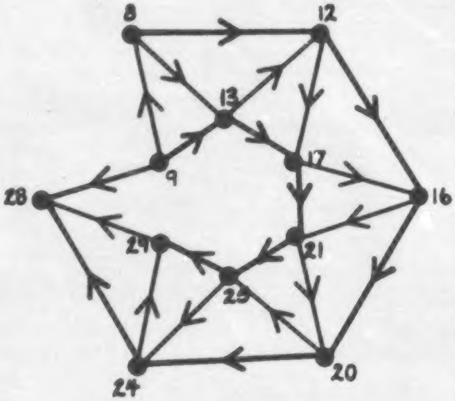
$14 - 4 = \boxed{10}$

$16 - \boxed{6} = 10$

$\boxed{18} - 8 = 10$

Label the dots.
Draw all the missing green arrows.

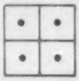
+5 −1 +4



There should be ten green arrows.
What could the black arrow be for? +19

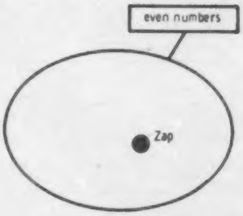
Other answers may be correct.

Zap is a secret number.
Zap can be put on this Minicomputer by removing one checker.

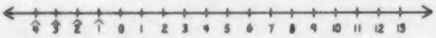


Zap could be 7, 11, 13, or 14.

Zap is in this string picture.



Who is Zap? 14



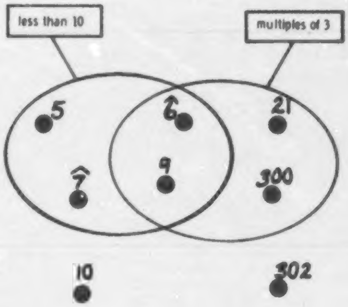
Complete.

$\begin{array}{r} 7 \\ -3 \\ \hline 4 \end{array}$ $\begin{array}{r} 3 \\ -7 \\ \hline \hat{4} \end{array}$ $\begin{array}{r} 5 \\ -8 \\ \hline \hat{3} \end{array}$ $\begin{array}{r} 8 \\ -5 \\ \hline 3 \end{array}$ $\begin{array}{r} 4 \\ -3 \\ \hline 1 \end{array}$ $\begin{array}{r} 3 \\ -4 \\ \hline \hat{1} \end{array}$

$\begin{array}{r} 10 \\ -7 \\ \hline 3 \end{array}$ $\begin{array}{r} 7 \\ -10 \\ \hline \hat{3} \end{array}$ $\begin{array}{r} 12 \\ -9 \\ \hline 3 \end{array}$ $\begin{array}{r} 9 \\ -12 \\ \hline \hat{3} \end{array}$ $\begin{array}{r} 10 \\ -13 \\ \hline \hat{3} \end{array}$ $\begin{array}{r} 13 \\ -10 \\ \hline 3 \end{array}$

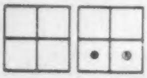
$\begin{array}{r} 6 \\ -4 \\ \hline 2 \end{array}$ $\begin{array}{r} 5 \\ -9 \\ \hline \hat{4} \end{array}$ $\begin{array}{r} 1 \\ -3 \\ \hline \hat{2} \end{array}$ $\begin{array}{r} 11 \\ -8 \\ \hline 3 \end{array}$ $\begin{array}{r} 8 \\ -10 \\ \hline \hat{2} \end{array}$ $\begin{array}{r} 12 \\ -4 \\ \hline 8 \end{array}$

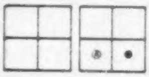
Label the dots in this string picture.
Many answers are possible.

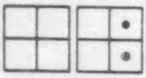


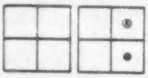
Many solutions are possible.

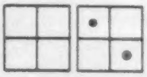
What number is on the Minicomputer?

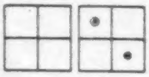
 = 1

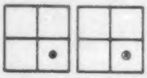
 = $\hat{1}$

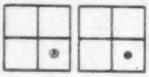
 = 3

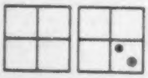
 = $\hat{3}$

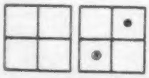
 = 7

 = $\hat{7}$

 = 9

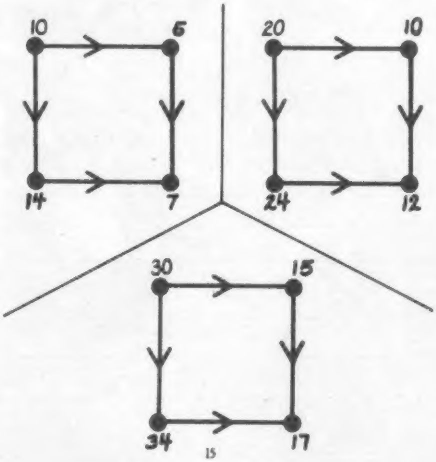
 = $\hat{9}$

 = 0

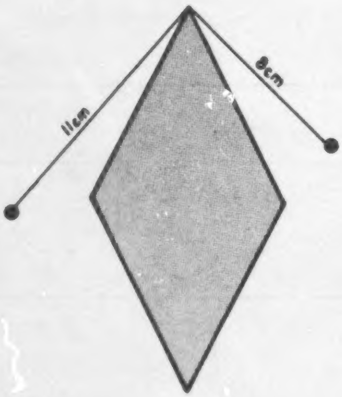
 = 2

Label the dots.
Fill in the box for the green arrows.

$\frac{1}{2}\times$ $+4$ $+\boxed{2}$



Connect the dots with a zigzag path, but do not go through the building. Try to make your zigzag path shorter than 20 cm.



Length of zigzag path = 19 cm

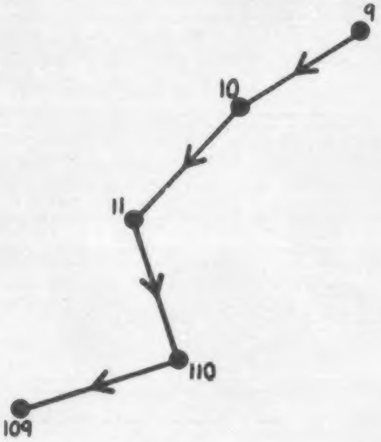
Is the length of your zigzag path less than 20 cm? Yes

Complete this table.

25¢	10¢	5¢	1¢	TOTAL AMOUNT
2	2	0	2	$0.72
1	3	1	3	$0.63
3	2	1	0	$1.00
1	1	1	9	$0.49
4	2	2	15	$1.45
3	1	0	2	$0.87

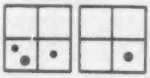
Build a red-blue-green road between 9 and 109. Try to use as few arrows as possible.

10 ×   +1   −1

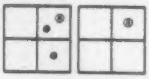


Many solutions are possible; however, this is a shortest road.

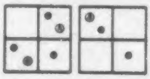
What number is on the Minicomputer?



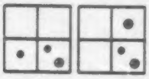
$30 - 21 = \underline{9}$



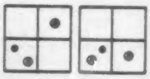
$50 - 44 = \underline{6}$



$79 - 68 = \underline{11}$



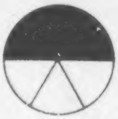
$31 - 15 = \underline{16}$



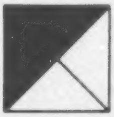
$22 - 63 = \underline{\widehat{41}}$

Color one-half of each shape red. In each case, write another name for $\frac{1}{2}$ as suggested by the picture.

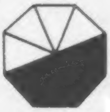
EXAMPLE



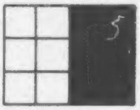
$\frac{1}{2} = \frac{3}{6}$



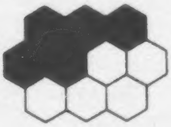
$\frac{1}{2} = \frac{2}{4}$



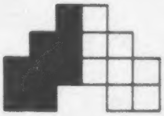
$\frac{1}{2} = \frac{4}{8}$



$\frac{1}{2} = \frac{6}{12}$



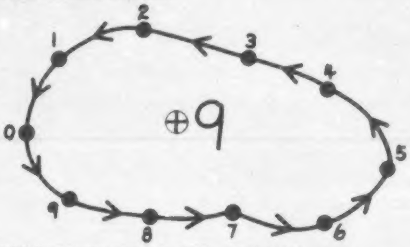
$\frac{1}{2} = \frac{5}{10}$



$\frac{1}{2} = \frac{8}{16}$

Other colorings are possible.

Ten number friends: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 are playing the ⊕ 9 game. Put these numbers in the picture.



Complete these number stories about the ten friends' addition games.

$6 \oplus 6 = \boxed{2}$

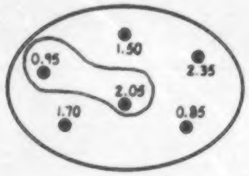
$\boxed{5} \oplus 8 = 3$

$9 \oplus \boxed{5} = 4$

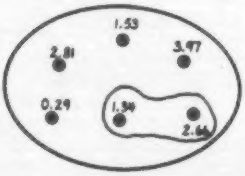
$2 \oplus 4 \oplus 6 \oplus 8 = \boxed{0}$

$5 \oplus \boxed{7} \oplus 9 = 1$

Dana buys two models and spends exactly $3.
The prices of these two models are in this string. Draw one string around these two prices.

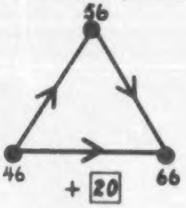

1.50
0.95
2.35
2.05
1.70
0.85

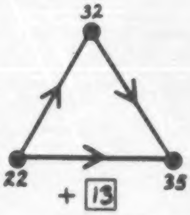
Gina buys two science kits and spends exactly $4.
The prices of these two science kits are in this string. Draw one string around these two prices.

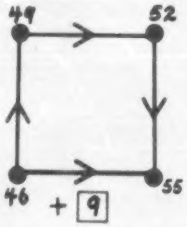

1.53
2.81
3.97
0.29
1.34
2.66

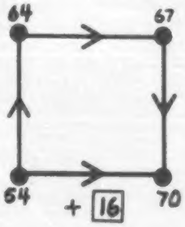
+10 +3

Label the dots.
Fill in the box for each green arrow.





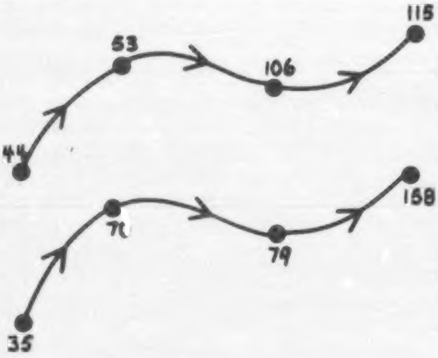



64
67
64
+ 16
70

Put these numbers in this arrow picture.

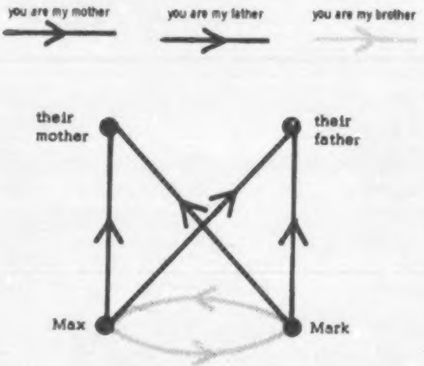
35 44 53 70 79 106 115 158

+9 2×


115
53
106
44
158
70
79
35

Max and Mark are twins. Draw an arrow picture showing the two boys, their mother, and their father.

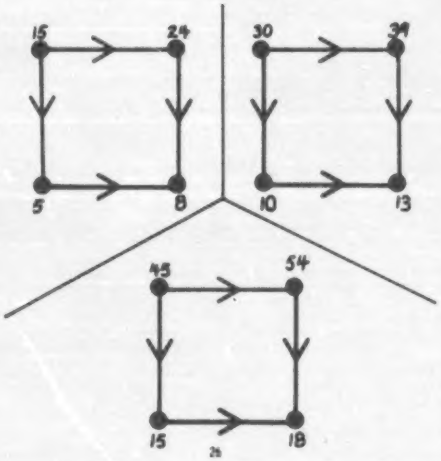
Draw all the possible red, blue, and gray arrows.


you are my mother
you are my father
you are my brother
their mother
their father
Max
Mark

Label the dots.

Fill in the box for the green arrows.

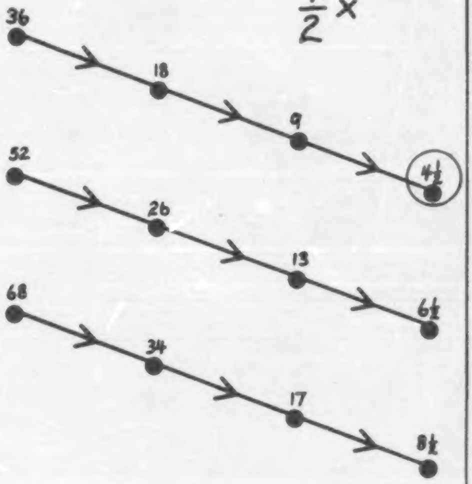
$\frac{1}{3}\times$ $\quad +9 \quad + \boxed{3}$


15
24
30
39
5
8
10
13
45
54
15
18

Label the dots.

Circle the smallest number in this arrow picture.

$\frac{1}{2}\times$


36
18
9
4½
52
26
13
6½
68
34
17
8½

"4.5", "6.5", and "8.5" are other names for $4\frac{1}{2}$, $6\frac{1}{2}$, and $8\frac{1}{2}$, respectively.

Put parentheses in these number stories to make them TRUE.

$3 + (5 \times 4) = 23$

$(3 \times 5) - 4 = 11$

$(3 \times 5) + 4 = 19$

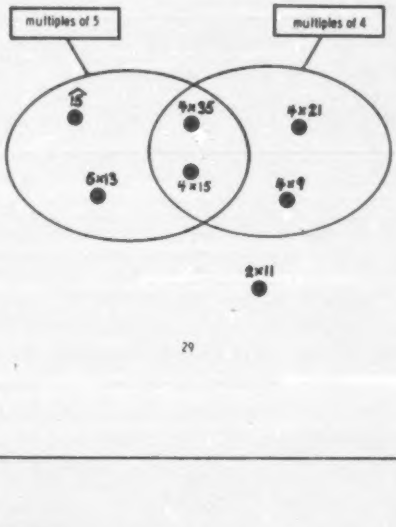
$(3 + 5) \times 4 = 32$

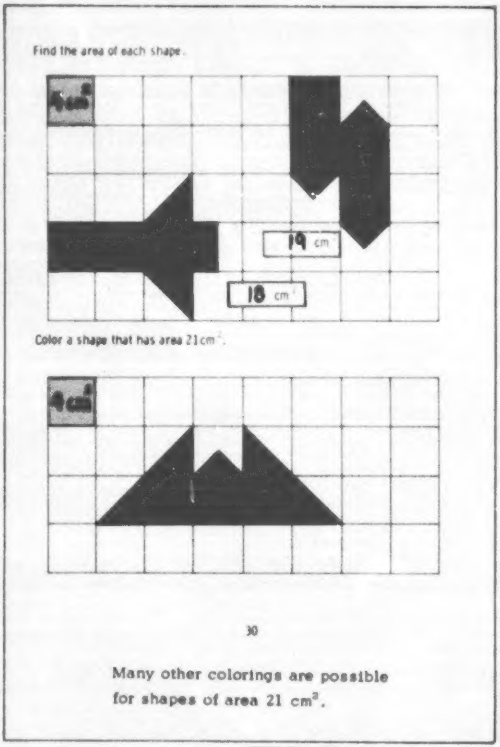
$3 \times (4 + 5) = 27$

Put these numbers in the string picture.

$\widehat{15}$ $\quad 4\times9 \quad 2\times11$

$4\times21 \quad 5\times13 \quad 4\times35$


multiples of 5
multiples of 4
15̂
4×35
4×21
5×13
4×15
4×9
2×11

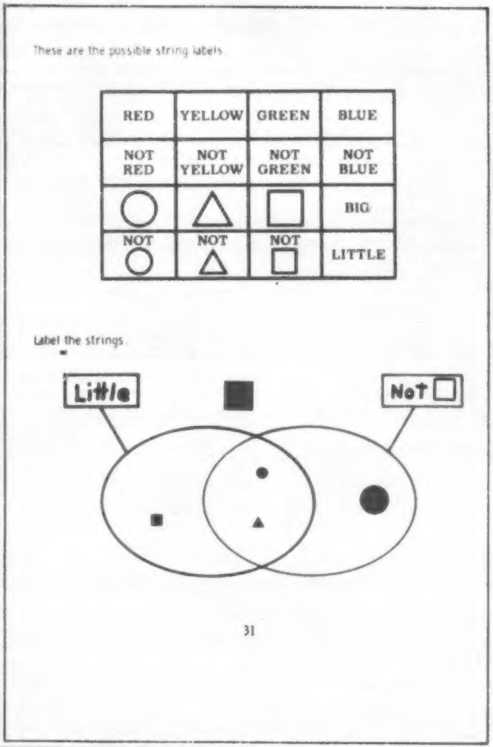


Find the area of each shape.

Color a shape that has area 21 cm².

30

Many other colorings are possible for shapes of area 21 cm².

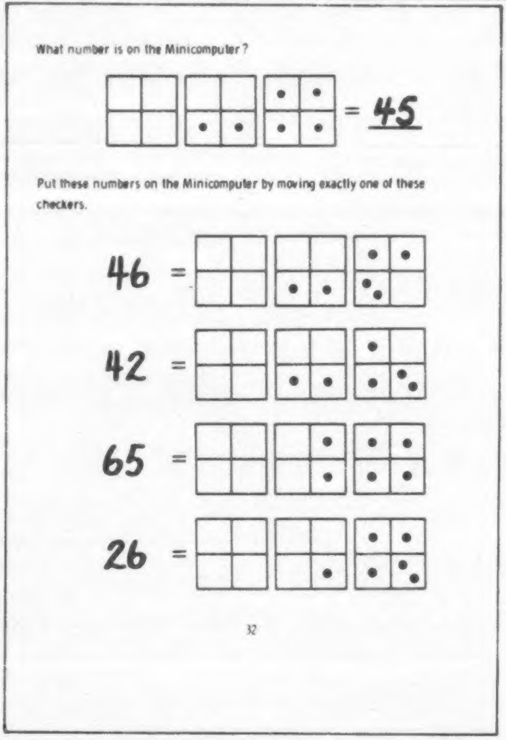


These are the possible string labels

RED	YELLOW	GREEN	BLUE
NOT RED	NOT YELLOW	NOT GREEN	NOT BLUE
○	△	□	BIG
NOT ○	NOT △	NOT □	LITTLE

Label the strings.

31



What number is on the Minicomputer?

= 45

Put these numbers on the Minicomputer by moving exactly one of these checkers.

46 =

42 =

65 =

26 =

32

W11 Spinner Game Detective Story

CAPSULE LESSON SUMMARY

Review the Spinner Game. Solve a detective story in which the clues involve the Minicomputer, a string picture, an arrow picture, and the Spinner Game.

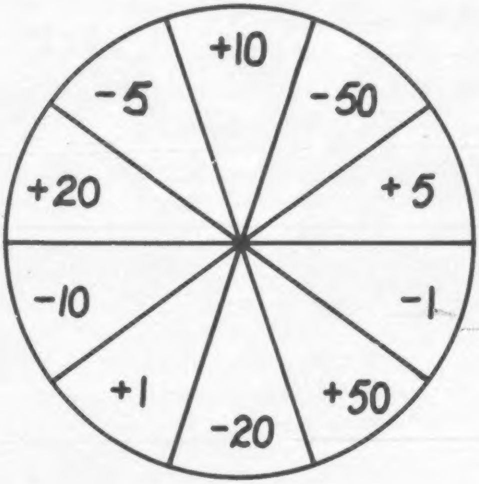
MATERIALS

Teacher:	Colored chalk; a string 45 cm long (optional); Minicomputer set
Student:	Paper; pencil

DESCRIPTION OF LESSON

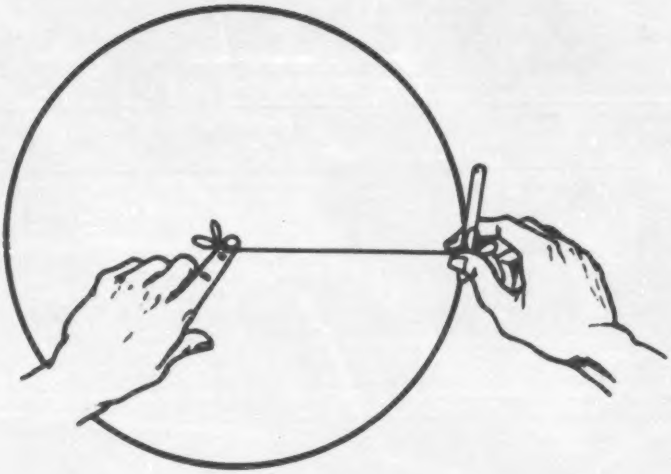
Exercise 1

Before the lesson begins, draw this spinner face very carefully on the board. The circle should have a radius of 40 cm approximately.



One way to draw this circle is to use a piece of string about 45 cm long with loops tied at both ends. Put a piece of chalk through one loop and secure it tightly; put an index finger through the other loop and draw the circle as shown in the next illustration.

W 11



NOTE : Try to keep the lesson moving quickly during this exercise so your students will have enough time to consider carefully each of the clues in the detective story in Exercise 2.

T: Do you remember the Spinner Game? What can you tell me about the Spinner Game?

Encourage the students to recall the important facts about the Spinner Game.

- You spin four times, start at 0, and follow the results of the four spins to get your score.
- The highest possible score is 200.
- The lowest possible score is $\widehat{200}$.
- There are some numbers between $\widehat{200}$ and 200 which are impossible to get as scores when you play this game.

T: What are some of the scores which we can get if we play this game?

Record the students' suggestions on the board. Some of the possibilities are :

Score

$$0 + 10 + 20 - 10 + 50 = 70$$

$$0 + 50 + 50 + 50 + 50 = 200$$

$$0 - 50 - 50 - 50 - 50 = \widehat{200}$$

$$0 + 20 + 20 + 20 - 20 = 40$$

$$0 + 20 + 20 + 10 + 10 = 60$$

$$0 - 20 - 20 - 20 + 50 = \widehat{10}$$

T: Is it possible to get a score of 150?

S: I can get 150 in five spins.

S: I can get 151 in four spins.

S: We need +30 on the spinner.

S: It is impossible to get a score of 150.

T: Is it possible to get a score of 149?

S: Yes, 0 + 50 + 50 + 50 - 1 = 149.

T: I am going to write some numbers on the board. I want you to try to find a way to get each of them as a score in the Spinner Game.

W 11

Write these numbers on the board.

16
29
51
52
53

Ask the students to record on their papers a sequence of spins that would result in each score. Allow a few minutes for individual work and then invite some students to record on the board one possible sequence of spins for each of the numbers 16, 29, 51, and 53.

Only four of the five numbers are possible scores. One way to get each of those four numbers is given below.

$$16 = 0 + 5 + 5 + 5 + 1$$
$$29 = 0 + 10 + 10 + 10 - 1$$
$$51 = 0 + 50 + 1 + 1 - 1$$
$$52$$
$$53 = 0 + 50 + 1 + 1 + 1$$

T: Is it possible to get a score of 52? (No)
Why not?

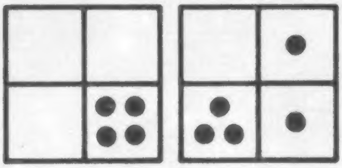
Perhaps a student can explain why a score of 52 is impossible, but do not expect a well-formed explanation.

NOTE : Although it is possible to get a score of 50 after only one spin, it is not possible to increase that score by 2 in exactly three more spins. It is also possible to get a score of 50 after three spins but impossible to increase that score by 2 in only one more spin. This is for your information only.

Exercise 2 : Detective Story

First clue

Put this configuration on the Minicomputer.

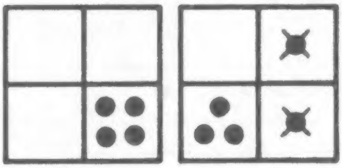


T : What number is on the Minicomputer?

S : 51, because 4 × 10 = 40 and 4 + 6 + 1 = 11.

T : Today you will have another opportunity to be detectives. The first clue is that the secret number can be put on this Minicomputer by removing exactly two checkers. What could the secret number be?

Invite a student to remove two checkers from the configuration on the Minicomputer. The student might remove the checker on the 4-square and the checker on the 1-square.



An "×" through a checker indicates that the checker is removed.

W 11

T: Which number is this?

S: 46, because 4 × 10 = 40 and 3 × 2 = 6.

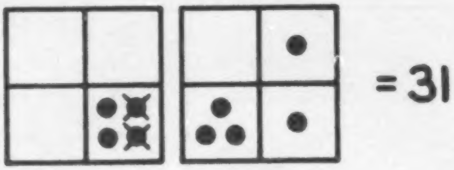
T: The secret number could be 46, because we can get 46 on this Minicomputer by removing exactly two checkers.

Record "46" on the board and then put back the checkers on the 4-square and the 1-square.

T: What is the smallest number we can get on this Minicomputer by removing two checkers?

S: 31.

Invite someone to put 31 on the Minicomputer by removing exactly two checkers.

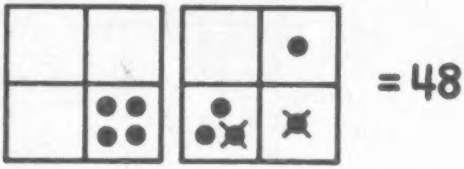


Record "31" on the board and then put back the checkers on the 10-square.

T: What is the largest number we can get on this Minicomputer by removing two checkers?

S: 48.

Invite someone to put 48 on the Minicomputer by removing exactly two checkers.



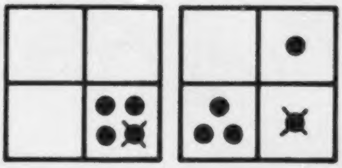
769

Record "48" on the board, leaving enough space between it and "31" so that several other numbers can be recorded between them. Put back the checkers on the 1-square and the 2-square.

31 46 48

T: The smallest number we can get by removing exactly two checkers is 31 and the largest is 48. We can also get 46. Are there any other numbers between 31 and 48 that we can get? Can we get 40? (Yes)

Invite a student to show 40 by removing two checkers. Add "40" to the list of possibilities for the secret number.



31 40 46 48

Let your students experiment to find the other possibilities between 40 and 48.

NOTE: Removing two checkers from the tens' board gives 31 as a possibility. Removing two checkers from the ones' board gives the possibilities larger than 40. Removing one checker from each board gives the other possibilities (37, 39, 40).

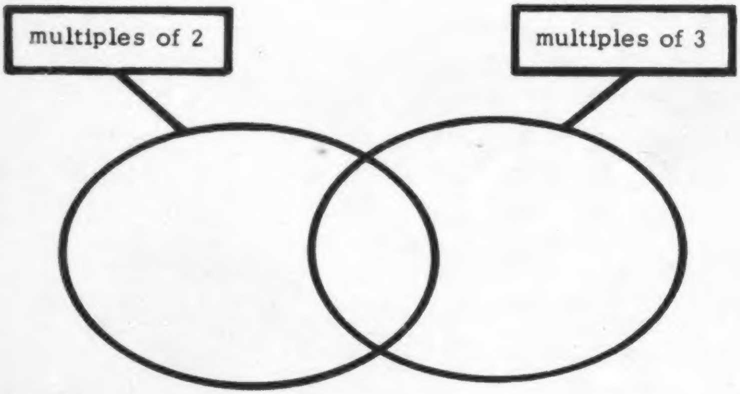
31 37 39 40 45 46 47 48

T: The secret number is one of these numbers. We need another clue.

W 11

Second clue

Draw this string picture on the board.



T: Which numbers belong inside the blue string?

S: The multiples of 2.

S: 0, 2, 4, 6, 8, 10 . . . .

S: $\hat{2}$, $\hat{4}$, $\hat{6}$, $\hat{8}$, ...

T: Which numbers belong inside the red string?

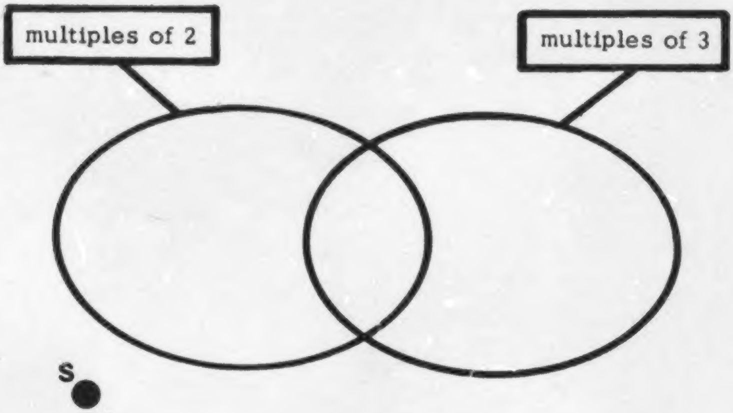
S: The multiples of 3.

S: 0, 3, 6, 9, 12 . . . and $\hat{3}$, $\hat{6}$, $\hat{9}$, $\widehat{12}$ . . . .

T: Which numbers are in both the blue string and the red string?

Encourage the students to suggest several numbers that are multiples of 2 and multiples of 3. Perhaps a student will comment that the numbers in the middle region are the multiples of 6.

Draw and label a dot for the secret number outside of both strings.



T: What information does this give us about the secret number?

S: The secret number is not a multiple of 2.

S: The secret number is an odd number.

S: The secret number is not a multiple of 3.

T: Which of the numbers in the list can we cross out because they are multiples of 2?

The students should tell you to cross out 40, 46, and 48.

T: Which of these numbers can we cross out because they are multiples of 3?

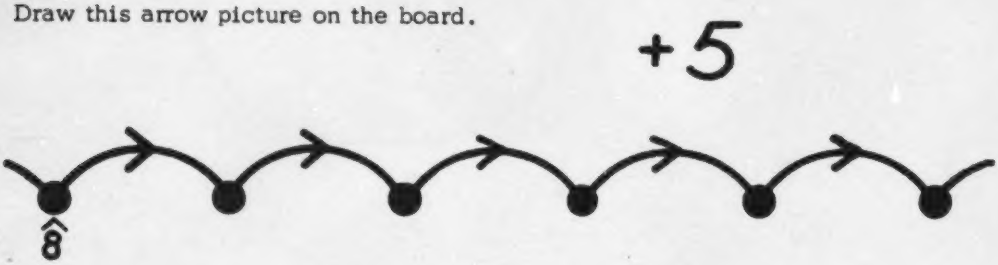
The students should tell you to cross out 39 and 45. Someone might observe that 48 is a multiple of 3 and a multiple of 2. Your class should conclude that the secret number could be 31, 37, or 47. Erase all other numerals in the list and erase the string picture.

W 11

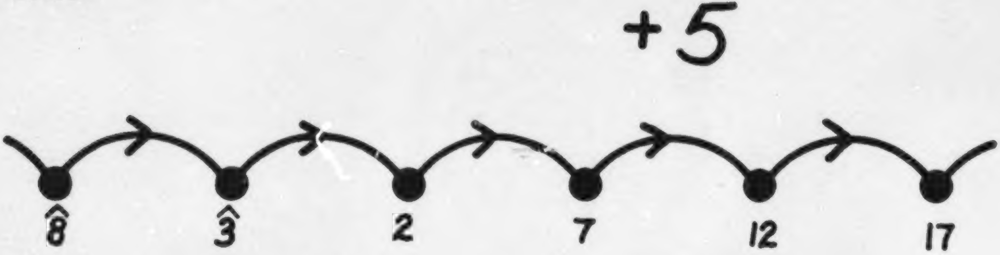
Third clue

T: The secret number is one of these 31, 37, or 47. I will give you another clue. The secret number is on the same +5 arrow road as $\hat{8}$. If you think you know the secret number, write it on a piece of paper.

Draw this arrow picture on the board.



Walk around the room and look at several of the students' papers. Invite a student to label the dots in the arrow picture on the board. If necessary, remind the students that the unfinished arrows tell us that the +5 arrow road continues in both directions and that only part of the road is drawn on the board.



Point to the list of possible numbers and ask which of them are on this +5 arrow road.

S: 37 and 47 are on this arrow road because they both end in 7.

T: All whole numbers that end in 2 or 7 are on this arrow road, so 37 and 47 are on this +5 arrow road. Is 31 on this arrow road?

S: No, 31 is on the +5 arrow road that meets the number 1.

Erase "31" and conclude that the secret number is either 37 or 47.

Fourth clue

T: The last clue is that the secret number is one of the scores we can get when we play the Spinner Game. Remember that we spin exactly four times when we play that game. What is the secret number? Write it on your paper.

Look at several of the students' papers and then ask someone to answer aloud.

S: 47 is the secret number because 0 + 50 - 1 - 1 - 1 = 47.

Record "0 + 50 - 1 - 1 - 1 = 47" on the board.

T: Could we get 37 as a score when we play the Spinner Game? (No)

Perhaps one of the students will be able to explain why it is not possible to get a score of 37 when you play the Spinner Game, but do not expect a well-formed answer.

NOTE: A score of 37 is impossible. Neither "30" nor "40" is on the spinner face (so it takes at least two spins to get either). It takes at least three spins to increase a score of 30 to 37 or to decrease a score of 40 to 37.

Conclude that 47 is the secret number.

W 12

W12 GALAXY OF PROBLEMS #4 (Lesson 1)

CAPSULE LESSON SUMMARY

Begin the Workbook GALAXY OF PROBLEMS #4. Play the Number Line Game with non-integer decimal numbers as the secret numbers.

MATERIALS

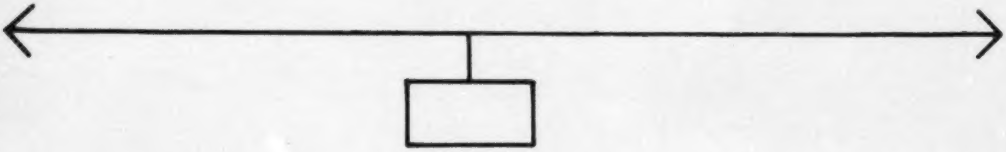
Teacher:	None
Student:	Workbook GALAXY OF PROBLEMS #4; colored pencils; pencil

DESCRIPTION OF LESSON

Distribute copies of the Workbook GALAXY OF PROBLEMS #4 and let the students begin working individually. If the questions you are receiving indicate that many of your students are having difficulty with a particular page, hold a short collective discussion of the problem presented on that page.

Allow 30 to 40 minutes of the lesson for the workbook and then collect them for your review and for use in Lesson W13. Spend the remaining time playing the Number Line Game with secret numbers that are non-integer decimal numbers. An example of a possible game is given here. The secret number is 0.57.

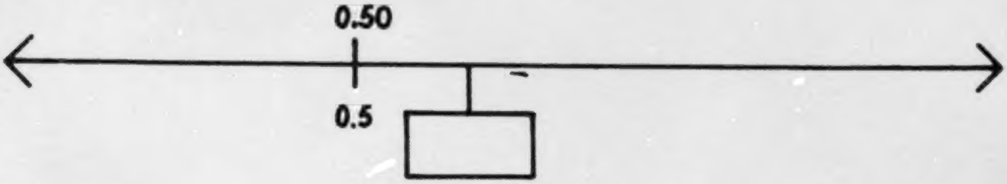
T: I am thinking of a secret number which is between 0 and 1. The secret number is hidden in this box.



T: Guess what my secret number is. It may help you to think of the money line between \$0 and \$1.

S: 0.5.

T: That's the same number as $\frac{1}{2}$ and 0.50. I'll show you were 0.5 is on this number line.

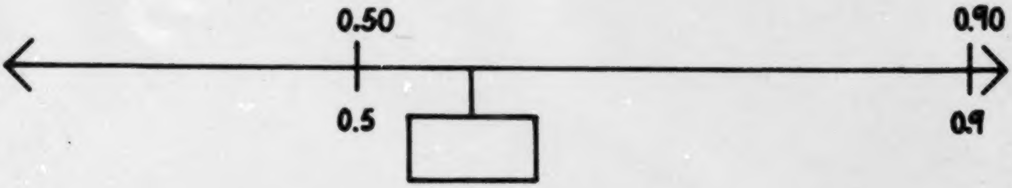


T: 0.5 is less than my number.

S: 0.9.

T: What is another name for 0.9?

S: 0.90.

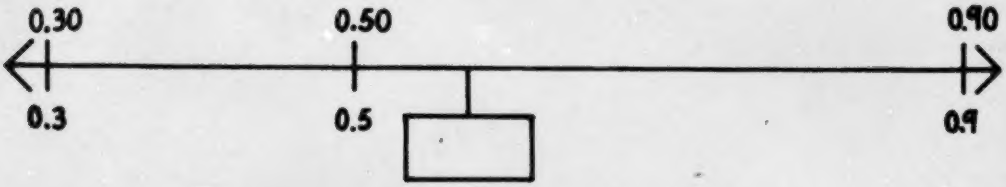

0.50
0.90
0.5
0.9

T: 0.9 might not be in exactly the right place, but I've tried to make it as close as possible. 0.9 is more than my number.

S: 0.30 (or 0.3).

T: How much money would that be?

S: 30 cents.

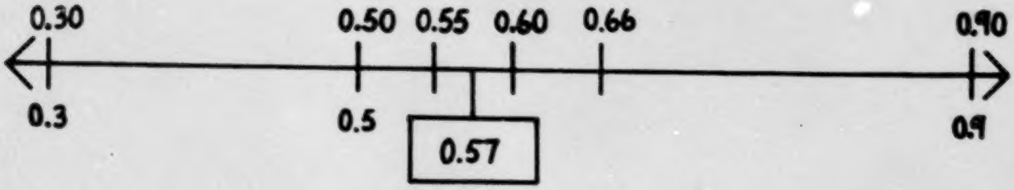

0.30
0.50
0.90
0.3
0.5
0.9

W 12

T: 0.30 is less than my secret number.

NOTE: Although a guess may give no new information, such as 0.30 in this sample game, do not criticize or comment except to say that it is more or less than your number.

The game continues with students guessing 0.55; 0.60; 0.66; and 0.57.



T: 0.57 is my secret number.

Continue this activity with a secret number between 0 and 2.

W13 GALAXY OF PROBLEMS #4 (Lesson 2)

CAPSULE LESSON SUMMARY

Continue working in the Workbook GALAXY OF PROBLEMS #4. (This is the second of two lessons using this workbook.)

MATERIALS

Teacher: None

Student : Workbook GALAXY OF PROBLEMS #4; colored pencils

DESCRIPTION OF LESSON

Your review of the students' workbooks may have indicated that a short collective discussion is needed about a particular page in the workbook. If appropriate, solve a problem similar to the problem on that page in the workbook before distributing the students' copies of the Workbook GALAXY OF PROBLEMS #4.

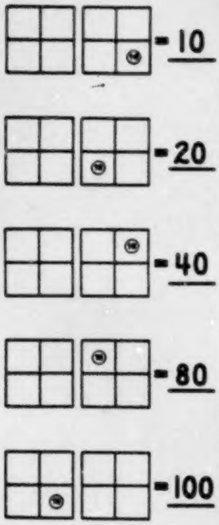
After 30 to 40 minutes of individual work, collect the workbooks for your review. Spend any time that remains playing Minicomputer Golf (see Lesson N15 for a description of this game).

GALAXY OF PROBLEMS

#4

Galaxy of Problems #4

Complete.


= 10
= 20
= 40
= 80
= 100

10 × 1 = 10

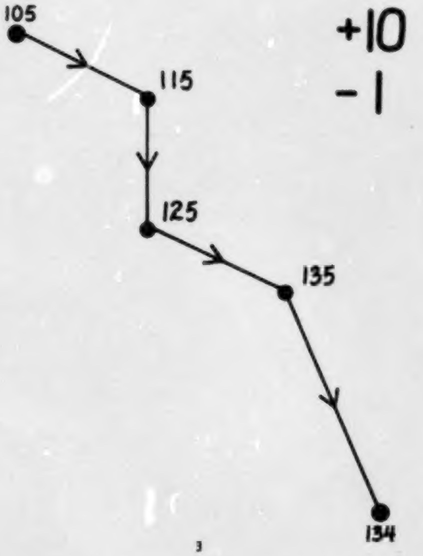
10 × 2 = 20

10 × 4 = 40

10 × 8 = 80

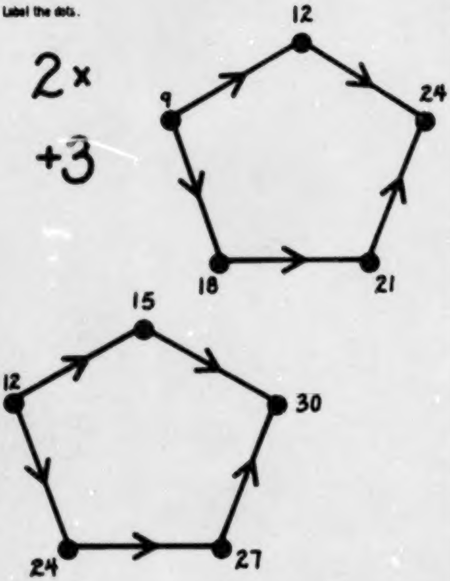
10 × 10 = 100

Build a red-blue road from 105 to 134. Use less than five arrows in your road.

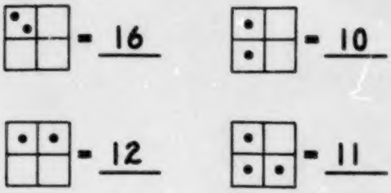

105
115
125
135
134
+10
-1

Other solutions are possible; however, a shortest road has three red arrows and one blue arrow.

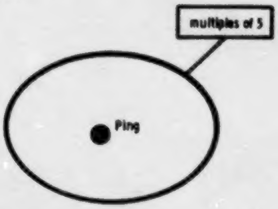
Label the dots.


2×
+3
12
9
24
18
21
15
12
30
24
27

Ping is one of these numbers.



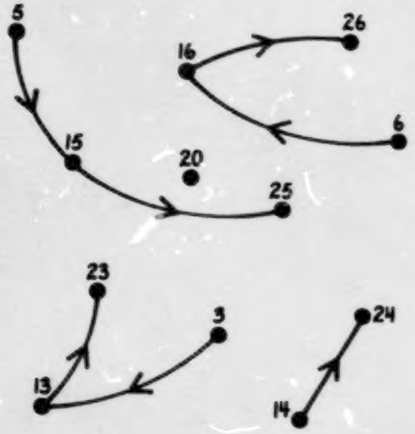
Ping is in this string picture.


multiples of 5
Ping

Who is Ping? 10

Draw all the missing blue arrows. One is drawn for you.

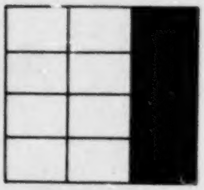
+10



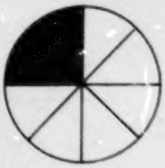
Complete this subtraction table.

−	3	5
10	7	5
20	17	15
30	27	25
40	37	35
50	47	45

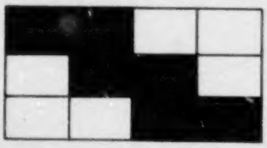
What fractional part of each shape is colored red?



$\frac{1}{3}$



$\frac{1}{4}$



$\frac{1}{2}$

Complete these number stories.

2 × (7 + 1) = 16

(2 × 7) + 1 = 15

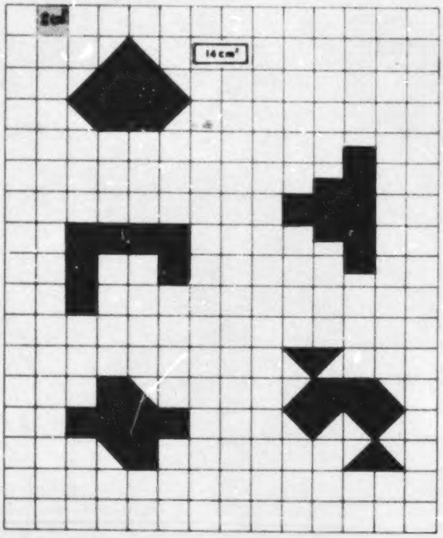
2 + (7 × 1) = 9

(2 + 7) × 1 = 9

(2 × 1) + 7 = 9

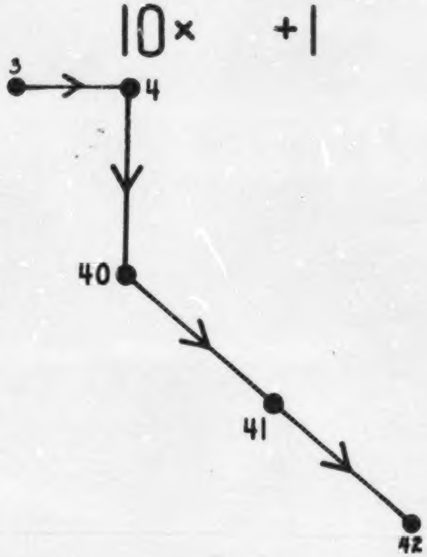
(2 + 1) × 7 = 21

The blue shape has area 14 cm ^{2}.
Color four more shapes with area 14 cm ^{2}.



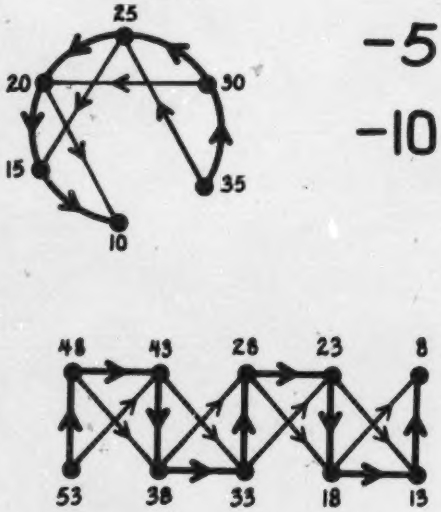
Other colorings are possible.

Build a red-blue road between 3 and 42. Use less than five arrows in your road.

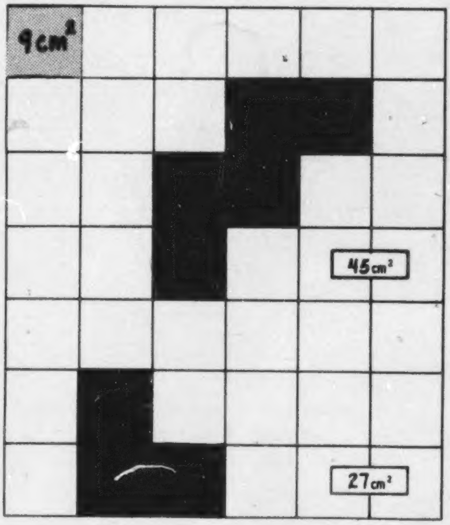

10×
+1
3
4
40
41
42

Other solutions are possible; however, this is the shortest road.

Label the dots.
Draw all the missing red arrows.


25
20
30
15
35
10
−5
−10
48
43
28
23
8
53
38
33
18
13

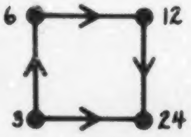
Find the area of the red shape.
Color a blue shape whose area is 27 cm ^{2}.


9cm²
45cm²
27cm²

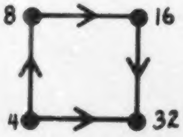
Other colorings are possible.

Label the dots and complete the number stories.

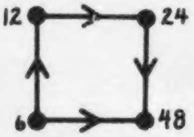
2× 8×



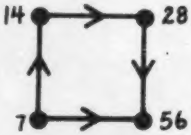
8×3=24



8×4=32



8×6=48



8×7=56

Complete this table.

Many answers are possible. One is done for you.

25¢	10¢	5¢	1¢	TOTAL AMOUNT
2	0	1	1	$0.56
0	1	1	1	$0.16
1	0	4	0	$0.45
2	1	0	8	$0.68
3	2	1	0	$1.00
4	3	0	1	$1.31

Other solutions are possible.

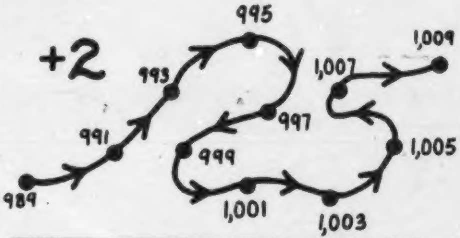
Complete this multiplication table.

×	6	9	7
2	12	18	14
3	18	27	21
4	24	36	28

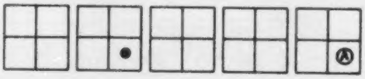
Complete this subtraction table.

−	5	7	12	20
10	5	3	$\hat{2}$	$\widehat{10}$
12	7	5	0	$\hat{8}$

Pang is in this arrow picture.


+2
989
991
993
995
997
999
1,001
1,003
1,005
1,007
1,009

Pang can be put on this Minicomputer using exactly one regular checker and one negative checker.



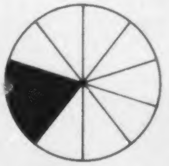
Who is Pang? 999

Put Pang on the Minicomputer.

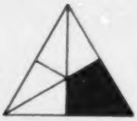
Complete.

20 -18 2	50 -18 32	90 -18 72
45 -15 30	45 -17 28	45 -19 26
31 -14 17	67 -29 38	73 -32 41

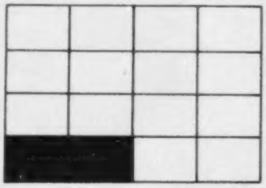
What fractional part of each shape is colored blue?



$\frac{1}{5}$



$\frac{1}{3}$

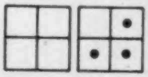


$\frac{1}{8}$

Tip is a secret number.

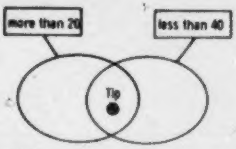
First clue

Tip can be put on this Minicomputer by moving one of these checkers to the tens' board.



Which numbers could Tip be? 15, 25, 45, 85, 13, 23, 43, 83, 16, 26, 46, or 86.

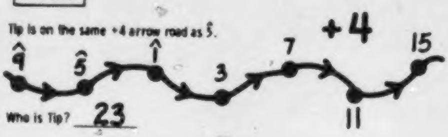
Second clue



Tip could be 23, 25, or 26.

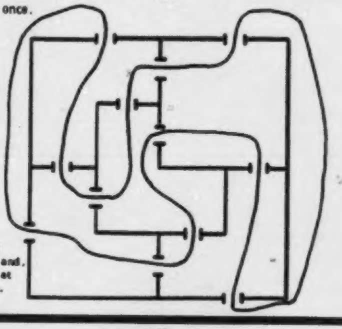
Third clue

Tip is on the same +4 arrow road as $\hat{5}$.

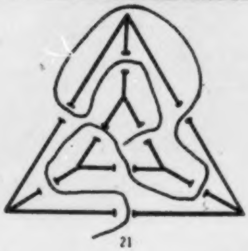


Who is Tip? 23

Find walks through these houses using all the doors, but do not use a door more than once.



This tour could start anywhere and, of course, end at the same place.



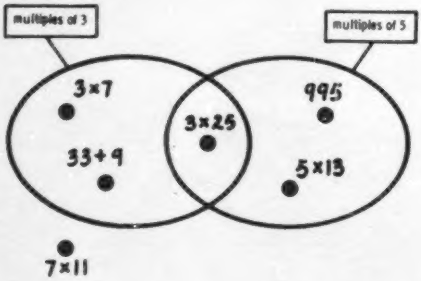
This tour starts in the middle room and ends outside, or vice versa.

784

Put these numbers in the string picture.

3 × 7 7 × 11 995

5 × 13 3 × 25 33 + 9


multiples of 3
multiples of 5
3×7
33+9
3×25
995
5×13
7×11

Share 72 candies fairly among Nick, Cathy, and John.

For Nick	For Cathy	For John
20	20	20
4	4	4

Write a number story about this sharing. 3 × 24 = 72
72 ÷ 3 = 24

Share 105 marbles fairly among Nick, Cathy, and John.

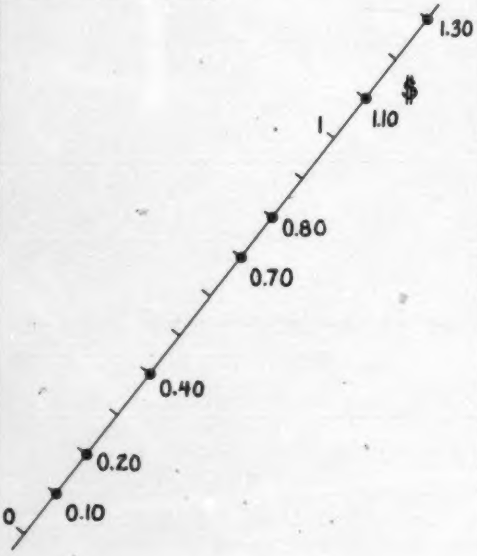
For Nick	For Cathy	For John
30	30	30
5	5	5

Write a number story about this sharing.

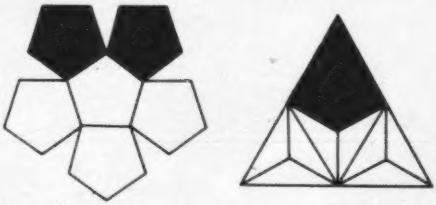
3 × 35 = 105
105 ÷ 3 = 35

Other methods may be used to do the sharing, and other number stories may be suggested.

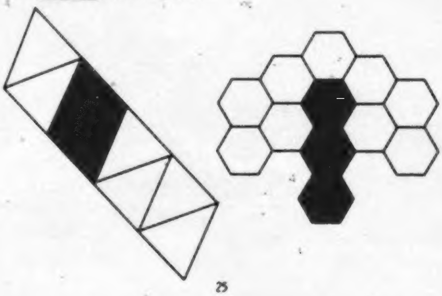
Label the dots on this money number line.


0
0.10
0.20
0.40
0.70
0.80
1
1.10
$
1.30

Color one-third of each shape red.



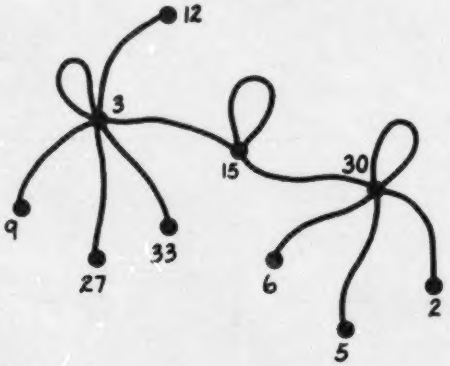
Color one-fourth of each shape blue.



Other colorings are possible.

Label the dots. Many solutions are possible.

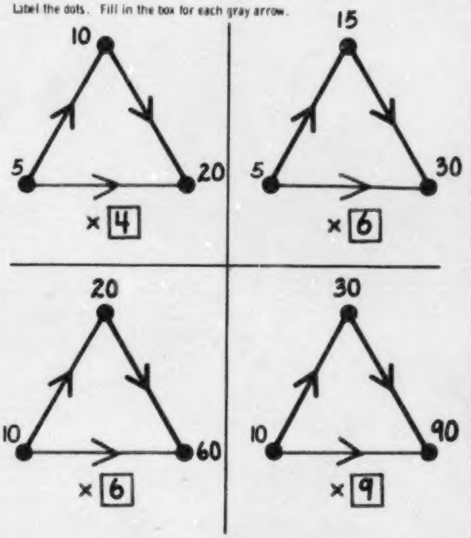
Two numbers may talk to each other
if and only if
one number is a multiple of the other.


12
3
15
30
9
27
33
6
5
2

Other solutions are possible.

×2 ×3

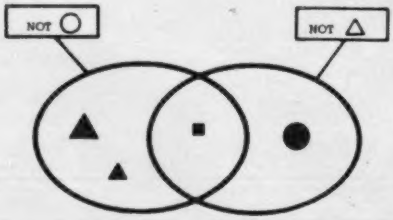
Label the dots. Fill in the box for each gray arrow.

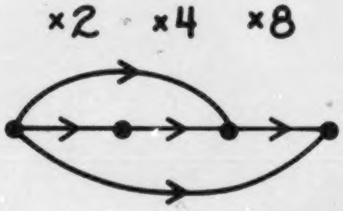

10
5
20
× 4
15
5
30
× 6
20
10
60
× 6
30
10
90
× 9

These are the possible string labels.

RED	YELLOW	GREEN	BLUE
NOT RED	NOT YELLOW	NOT GREEN	NOT BLUE
○	△	□	BIG
NOT ○	NOT △	NOT □	LITTLE

Label the strings.

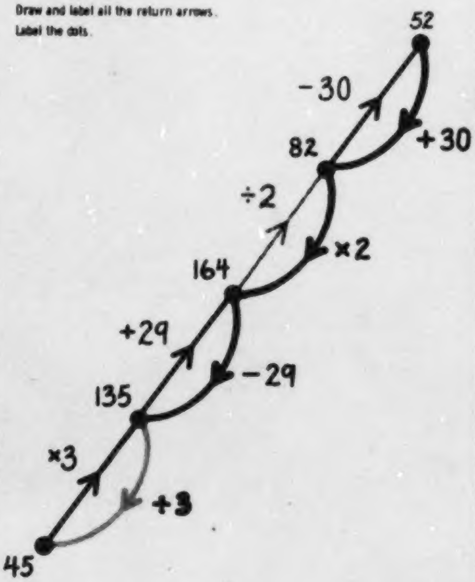




Complete this table.

Starting Number	×2	×4	×8
12	24	48	96
16	32	64	128
25	50	100	200
50	100	200	400
49	98	196	392

Draw and label all the return arrows.
Label the dots.



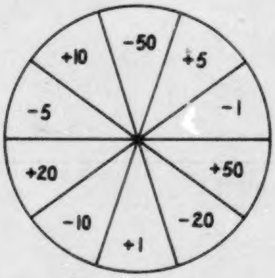
30

Complete.

$$\begin{array}{r}154\\+505\\\hline 659\end{array}\qquad\begin{array}{r}39\\+\ 52\\\hline 91\end{array}$$

$$\begin{array}{r}408\\+357\\\hline 765\end{array}\qquad\begin{array}{r}273\\+127\\\hline 400\end{array}$$

31



Some of these scores are possible after four spins.
Show how you could get them.
Some of these scores are not possible after four spins. Cross them out.

75 50 + 10 + 10 + 5
76 50 + 20 + 5 + 1
~~77~~

100 50 + 10 − 10 + 50
~~101~~
102 50 + 50 + 1 + 1

32

Other solutions are possible.

W14 Who are Zin and Zan?

CAPSULE LESSON SUMMARY

Solve a detective story in which there are two secret numbers and in which the clues involve the Minicomputer, a string picture, and an arrow picture.

MATERIALS

Teacher: Minicomputer set; colored chalk

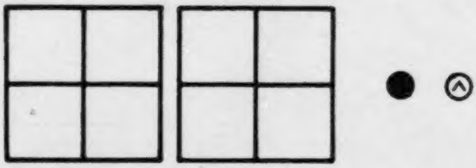
Student : Paper; pencil

DESCRIPTION OF LESSON

T: Today we are going to investigate two secret numbers. Their code names are "Zin" and "Zan".

First clue

Display two Minicomputer boards, one regular checker, and one negative checker.



T: Your first clue is that Zin and Zan can each be put on this Minicomputer using exactly one regular checker and one negative checker. Which numbers could Zin and Zan be?

Invite several students to put numbers on the Minicomputer using exactly two checkers, one regular and one negative. For each number, ask if it is even or odd and record it on the board.

After several possibilities for Zin and Zan have been found, ask for:

- the largest possible number (79);
- the smallest possible number ($\widehat{79}$);
- an even number less than 50;
- the largest possible even number (78);
- an odd number between 30 and 40 (39);
- an odd number less than 0.

Continue this activity until your list includes at least ten numbers.

T: The largest number we can put on this Minicomputer using these two checkers is 79 and the smallest number is $\widehat{79}$. Therefore, we know that Zin and Zan are both between 80 and $\widehat{80}$.

Record this information on the board.

Zin and Zan are between $\widehat{80}$ and 80.

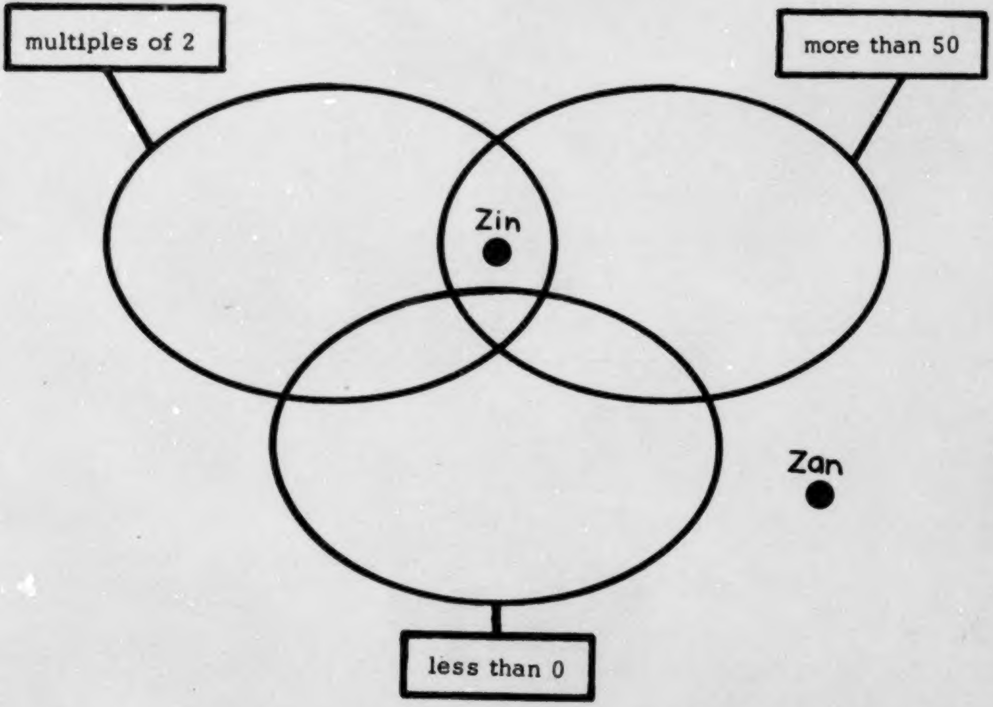
T: Some of the numbers between $\widehat{80}$ and 80 which could be Zin and Zan are in our list. How many numbers do you think would be in our list if we found all the numbers that can be put on this Minicomputer using one regular checker and one negative checker?

Let the students guess but do not try to reach any conclusions about the number of possibilities. For your information, there are fifty-three numbers that can be put on two Minicomputer boards using exactly two checkers, one regular and one negative.

T: Remember that only a few of the numbers which could be Zin or Zan are in our list. I will give you another clue.

Second clue

Draw this string picture on the board.


multiples of 2
more than 50
Zin
Zan
less than 0

T (tracing the blue string): This string is for numbers that are multiples of 2. What is another name for those numbers?

S: "Even numbers".

T: What does this string picture tell us about Zin and Zan?

S: Zin is an even number more than 50.

S: Zan is an odd number between 0 and 50.

NOTE: Since Zan is an odd number, Zan cannot be 0 or 50.

T: Which of the secret numbers is larger?

S: Zin.

T: Let's make a list of the numbers which Zin could be.

Point to the list of numbers resulting from the first clue.

T: Now that we know where Zin is in this string picture, could Zin be any of these numbers?

With your students, identify numbers in this list which could be Zin and record them on the board. Perhaps 78 is the only number in your original list which could be Zin.

Zin
78

Ask for other numbers that Zin could be. When a number is suggested, first ask if it is an even number larger than 50. If the answer is yes, invite a volunteer to put the number on the Minicomputer using exactly one regular checker and one negative checker. Record any numbers which Zin could be on the board. Continue until you have the following list for Zin.

Zin	
78	$(80 + \hat{2})$
76	$(80 + \hat{4})$
72	$(80 + \hat{8})$
70	$(80 + \widehat{10})$
60	$(80 + \widehat{20})$

T: Now we know that Zin is one of these five numbers. Let's make a list of the numbers that Zan could be.

Point to the list of numbers resulting from the first clue.

T: Could Zan be any of these numbers? Remember that the string tells us that Zan is an odd number between 0 and 50.

With your students, identify numbers in this list which Zan could be and record them on the board. Perhaps 39 will be the only number in the original list which Zan could be.

Zan	Zin
39	78
	76
	72
	70
	60

Erase the list of numbers resulting from the first clue. Ask for other numbers which Zan could be. When a number is suggested, ask first if it is an odd number between 0 and 50. If the answer is yes, invite a student to try to put the number on the Minicomputer using exactly one regular checker and one negative checker. Record on the board any numbers which Zan could be.

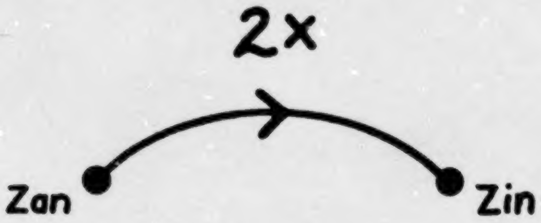
NOTE: It is possible that a student will observe that for Zan to be an odd number and to be on the Minicomputer, the negative checker must be on the 1-square. This observation, if made, will help your students find the possibilities for Zan.

Continue until your students have found all possibilities for Zan. Erase everything except the lists of numbers for Zin and Zan.

	Zan	Zin
$(40 + \hat{1})$	39	78
$(20 + \hat{1})$	19	76
$(10 + \hat{1})$	9	72
$(8 + \hat{1})$	7	70
$(4 + \hat{1})$	3	60
$(2 + \hat{1})$	1	

Third clue

Draw this arrow picture on the board.



T: 2 × Zan = Zin.

Zin is twice as much as Zan.

If you think you know which numbers are Zan and Zin, write them on your paper.

Look at several of the students' papers and then ask someone to answer aloud.

S: Zan is 39 and Zin is 78.

Find that 39 is in the list for Zan and 78 is in the list for Zin, label the dots in the arrow picture, and calculate "2 × 39" before concluding that Zan could be 39 and Zin could be 78.

T: Could any of these other numbers be Zan?

A possible dialogue is given here.

S: 39 is the largest number in the list for Zan and 19 is the next largest. 2 × 19 is less than 50, so none of the numbers except 39 can be Zan.

T: What number is 2 × 19?

S: 38.

T: All the numbers Zin could be are 50 or more, so it would be impossible to draw a 2 × arrow from any of these numbers (point to "1", "3", "7", "9", and "19") to any of the numbers which Zin could be.

Conclude that Zan is 39 and Zin is 78.

W15 FISHING FOR NUMBERS (Lesson 1)

CAPSULE LESSON SUMMARY

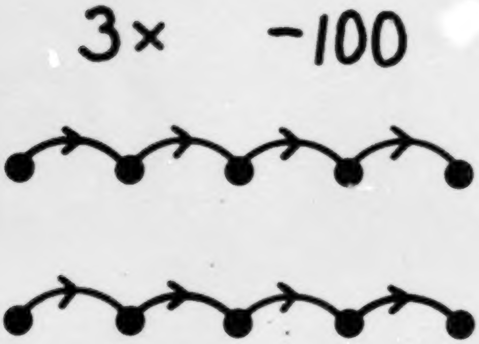
Put the numbers from a string picture in their proper places in an arrow picture which has 3× and -100 arrows. Begin the Workbook FISHING FOR NUMBERS.

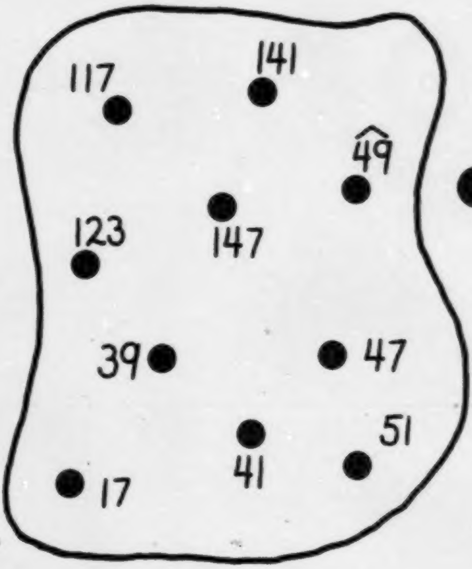
MATERIALS

Teacher: Colored chalk
Student : Workbook FISHING FOR NUMBERS; pencil

DESCRIPTION OF LESSON

Draw these pictures on the board.





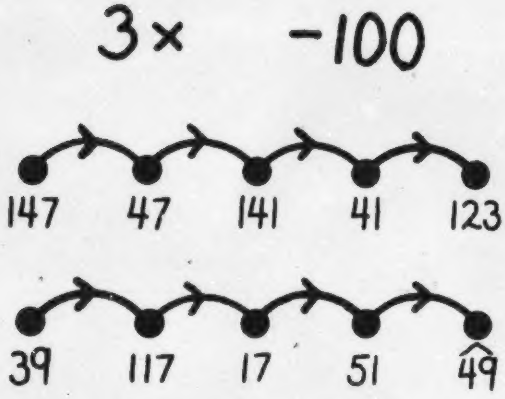
Remind your students of the "Fishing for Numbers" activities that they have done previously. They should tell you that the problem is to put all of the numbers shown in the string picture in their proper places in the arrow picture.

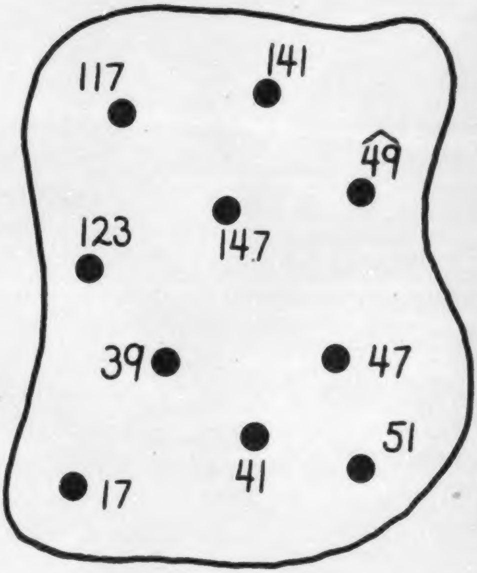
Ask a student to suggest a number for the starting dot of one of the arrow roads. Do not try to get a correct choice immediately as it would be more helpful for your class to see an example that doesn't work. Pursue this choice and see where it leads. If it leads to an incorrect situation, let the students discover this and discuss why it is wrong and what they should do.

In the course of solving this problem collectively, the opportunity to discuss various strategies should arise. Allow ample opportunity for the students to discover some good strategies themselves. A student may suggest identifying the smallest (largest) number in one of the arrow roads but this would be difficult because multiplying a whole number by 3 results in a larger number and subtracting 100 will always result in a smaller number.

The students might suggest crossing out the numbers inside the string after they have been placed in the arrow picture. This procedure is questionable since in a trial and error method many mistakes may be made and new starts may be necessary.

Continue until all the dots in the arrow picture are labeled.





W 15

Distribute copies of the Workbook FISHING FOR NUMBERS and let the students begin working individually. Emphasize that very likely they will need to use their erasers, and that they should always try more than one starting number for a road before asking for help. At the end of the lesson, collect the workbooks for your review and for use in Lesson W16.

W16 FISHING FOR NUMBERS (Lesson 2)

CAPSULE LESSON SUMMARY

Put the numbers from a string picture in their proper places in an arrow picture which has "I am more than you" and "I am less than you" arrows. Continue working in the Workbook FISHING FOR NUMBERS. (This is the second of two lessons using this workbook.)

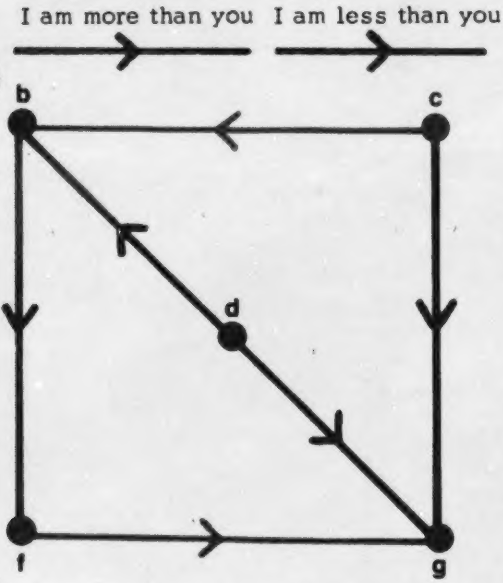
MATERIALS

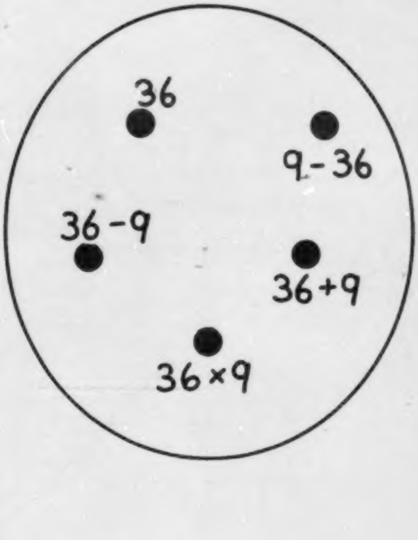
Teacher: Colored chalk
Student : Workbook FISHING FOR NUMBERS; pencil

DESCRIPTION OF LESSON

Draw these pictures on the board.

Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.





Remind the class that the problem is to put all the numbers in the string picture in their proper places in the arrow picture.

W 16

T: What is the smallest number in this string picture? How do you know?

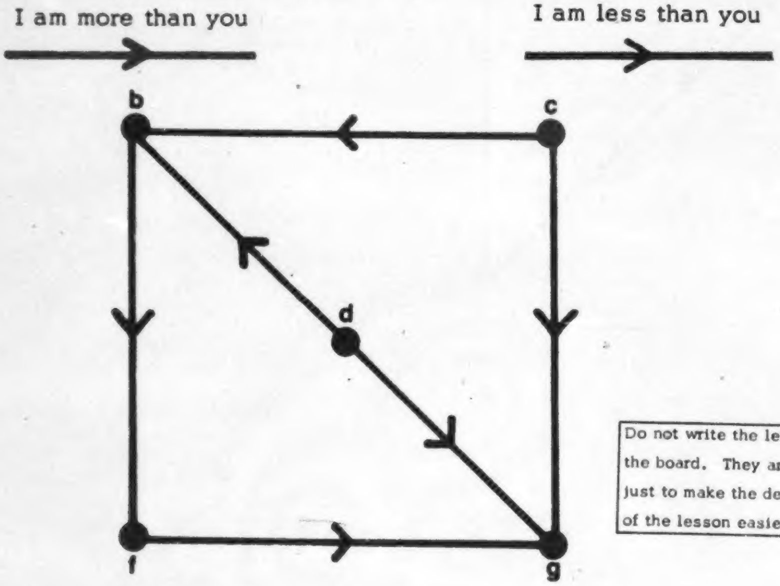
S: 9 - 36 is the smallest number because it is a negative number and all the other numbers are more than 0.

T: What is the largest number in this string picture? How do you know?

S: 36 × 9 is the largest number because 36 + 36 + 36 + 36 + 36 + 36 + 36 + 36 + 36 is more than 36 + 9 and the other numbers are even smaller.

T: Where is the largest number in this arrow picture? How do you know?

Encourage the students to express themselves freely. If necessary, direct the discussion by pointing to a dot in the arrow picture and asking if there is another number in the arrow picture which is larger than this one. Continue this activity until your class concludes that the largest number is at d in the arrow picture. A possible dialogue is given here.



Do not write the letters on the board. They are here just to make the description of the lesson easier to follow.

S (tracing the arrow from b to f): We know that this number (b) is more than this number (f).

S (tracing the arrow from d to b): And this number (d) is even more than this number (b).

T (pointing to b, d, and f): Of these three numbers, which is the largest?

Invite a student to point to the dot for the largest of the three numbers. (d)

T: Which is the smallest (of the three)?

Invite a student to point to the dot for the smallest of the three numbers. (f)

S: This number (pointing to d) is also larger than this number (pointing to g).

S: Either this number (d) or this number (c) is the largest number in this arrow picture.

S: This number (c) cannot be the largest number in this arrow picture because it is less than this number (b).

The students should conclude that the largest number is at d in this arrow picture.

T: Where is the smallest number in this arrow picture?
How do you know?

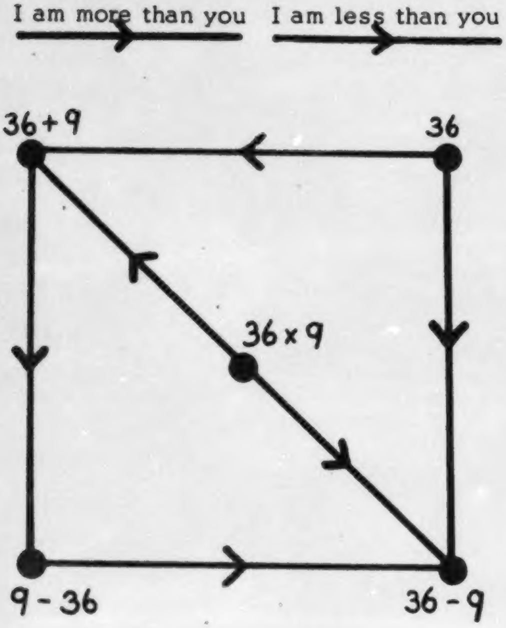
Encourage the students to use the red and blue arrows in determining where the smallest number is, as they did in locating the largest number in the arrow picture. Continue this activity until your class concludes that the smallest number is at f in this arrow picture.

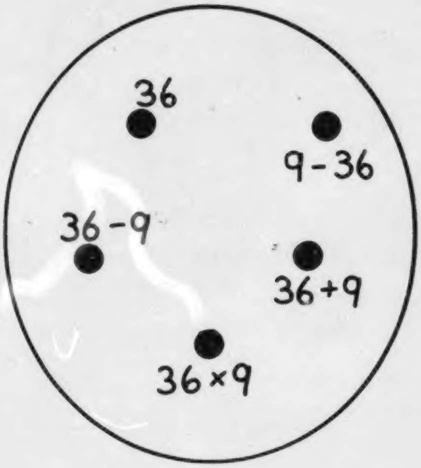
W 16

Invite a student to label one of the dots in the arrow picture. Discourage the students from doing any calculations. They should label the dots with the same number names as those in the string picture.

When a dot has been labeled, encourage the class to comment and to explain why the dot could or could not be for that number. If your class concludes that the dot has been labeled incorrectly, erase the label and invite another student to label a dot.

During the discussion the students should conclude that d should be labeled "36 × 9" because it is the largest number in this arrow picture, and f should be labeled "9 - 36" because it is the smallest number in this arrow picture. The three remaining dots may need to be labeled with a trial-and-error method. Continue this activity until all the dots are correctly labeled.





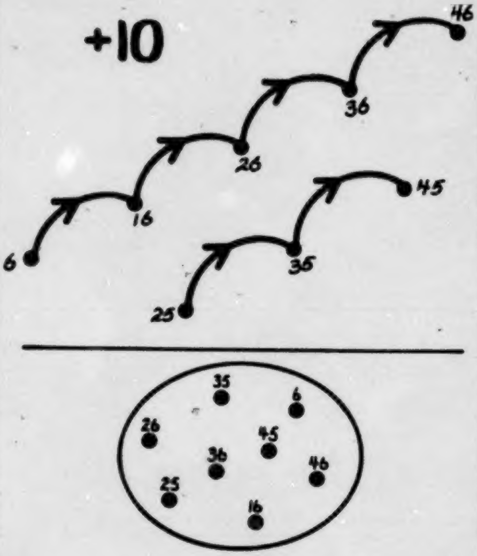
T: Look at each of these arrows carefully and see if each of them tells a true number story with the dots labeled this way.

The students should conclude that the dots are labeled correctly.

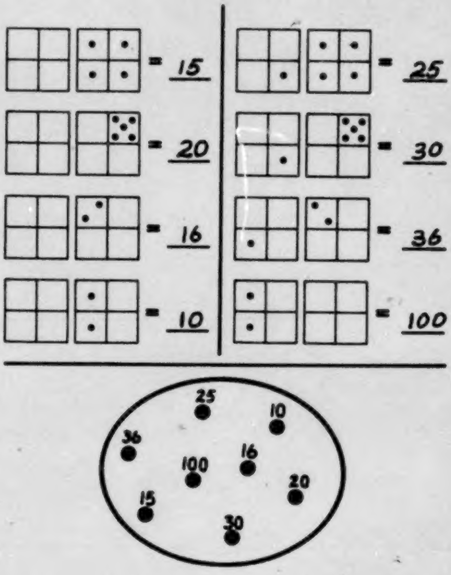
Distribute the students' copies of FISHING FOR NUMBERS and let them continue working in this workbook individually. At the end of the lesson, collect the workbooks for your review.

FISHING FOR NUMBERS
(Part IV)

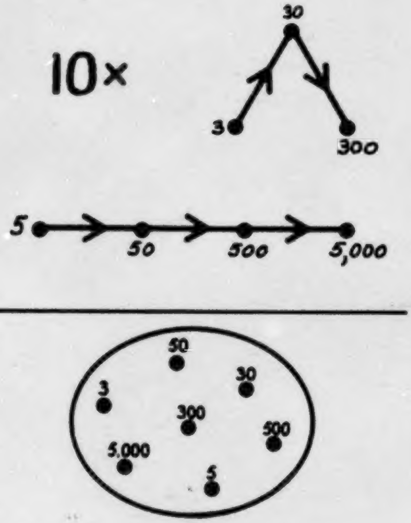
Fishing for Numbers (Part IV)


+10
6
16
26
36
46
25
35
45
35
6
26
45
36
46
25
16

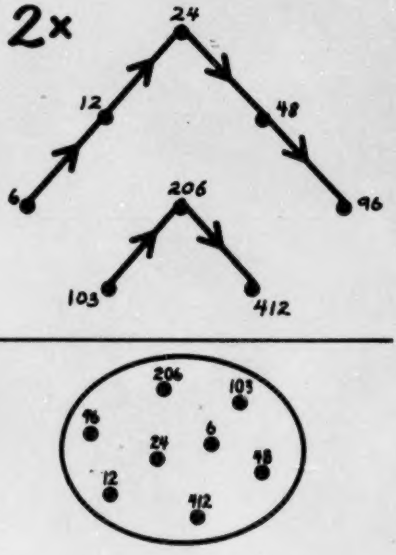
2


= 15
= 20
= 16
= 10
= 25
= 30
= 36
= 100
25
10
36
100
16
20
15
30

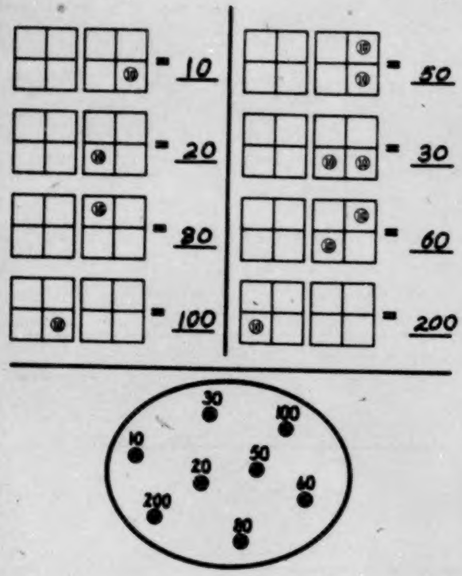
3


10×
3
30
300
5
50
500
5,000
50
30
3
300
500
5,000
5

4

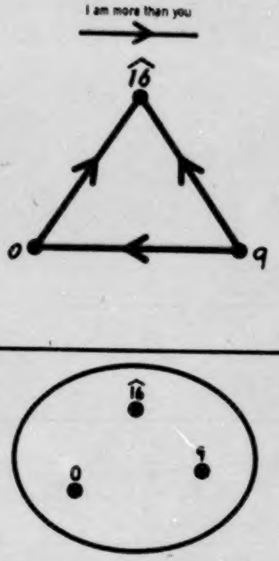

2×
6
12
24
48
96
103
206
412
206
103
96
24
6
48
12
412

5



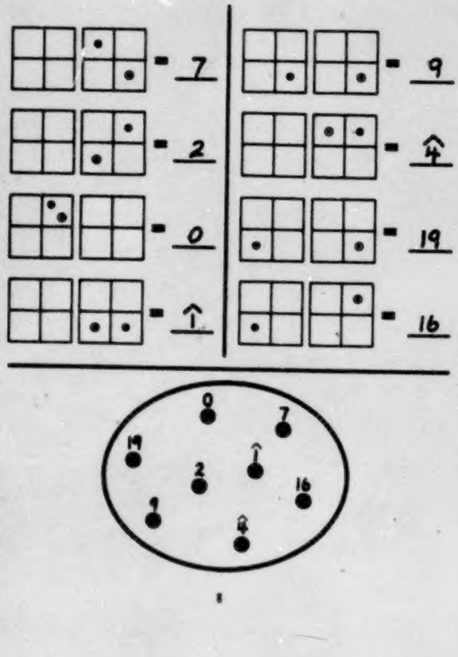
= 10
= 50
= 20
= 30
= 80
= 60
= 100
= 200
30
100
10
20
50
60
200
80

6



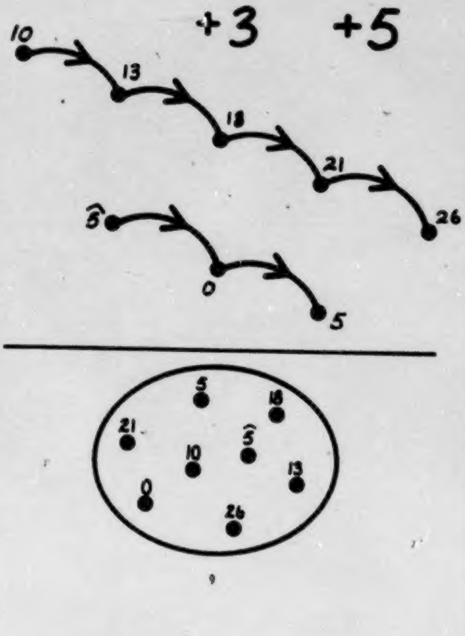
I am more than you
16̂
0
9
16̂
9
0

7



= 7
= 9
= 2
= 4̂
= 0
= 19
= 1̂
= 16
0
7
19
1̂
2
16
9
4̂

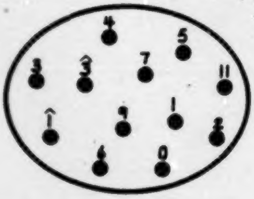
8



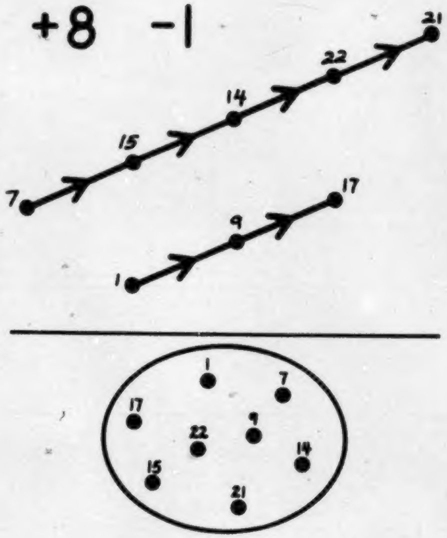
+3
+5
5̂
5̂

9

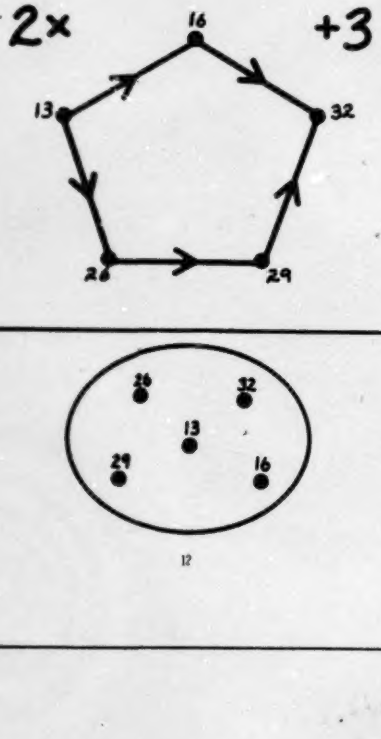
−	5	9	11	7
8	3	$\hat{1}$	$\hat{3}$	1
11	6	2	0	4
16	11	7	5	9


4
5
3
$\hat{3}$
7
11
$\hat{1}$
9
1
2
6
0

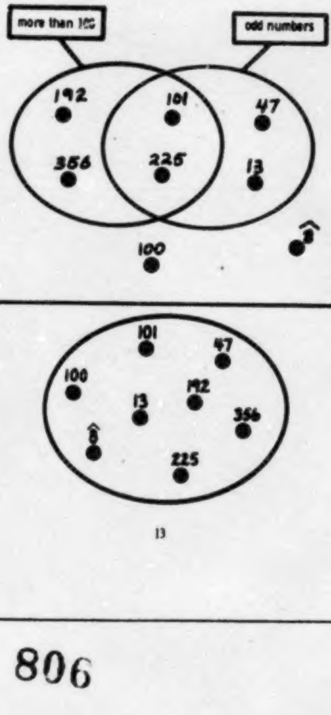
10


+8
-1
7
15
14
22
21
1
9
17
1
7
17
22
9
14
15
21

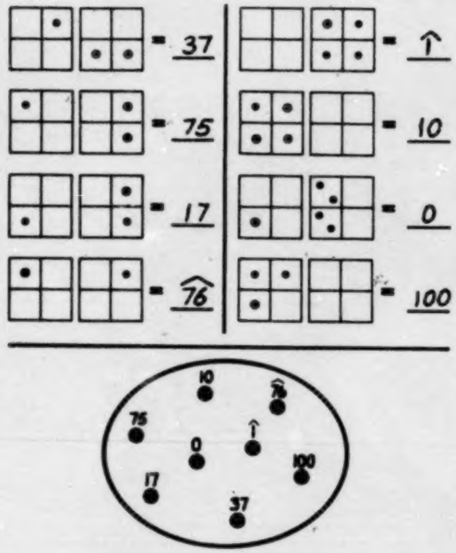
11


2×
+3
16
13
32
26
29
26
32
13
29
16

12

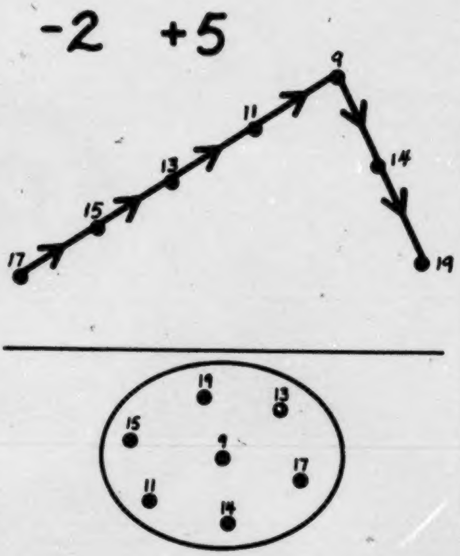

more than 100
odd numbers
192
101
47
356
225
13
100
$\hat{3}$
101
47
100
13
192
356
$\hat{3}$
225

13



= 37
= 75
= 17
= $\widehat{76}$
= $\hat{1}$
= 10
= 0
= 100
10
$\widehat{76}$
75
$\hat{1}$
0
100
17
37

14

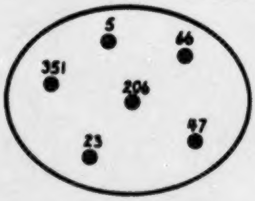


-2
+5
17
15
13
11
9
14
19
19
13
15
9
17
11
14

15

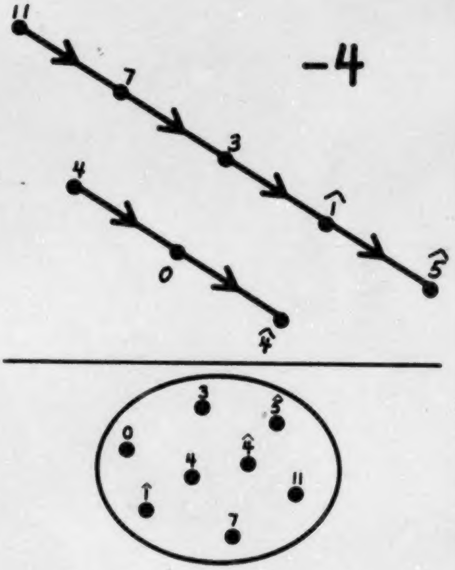
$$\begin{array}{r} 60 \\ -37 \\ \hline 23 \end{array} \qquad \begin{array}{r} 93 \\ -46 \\ \hline 47 \end{array} \qquad \begin{array}{r} 75 \\ -70 \\ \hline 5 \end{array}$$

$$\begin{array}{r} 285 \\ -79 \\ \hline 206 \end{array} \qquad \begin{array}{r} 604 \\ -253 \\ \hline 351 \end{array} \qquad \begin{array}{r} 124 \\ -58 \\ \hline 66 \end{array}$$



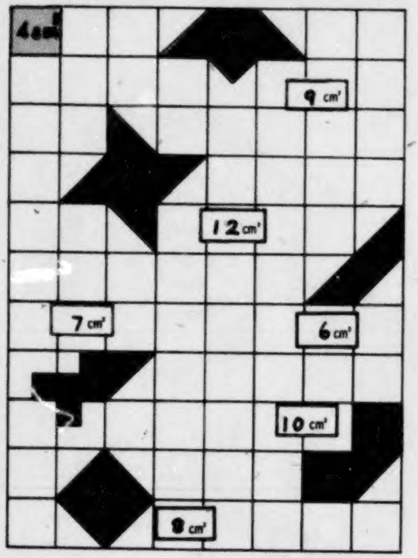
5
66
351
206
47
23

16

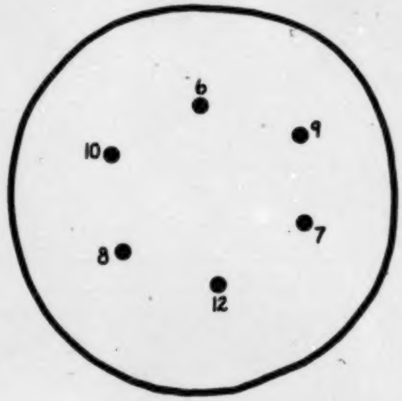


11
7
3
$\hat{1}$
$\hat{5}$
4
0
$\hat{4}$
-4
3
$\hat{5}$
0
$\hat{4}$
4
11
$\hat{1}$
7

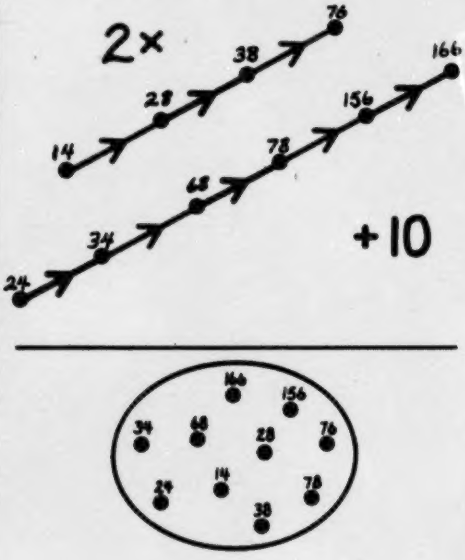
17


4 cm²
9 cm²
12 cm²
7 cm²
6 cm²
10 cm²
8 cm²

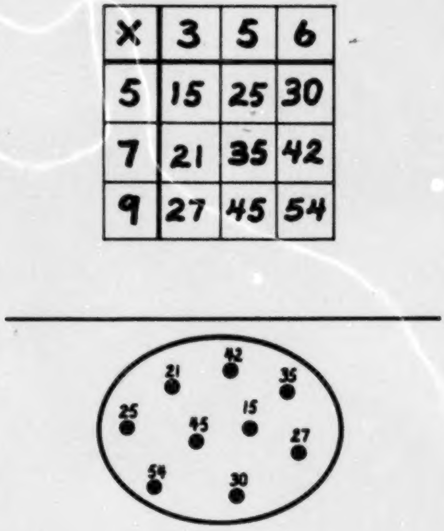
18



19


2×
76
38
28
14
166
156
78
68
34
24
+10
166
156
34
68
28
76
14
24
78
38

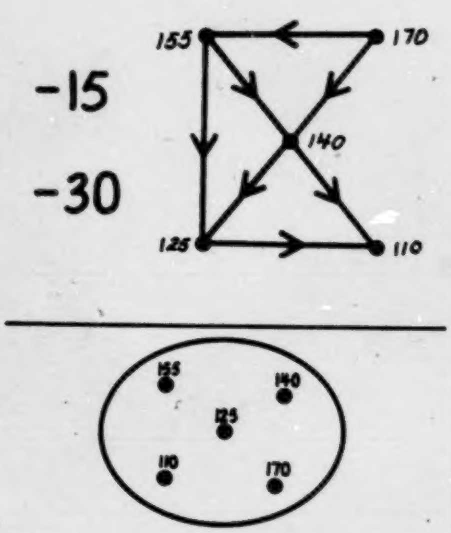
20



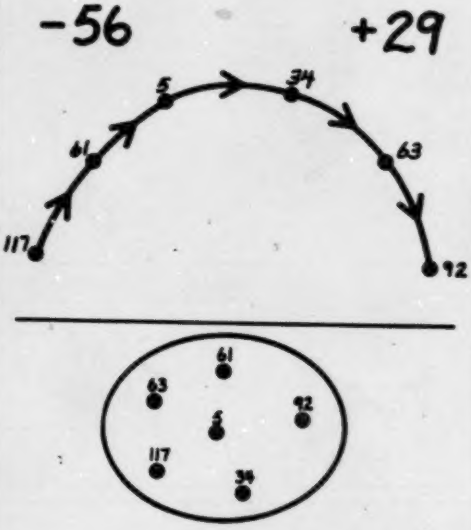
×	3	5	6
5	15	25	30
7	21	35	42
9	27	45	54

42
21
35
25
15
45
27
54
30

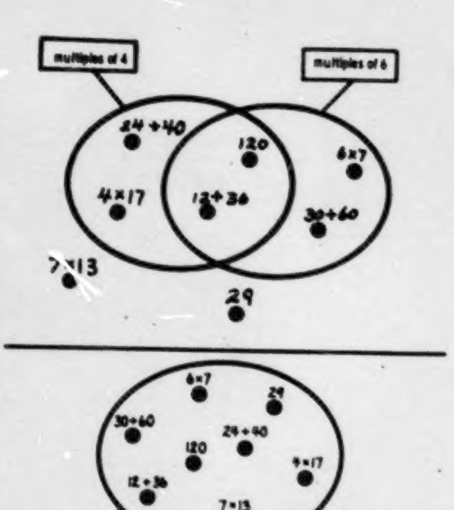
21


-15
-30
155
170
140
125
110
155
140
125
110
170

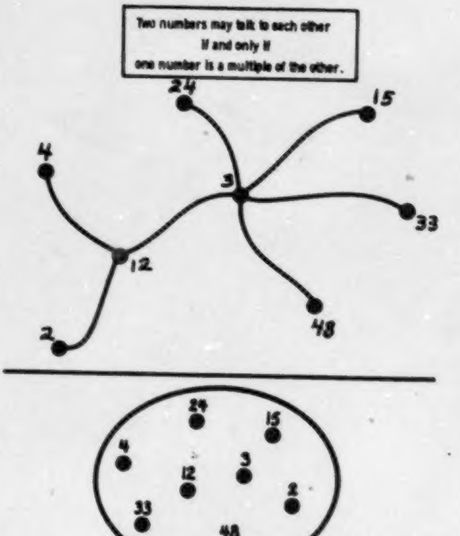
22


-56
+29
5
34
61
63
117
92
61
63
92
5
117
34

23

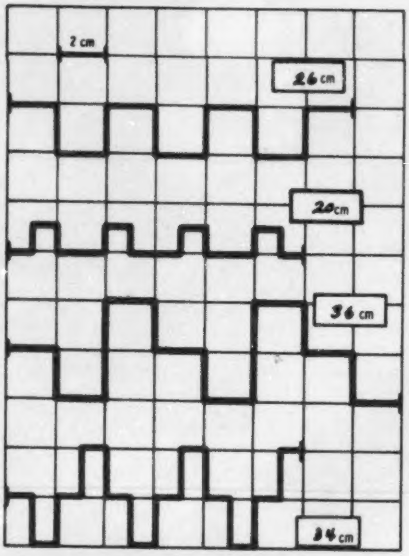

multiples of 4
multiples of 6
24+40
120
6×7
4×17
12+36
30+60
7×13
29
6×7
29
30+60
24+40
120
4×17
12+36
7×13

24

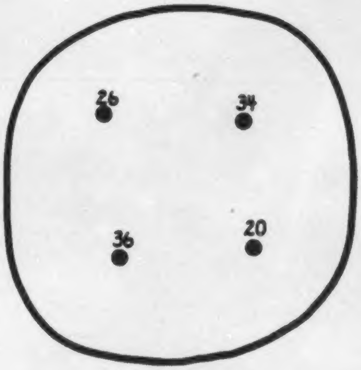

Two numbers may talk to each other
if and only if
one number is a multiple of the other.
24
15
4
3
33
12
48
2
24
15
4
3
12
2
33
48

25

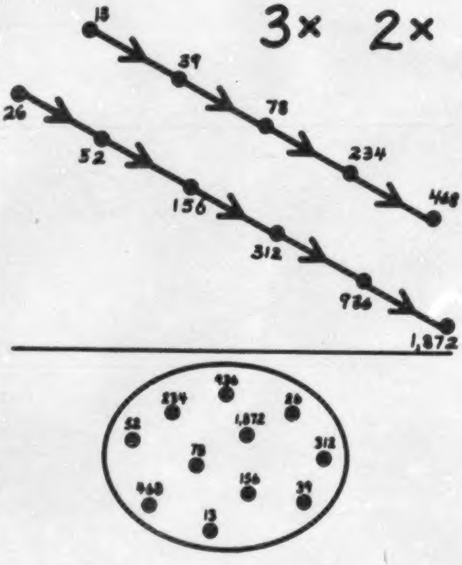
Other solutions are possible.


2 cm
26 cm
20 cm
36 cm
34 cm

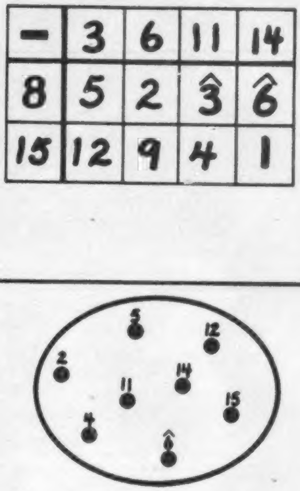
26



27


3× 2×
13
39
78
234
468
26
52
156
312
936
1,872
936
234
26
52
1,872
312
78
468
156
39
13

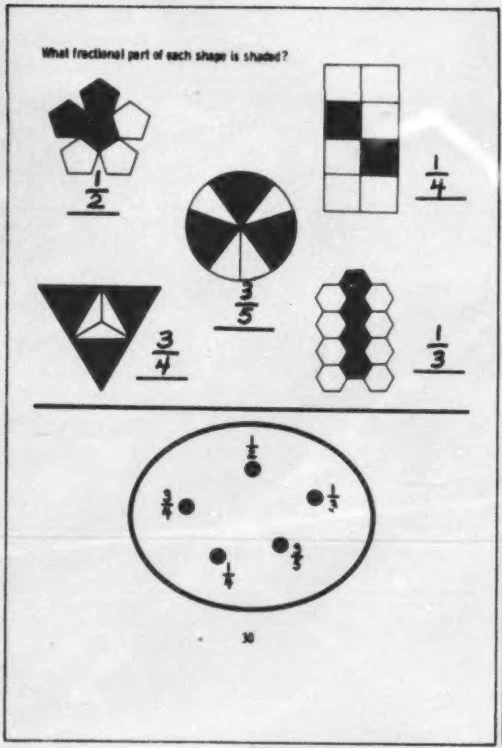
28



–	3	6	11	14
8	5	2	$\hat{3}$	$\hat{6}$
15	12	9	4	1

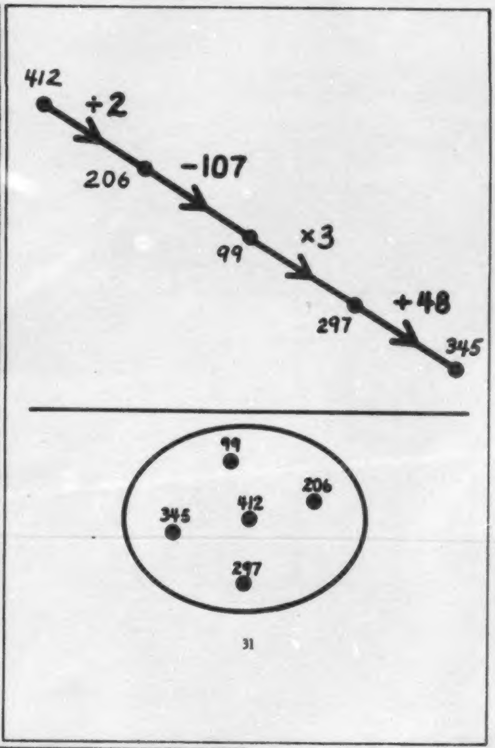
5
12
2
14
11
15
4
$\hat{6}$

29

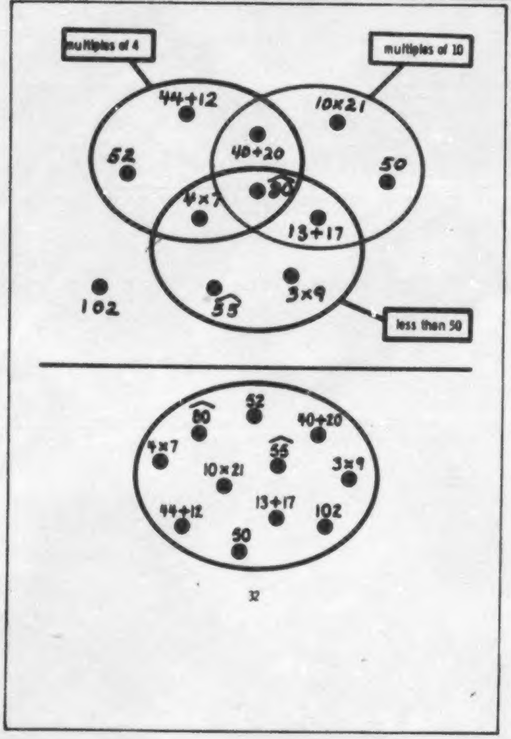

What fractional part of each shape is shaded?

$\frac{1}{2}$
$\frac{1}{4}$
$\frac{3}{5}$
$\frac{3}{4}$
$\frac{1}{3}$
$\frac{1}{2}$
$\frac{3}{4}$
$\frac{1}{3}$
$\frac{1}{4}$
$\frac{3}{5}$

30


412
÷2
206
−107
99
×3
297
+48
345
99
206
412
345
297

31


multiples of 4
multiples of 10
44+12
10×21
52
40+20
50
4×7
80
13+17
102
55
3×9
less than 50
80
52
40+20
4×7
55
10×21
3×9
44+12
13+17
102
50

32

W17 A Short Story about Three Friends

CAPSULE LESSON SUMMARY

Solve a detective story for three secret numbers. The first clue tells their positions in a +2 and +3 arrow picture, and the second clue says that one of the three numbers is 3 × another. Play the Number Line Game with non-integer decimal numbers as the secret numbers.

MATERIALS

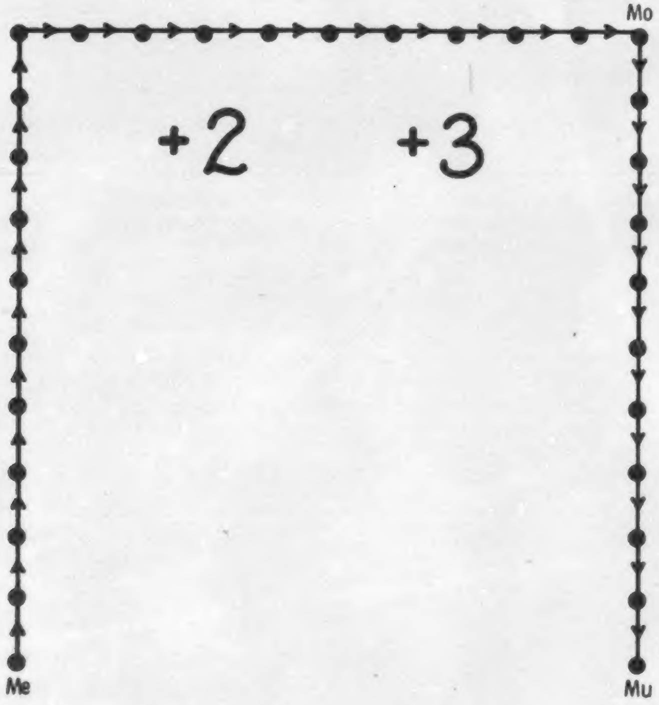
Teacher: Colored chalk

Student : Worksheet W17 ; pencil

DESCRIPTION OF LESSON

Exercise 1: Detective story

Draw this arrow picture on the board and distribute copies of Worksheet W17 to the students.



W 17

T: Today you are going to be detectives again. Your assignment is to discover the identities of three whole numbers whose code names are "Me", "Mo", and "Mu". The first clue is this arrow picture. What does this arrow picture tell us about the three secret numbers?

The students should observe that Me is the smallest and Mu is the largest of the three secret numbers. If necessary, ask where the largest and smallest numbers are in this arrow picture.

Perhaps a student will explain that Mo is 40 more than Me, and Mu is 30 more than Mo. Hence Mu is 70 more than Me. Commend any student who makes this observation but do not suggest this yourself.

T: This is the same arrow picture as the arrow picture on your worksheet. How many arrows are there of each color?

S: There are twenty red arrows and ten blue arrows.

T: If Me were 100, what would Mo be and what would Mu be?

Draw this table on the board near the arrow picture.

Me	Mo	Mu
100		

S: If Me were 100, Mo would be 140 and Mu would be 170.

Record this information in the table on the board and ask the students to record it in the table on their worksheets.

Me	Mo	Mu
100	140	170

T: If Me were 30, what numbers would Mo and Mu be?

Let the students write their answers in the table on their worksheets. Look at several of the students' answers before letting someone answer aloud.

S: If Me were 30, then Mo would be 70 and Mu would be 100.

Record this information in the table on the board. Repeat this activity assuming that Me is 52, 201, and 18, respectively.

Me	Mo	Mu
100	140	170
30	70	100
52	92	122
201	241	271
18	58	88

T: What do you notice about Me, Mo, and Mu?

W 17

Allow the students to express themselves freely. Your students very likely will observe that:

- Me + 40 = Mo
- Mo + 30 = Mu
- Me + 70 = Mu
- The ones digit of all three numbers is the same.

Trace an arrow from Me to Mo.

T: What could this arrow be for? How do you know?

S: It could be for +40 because Me + 40 = Mo.

S: Ten +2 arrows are the same as one +20 arrow, so twenty +2 arrows are the same as one +40 arrow.

Trace an arrow from Mo to Mu.

T: What could this arrow be for? How do you know?

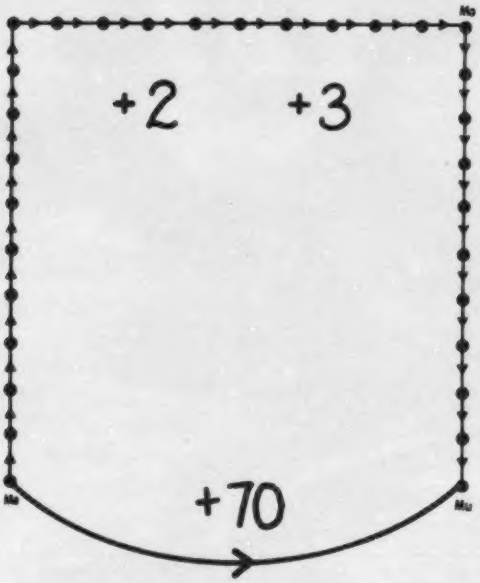
S: +30, because there are ten +3 arrows.

Draw an arrow in green from Me to Mu.

T: What could this arrow be for? How do you know?

S: +70, because 40 + 30 = 70.

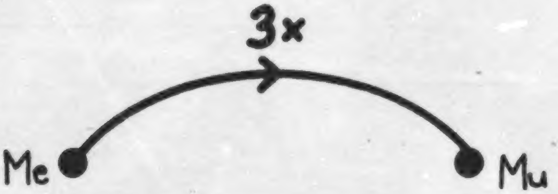
Label the arrow from Me to Mu "+70".



T: If Me were 10, what number would Mu be? (80)
If Me were 105, . . . ? (175)
If Me were 37, . . . ? (107)
If Me were 99, . . . ? (169)

Ask one or two students to explain how they calculated "99 + 70". If no one suggests that 99 = 100 - 1 so 99 + 70 = (100 + 70) - 1, suggest this yourself.

Draw this arrow picture on another part of the board.



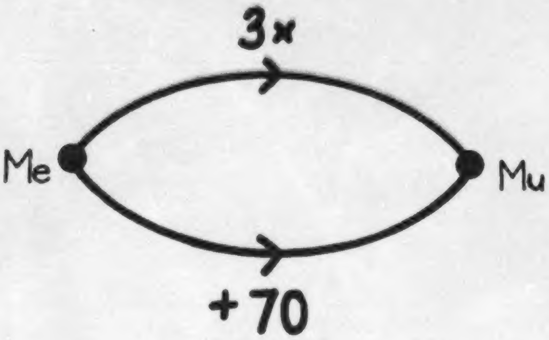
W 17

T: What does this clue tell us about Me and Mu?

S: Mu is three times as large as Me.

T: If Me were 30, what number would Mu be? (90)
If Me were 25, . . . ? (75)
If Me were 40, . . . ? (120)
If Me were 50, . . . ? (150)
If Me were 75, . . . ? (225)

Draw a green +70 arrow from Me to Mu.



T: Our first clue told us that Mu is 70 more than Me.
Now we know that Mu is also three times Me.
Could Me be 20? (No)
Why not?

S: Because 3 × 20 = 60 and 20 + 70 = 90. Mu cannot be both 60 and 90 at the same time.

T: Could Me be 30? (No)
Why not?

S: Because 3 × 30 = 90 and 30 + 70 = 100.

T: Could Me be 40? (No)
Why not?

S: Because 3 × 40 = 120 and 40 + 70 = 110.

T: Who are Me and Mu? Write your answer on your worksheet.

Look at many of the students' answers before asking a student to answer aloud.

S: Me is 35 and Mu is 105.

T: What number is 3 × 35? (105)
What number is 35 + 70? (105)

Label the dots for Me and Mu in both arrow pictures.

T: If Me is 35 and Mu is 105, what number is Mo?

S: 75.

Label the dot for Mo and conclude that 35, 75, and 105 are the secret numbers.

Exercise 2: Number line game

Play the Number Line Game with your class. Choose non-integer decimal numbers to be your secret numbers. Begin with a secret number between 0 and 1. If you play a second game, choose a secret number between 0 and 10. Adjust the difficulty of the game to the abilities of your students.